Bi-Slant pseudo-Riemannian submersions from indefinite almost Hermitian manifolds onto pseudo-Riemannian manifolds

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Abstract

In this paper, we introduce the notion of a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold onto a pseudo-Riemannian manifold as a generalization of slant and semi-slant submersions. We investigate the geometry of foliations determined by horizontal and vertical distributions and provide a non-trivial example. We also obtain a necessary and sufficient condition for submersions to be totally geodesic and check the harmonicity of such submersions.

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1 Introduction

The theory of Riemannian submersions was independently introduced by O' Neill [13] in 1966 and Gray [8] in 1967. Later such submersions have been studied by several geometers ([7], [12], [13], [16]). It is known that Riemannian submersions are related with physics and have their applications in Kaluza-Klein theory ([5], [10]), Yang-Mills theory ([4], [22]), the theory of robotics ([1]), supergravity and superstring theories ([10], [11]).
In 1976, B. Watson defined almost Hermitian submersions between almost Hermitian manifolds and proved that the base manifold and each fibre have the same kind of structure as the total space, in most cases [21]. In 2010, Sahin introduced anti-invariant and semi-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds [17], [18]. He also gave the notion of a slant submersion as a generalization of Hermitian and anti-invariant submersions [19]. In 2012, K. S. Park also studied H-slant and V-slant

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submersions and investigated interesting geometric properties of such submersions [14], [15].
In the present paper, our aim is to study bi-slant pseudo-Riemannian submersions from indefinite almost Hermitian manifolds onto pseudo-Riemannian manifolds.
The composition of the paper is as follows. In section 2, we collect some basic definitions, formulas and results on indefinite almost Hermitian manifolds and pseudo-Riemannian submersions. In section 3, we define bi-slant pseudo-Riemannian submersions from indefinite almost Hermitian manifolds onto pseudo-Riemannian manifolds. We investigate the geometry of foliations determined by horizontal and vertical distributions and provide a non-trivial example. We also obtain a necessary and sufficient condition for submersions to be totally geodesic and check the harmonicity of such submersions.

2 Preliminaries:

2.1 Indefinite Almost Hermitian Manifolds

A $(1,1)$-type tensor field J on a $2m$-dimensional smooth manifold M is said to be an almost complex structure if $J^2 = -I$ and then (M, J) is called an almost complex manifold.
An almost complex manifold (M, J) is such that the two eigen bundles T^+M and T^-M corresponding to respective eigen values $+1$ and -1 of J have the same rank.
An indefinite almost Hermitian manifold (M, J, g) is a smooth manifold endowed with an almost complex structure J and a pseudo-Riemannian metric g such that

$$g(JX, JY) = g(X, Y), \qquad \text{for all } X, Y \in \Gamma(TM). \tag{2.1}$$

Here, the metric g is neutral, i.e., g has signature (m, m).
The fundamental 2-form of the almost Hermitian manifold is defined by

$$F(X, Y) = g(X, JY), \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$.
For an almost Hermitian manifold (M, J, g), we have

$$g(JX, Y) = -g(X, JY), \tag{2.3}$$

$$F(X, Y) = -F(Y, X), \tag{2.4}$$

$$F(JX, JY) = F(X, Y), \tag{2.5}$$

$$\begin{aligned} 3dF(X, Y, Z) = {} & X(F(Y, Z)) - Y(F(X, Z)) + Z(F(X, Y)) \\ & - F([X, Y], Z) + F([X, Z], Y) - F([Y, Z], X), \end{aligned} \tag{2.6}$$

$$(\nabla_X F)(Y, Z) = g(Y, (\nabla_X J)Z) = -g(Z, (\nabla_X J)Y), \tag{2.7}$$

$$3dF(X, Y, Z) = (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y), \tag{2.8}$$

$$\text{The co-differential, } (\delta F)(X) = \sum_{i=1}^{2m} \varepsilon_i (\nabla_{e_i} F)(e_i, X). \tag{2.9}$$

for all $X, Y \in \Gamma(TM)$.
An indefinite almost Hermitian manifold (M, J, g) is called

(i) Hermitian, if $N_J = 0$, equivalently, $(\nabla_{JX} J)JY + (\nabla_X J)Y = 0$,

(ii) Kähler, if for any $X \in \Gamma(TM)$, $\nabla_X J = 0$, i.e., $\nabla J = 0$,

(iii) almost kähler, if $dF = 0$,

(iv) nearly Kähler, if $(\nabla_X J)X = 0$,

(v) almost semi-Kähler, if $\delta F = 0$,

(vi) semi para- Kähler, if $\delta F = 0$ and $N_J = 0$.

2.2 Pseudo-Riemannian Submersions:

Let $(\bar{M}^m,\ \bar{g})$ and $(M^n,\ g)$ be two connected pseudo-Riemannian manifolds of indices $\overline{s}$ $(0 \leq \overline{s} \leq m)$ and s $(0 \leq s \leq n)$ respectively, with $\overline{s} > s$.
A pseudo-Riemannian submersion is a smooth map $f : \bar{M}^m \longrightarrow M^n$, which is onto and satisfies the following conditions ([7], [8], [13], [16]):

(i) the derivative map $f_{*_p} : T_p\bar{M} \longrightarrow T_{f(p)}M$ is surjective at each point $p \in \bar{M}$;

(ii) the fibres $f^{-1}(q)$ of f over $q \in M$ are pseudo-Riemannian submanifolds of $\bar{M}$;

(iii) f_* preserves the length of horizontal vectors.

