Ricci Solitons on submanifolds of $(LCS)_n$-Manifolds.

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Abstract

The present paper deals with the study of Ricci solitons on invariant and anti-invariant submanifolds of $(LCS)_n$-manifolds with respect to Riemannian connection as well as quarter symmetric metric connection.

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1 Introduction

In 1982, Hamilton [7] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for manifolds with positive curvature. Perelman [22] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism

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and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [8]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [29] Sharma studied the Ricci solitons in contact geometry. Thereafter, Ricci solitons in contact metric manifolds have been studied by various authors such as Bejan and Crasmareanu [1], Hui et al. ([2],[13]-[15], [17]), Chen and Deshmukh [3], Deshmukh et al. [4], He and Zhu [10], Tripathi [30] and many others.

In 2003, Shaikh [23] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [19] and also by Mihai and Rosca [20]. Then Shaikh and Baishya ([25], [26]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds is also studied by Hui [11], Hui and Atceken [12], Shaikh and his co-authors ([24]-[28]) and many others.

In modern analysis, the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and theoretical physics. The present paper deals with the study of Ricci solitons on submanifolds of $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 is devoted to the study of Ricci solitons on invariant and anti-invariant submanifolds of $(LCS)_n$-manifolds.

In 1924, Friedman and Schouten [5] introduced the notion of semi-symmetric linear connection on a differentiable manifold. In 1932, Hayden [9] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, Yano [31] studied some curvature tensors and conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Golab [6] defined and studied quarter symmetric linear connection on a differentiable manifold. A linear connection $\overline{\nabla}$ in an n-dimensional Riemannian manifold is said to be a quarter symmetric connection [6] if torsion tensor T is of the form

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] = A(Y)K(X) - A(X)K(Y) \tag{1.2}$$

where A is an 1-form and K is a tensor of type (1,1). If a quarter symmetric linear connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y, Z) = 0$$

for all X, Y, $Z \in \chi(M)$, where $\chi(M)$ is a Lie algebra of vector fields on the manifold M, then $\overline{\nabla}$ is said to be a quarter symmetric metric connection. For a contact metric manifold admitting quarter symmetric connection, we can take $A = \eta$ and $K = \phi$ and hence (1.2) takes in the form:

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{1.3}$$

The relation between Levi-Civita connection ∇ and quarter symmetric metric connection $\overline{\nabla}$ of a contact metric manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.4}$$

Recently Hui, Piscoran and Pal [16] studied invariant submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric metric connection. Ricci solitons on invariant and anti-invariant submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric metric connections are studied in section 4 of the paper.

2 preliminaries

An n-dimensional Lorentzian manifold $\widetilde{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, $\widetilde{M}$ admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in \widetilde{M}$, the tensor $g_p : T_p\widetilde{M} \times T_p\widetilde{M} \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where $T_p\widetilde{M}$ denotes the tangent vector space of $\widetilde{M}$ at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p\widetilde{M}$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, $=0$, > 0) [21].

Definition 2.1. *In a Lorentzian manifold $(\widetilde{M}, g)$ a vector field P defined by*

$$g(X, P) = A(X)$$

for any $X \in \Gamma(T\widetilde{M})$, is said to be a concircular vector field [32] if

$$(\widetilde{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form and $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let $\widetilde{M}$ be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\widetilde{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad (\alpha \neq 0) \tag{2.3}$$

$$\widetilde{\nabla}_X\xi = \alpha\{X + \eta(X)\xi\}, \quad \alpha \neq 0, \tag{2.4}$$

for all vector fields X, Y, where $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\widetilde{\nabla}_X\alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. Let us take

$$\phi X = \frac{1}{\alpha}\widetilde{\nabla}_X\xi, \tag{2.6}$$

then from (2.4) and (2.6) we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.8}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold $\widetilde{M}$ together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [23]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [19]. In a $(LCS)_n$-manifold $(n > 2)$, the following relations hold [23]:

$$\eta(\xi) = -1, \ \ \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.9}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.10}$$

$$\widetilde{S}(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.11}$$

$$\widetilde{R}(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.12}$$

$$\widetilde{R}(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.13}$$

$$(\widetilde{\nabla}_X\phi)Y = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.15}$$

$$\widetilde{R}(X, Y)Z = \phi\widetilde{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \tag{2.16}$$

for all $X,\ Y,\ Z \in \Gamma(T\widetilde{M})$ and $\beta = -(\xi\rho)$ is a scalar function, where $\widetilde{R}$ is the curvature tensor and $\widetilde{S}$ is the Ricci tensor of the manifold.

Let M be a submanifold of dimension m of a $(LCS)_n$-manifold $\widetilde{M}$ $(m < n)$ with induced metric g. Also let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.17}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.18}$$

for all $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\widetilde{M}$. The second fundamental form h and the shape operator A_V are related by [33]

$$g(h(X,Y),V) = g(A_V X, Y), \tag{2.19}$$

for any $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. We note that $h(X,Y)$ is bilinear and since $\nabla_{fX}Y = f\nabla_X Y$ for any smooth function f on a manifold, we have

$$h(fX,Y) = fh(X,Y). \tag{2.20}$$

The mean curvature vector H on M is given by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i,e_i)$, where $\{e_1, e_2, \cdots, e_m\}$ is a local orthonormal frame of vector fields on M. A submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H \tag{2.21}$$

for any vector fields $X,\ Y \in TM$. Moreover if $h(X,Y) = 0$ for all $X,\ Y\ \in TM$, then M is said to be totally geodesic and if $H = 0$ then M is minimal in $\widetilde{M}$.

Analogous to almost Hermitian manifolds, the invariant and anti-invariant submanifols are depend on the behaviour of almost contact metric structure ϕ.

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for every vector field X tangent to M at evey point of M. i.e. $\phi(TM) \subset TM$ at evey point of M.

On the other hand, M is said to be anti-invariant if for any X tangent to M, ϕX is normal to M, i.e., $\phi(TM) \subset T^{\perp}M$ at every point of M, where $T^{\perp}M$ is the normal bundle of M.

Let $\overline{\widetilde{\nabla}}$ be a linear connection and $\widetilde{\nabla}$ be the Levi-Civita connection of a $(LCS)_n$-manifold $\widetilde{M}$ such that

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + U(X,Y), \tag{2.22}$$

where U is a (1,1) type tensor and $X,\ Y \in \Gamma(T\widetilde{M})$. For $\overline{\widetilde{\nabla}}$ to be a quarter symmetric metric connection on $\widetilde{M}$, we have

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{2.23}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{2.24}$$

From (1.3) and (2.24) we get

$$T'(X,Y) = \eta(X)\phi Y - g(Y,\phi X)\xi. \tag{2.25}$$

So,

$$U(X,Y) = \eta(Y)\phi X - g(Y,\phi X)\xi. \tag{2.26}$$

Therefore a quarter symmetric metric connection $\overline{\widetilde{\nabla}}$ in a $(LCS)_n$-manifold $\widetilde{M}$ is given by

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.27}$$

Let $\overline{\widetilde{R}}$ and $\widetilde{R}$ be the curvature tensors of a $(LCS)_n$-manifold $\widetilde{M}$ with respect to the quarter symmetric metric connection $\overline{\widetilde{\nabla}}$ and the Levi-Civita connection $\widetilde{\nabla}$ respectively. Then we have

$$\begin{aligned} \overline{\widetilde{R}}(X,Y)Z \;=\; & \widetilde{R}(X,Y)Z + (2\alpha - 1)\left[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X\right] \\ & +\alpha\left[\eta(Y)X - \eta(X)Y\right]\eta(Z) \\ & +\alpha\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\xi, \end{aligned} \tag{2.28}$$

where $\overline{\widetilde{R}}(X,Y)Z = \overline{\widetilde{\nabla}}_X\overline{\widetilde{\nabla}}_Y Z - \overline{\widetilde{\nabla}}_Y\overline{\widetilde{\nabla}}_X Z - \overline{\widetilde{\nabla}}_{[X,Y]}Z$ and $X,\ Y,\ Z \in \chi(\widetilde{M})$.

By suitable contraction we have from (2.28) that

$$\begin{aligned} \overline{\widetilde{S}}(Y,Z) \;=\; & \widetilde{S}(Y,Z) + (\alpha - 1)g(Y,Z) + (n\alpha - 1)\eta(Y)\eta(Z) \\ & -(2\alpha - 1)ag(\phi Y,Z), \end{aligned} \tag{2.29}$$

where $\overline{\widetilde{S}}$ and $\widetilde{S}$ are the Ricci tensors of $\widetilde{M}$ with respect to $\overline{\widetilde{\nabla}}$ and $\widetilde{\nabla}$ respectively and $a = trace\phi$.

3 Ricci solitons on submanifolds of $(LCS)_n$-Manifolds

Let us take (g,ξ,λ) be a Ricci soliton on a submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$. Then we have

$$(\pounds_\xi g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{3.1}$$

From (2.6) and (2.17) we get

$$\alpha\phi X = \widetilde{\nabla}_X\xi = \nabla_X\xi + h(X,\xi). \tag{3.2}$$

If M is invariant in $\widetilde{M}$, then $\phi X, \xi \in TM$ and therefore equating tangential and normal components of (3.2) we get

$$\nabla_X\xi = \alpha\phi X \quad \text{and} \quad h(X,\xi) = 0. \tag{3.3}$$

From (2.1), (2.2), (2.7) and (3.3) we get

$$\begin{aligned} (\pounds_\xi g)(Y,Z) &= g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi) \\ &= 2\alpha[g(Y,Z) + \eta(Y)\eta(Z)]. \end{aligned} \tag{3.4}$$

In view of (3.4), (3.1) yields

$$S(Y,Z) = -(\alpha+\lambda)g(Y,Z) - \alpha\eta(Y)\eta(Z), \tag{3.5}$$

which implies that M is η-Einstein. Also from (2.20) and (3.3) we get $\eta(X)H = 0$, i.e., $H = 0$, since $\eta(X) \neq 0$.
Consequently M is minimal in $\widetilde{M}$. Thus we can state the following:

Theorem 3.1. *If (g,ξ,λ) is a Ricci soliton on an invariant submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$, then M is η-Einstein and also M is minimal in $\widetilde{M}$.*

From (3.3) and using the formula

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$$

we get

$$R(X,Y)\xi = (\alpha^2-\rho)[\eta(Y)X - \eta(X)Y]$$

from which it follows that

$$S(X,\xi) = (m-1)(\alpha^2-\rho)\eta(X) \quad \text{for all X.} \tag{3.6}$$

Putting $Z = \xi$ in (3.5) and using (2.2) and (3.6) we get $\lambda = -(m-1)(\alpha^2-\rho)$. This leads to the following:

Theorem 3.2. *A Ricci soliton (g,ξ,λ) on an invariant submanifold of a $(LCS)_n$-manifold is shrinking, steady and expanding according as $\alpha^2-\rho < 0$, $\alpha^2-\rho = 0$ and $\alpha^2-\rho > 0$ respectively.*

Again, if M is anti-invariant in $\widetilde{M}$, then for any $X \in TM$, $\phi X \in T^\perp M$ and hence from (3.2) we get $\nabla_X\xi = 0$ and $h(X,\xi) = \alpha\phi X$. Then

$$(\pounds_\xi g)(Y,Z) = g(\nabla_Y\xi, Z) + (Y, \nabla_Z\xi) = 0,$$

which means that ξ is a Killing vector field and consequently (3.1) yields

$$S(Y,Z) = -\lambda g(Y,Z),$$

which implies that M is Einstein. Thus we can state the following:

Theorem 3.3. *If (g, ξ, λ) is a Ricci soliton on an anti-invariant submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$, then M is Einstein and ξ is Killing vector field.*

Also, from $\nabla_X \xi = 0$ we get $R(X,Y)\xi = 0$ and hence $S(Y,\xi) = 0$. Again, we have $S(Y,\xi) = -\lambda\eta(Y)$. Therefore $\lambda = 0$ and hence the Ricci soliton (g, ξ, λ) is always steady. This leads to the following:

Theorem 3.4. *A Ricci soliton (g, ξ, λ) on an anti-invariant submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$ is always steady.*

4 Ricci solitons on submanifolds of $(LCS)_n$-Manifolds with respect to quarter symmetric metric connection

We now consider (g, ξ, λ) is a Ricci soliton on a submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$ with respect to quarter symmetric metric connection, where $\overline{\nabla}$ is the induced connection on M from the connection $\overline{\widetilde{\nabla}}$. Then we have

$$(\overline{\pounds}_\xi g)(Y,Z) + 2\overline{S}(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{4.1}$$

Let $\overline{h}$ be the second fundamental form of M with respect to induced connection $\overline{\nabla}$. Then we have

$$\overline{\widetilde{\nabla}}_X Y = \overline{\nabla}_X Y + \overline{h}(X,Y) \tag{4.2}$$

and hence by virtue of (2.17) and (2.27) we get

$$\overline{\nabla}_X Y + \overline{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{4.3}$$

If M is invariant submanifold of $\widetilde{M}$ then $\phi X,\ \xi \in TM$ for any $X \in TM$ and therefore equating tangential part from (4.1) we get

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{4.4}$$

which means M admits quarter symmetric metric connection.
Also from (4.4) we get $\overline{\nabla}_X \xi = (\alpha - 1)\phi X$ and hence

$$\begin{aligned} (\overline{\pounds}_\xi g)(Y,Z) &= g(\overline{\nabla}_Y \xi, Z) + g(Y, \overline{\nabla}_Z \xi) \\ &= 2(\alpha - 1)[g(Y,Z) + \eta(Y)\eta(Z)]. \end{aligned} \tag{4.5}$$

If $\overline{R}$ be the curvature tensor of submanifold M with respect to induced connection $\overline{\nabla}$ of a $(LCS)_n$-manifold $\widetilde{M}$ with respect to quarter symmetric metric connection $\overline{\widetilde{\nabla}}$. Then we have,

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + (2\alpha - 1)\left[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X\right] \\ &\quad + \alpha\left[\eta(Y)X - \eta(X)Y\right]\eta(Z) \\ &\quad + \alpha\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\xi, \end{aligned} \tag{4.6}$$

where $\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z$. Taking suitable contraction of above equation, we get

$$\overline{S}(Y,Z) = S(Y,Z) + [\alpha(1-2a)+a]g(Y,Z) + [\alpha(m-2a)+a-1]\eta(Y)\eta(Z). \tag{4.7}$$

Using (4.5) and (4.7) in (4.1), we get

$$S(Y,Z) = [2\alpha(a-1)+1-a-\lambda]g(Y,Z) + [\alpha(2a-m-1)+2-a]\eta(Y)\eta(Z), \tag{4.8}$$

which implies that M is η-Einstein.

Theorem 4.1. *Let (g,ξ,λ) be a Ricci soliton on an invariant submaniold M of a $(LCS)_n$-manifold $\widetilde{M}$ with respect to quarter symmetric metric connection $\overline{\overline{\nabla}}$. Let $\overline{\nabla}$ be the induced connection on M from the connection $\overline{\overline{\nabla}}$. Then M is η-Einstein with respect to Levi-Civita connection.*

Again, if M is an anti-invariant submanifold of $\widetilde{M}$ with respect to quarter symmetric metric connection, then from (4.3) we get $\overline{\nabla}_X\xi = 0$. Consequently we get

$$(\overline{\pounds}_\xi g)(Y,Z) = 0. \tag{4.9}$$

In view of (4.9), (4.1) yields

$$\overline{S}(Y,Z) = -\lambda g(Y,Z),$$

which implies that M is η-Einstein with respect to Riemannian connection by virtue of (4.7). Thus we can state the following:

Theorem 4.2. *Let (g,ξ,λ) be a Ricci soliton on an anti-invariant submaniold $\widetilde{M}$ of a $(LCS)_n$-manifold $\widetilde{M}$ with respect to quarter symmetric metric connection $\overline{\overline{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

5 Conclusion

In this paper, we have studied invariant and anti-invariant submanifolds of $(LCS)_n$-manifold $\widetilde{M}$ whose metric are Ricci solitons. From Theorem 3.1, Theorem 3.3, Theorem 4.1 and Theorem 4.2, we can state the following:

Theorem 5.1. Let (g,ξ,λ) be a Ricci soliton on a submanifold M of a $(LCS)_n$-manifold $\widetilde{M}$. Then the following holds:

nature of submanifold M	connection of $\widetilde{M}$	M
invariant	Riemannian	η-Einstein
anti-invariant	Riemannian	Einstein
invariant	quarter symmetric metric	η-Einstein
anti-invariant	quarter symmetric metric	η-Einstein

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