A vector field on $\bar{M}$ is called vertical if it is always tangent to fibres and it is called horizontal if it is always orthogonal to fibres. We denote by $\mathcal{V}$ the vertical distribution and by $\mathcal{H}$ the horizontal distribution. Also, we denote vertical and horizontal projections of a vector field E on $\bar{M}$ by vE and by hE respectively. A horizontal vector field $\bar{X}$ on $\bar{M}$ is said to be basic if $\bar{X}$ is f-related to a vector field X on M such that $f_*\bar{X} = X \circ f$. Thus, every vector field X on M has a unique horizontal lift $\bar{X}$ on $\bar{M}$.
We recall the following lemma for later use:

Lemma 2.1. *([7], [12]) If $f : \bar{M} \longrightarrow M$ is a pseudo-Riemannian submersion and $\bar{X}, \bar{Y}$ are basic vector fields on $\bar{M}$ that are f-related to the vector fields X, Y on M respectively, then we have the following properties:*

(i) $\bar{g}(\bar{X}, \bar{Y}) = g(X, Y) \circ f$,

(ii) $h[\bar{X}, \bar{Y}]$ is a vector field and $h[\bar{X}, \bar{Y}] = [X, Y] \circ f$,

(iii) $h(\bar{\nabla}_{\bar{X}}\bar{Y})$ is a basic vector field f-related to $\nabla_X Y$, where $\bar{\nabla}$ and ∇ are the Levi-Civita connections on $\bar{M}$ and M respectively,

(iv) $[E, U] \in \mathcal{V}$, for any vector field $U \in \mathcal{V}$ and for any vector field $E \in \Gamma(T\bar{M})$.

A pseudo-Riemannian submersion $f : \bar{M} \longrightarrow M$ determines tensor fields $\mathcal{T}$ and $\mathcal{A}$ of type $(1,2)$ on $\bar{M}$ defined by formulas ([7], [12], [13])

$$\mathcal{T}(E,F) = \mathcal{T}_E F = h(\bar{\nabla}_{vE} vF) + v(\bar{\nabla}_{vE} hF), \tag{2.10}$$

$$\mathcal{A}(E,F) = \mathcal{A}_E F = v(\bar{\nabla}_{hE} hF) + h(\bar{\nabla}_{hE} vF), \tag{2.11}$$

for any $E, F \in \Gamma(T\bar{M})$.
Let $\bar{X}, \bar{Y}$ be horizontal vector fields and U, V be vertical vector fields on $\bar{M}$. Then, we have

$$\mathcal{T}_U \bar{X} = v(\bar{\nabla}_U \bar{X}),\ \mathcal{T}_U V = h(\bar{\nabla}_U V), \tag{2.12}$$

$$\bar{\nabla}_U \bar{X} = \mathcal{T}_U \bar{X} + h(\bar{\nabla}_U \bar{X}), \tag{2.13}$$

$$\mathcal{T}_{\bar{X}} F = 0,\ \mathcal{T}_E F = \mathcal{T}_{vE} F, \tag{2.14}$$

$$\bar{\nabla}_U V = \mathcal{T}_U V + v(\bar{\nabla}_U V), \tag{2.15}$$

$$\mathcal{A}_{\bar{X}} \bar{Y} = v(\bar{\nabla}_{\bar{X}} \bar{Y}),\ \mathcal{A}_{\bar{X}} U = h(\bar{\nabla}_{\bar{X}} U), \tag{2.16}$$

$$\bar{\nabla}_{\bar{X}} U = \mathcal{A}_{\bar{X}} U + v(\bar{\nabla}_{\bar{X}} U), \tag{2.17}$$

$$\mathcal{A}_U F = 0,\ \mathcal{A}_E F = \mathcal{A}_{hE} F, \tag{2.18}$$

$$\bar{\nabla}_{\bar{X}} \bar{Y} = \mathcal{A}_{\bar{X}} \bar{Y} + h(\nabla_{\bar{X}} \bar{Y}), \tag{2.19}$$

$$h(\bar{\nabla}_U \bar{X}) = h(\bar{\nabla}_{\bar{X}} U) = \mathcal{A}_{\bar{X}} U, \tag{2.20}$$

$$\mathcal{A}_{\bar{X}} \bar{Y} = \frac{1}{2} v[\bar{X}, \bar{Y}], \tag{2.21}$$

$$\mathcal{A}_{\bar{X}} \bar{Y} = -\mathcal{A}_{\bar{Y}} \bar{X}, \tag{2.22}$$

$$\mathcal{T}_U V = \mathcal{T}_V U, \tag{2.23}$$

$\forall\ E, F \in \Gamma(T\bar{M})$.
It can be easily shown that a Riemannian submersion $f : \bar{M} \longrightarrow M$ has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically. By lemma (2.1), the horizontal distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A} = 0$. In view of equations (2.22) and (2.23), $\mathcal{A}$ is alternating on the horizontal distribution and $\mathcal{T}$ is symmetric on the vertical distribution.

Now, we recall the notion of harmonic maps between pseudo-Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and (M, g) be pseudo-Riemannian manifolds and let $f : \bar{M} \longrightarrow M$ be a smooth map. Then the second fundamental form of the map f is given by

$$(\bar{\nabla} f_*)(X, Y) = (\nabla^f_X f_* Y) \circ f - f_*(\bar{\nabla}_X Y) \tag{2.24}$$

for all $X, Y \in \Gamma(T\bar{M})$, where ∇^f denotes the pullback connection of ∇ with respect to f and the tension field τ of f is defined by

$$\tau(f) = trace(\bar{\nabla} f_*) = \sum_{i=1}^{m} (\bar{\nabla} f_*)(e_i, e_i), \tag{2.25}$$

where $\{e_1, e_2, \cdots e_m\}$ is an orthonormal frame on $\bar{M}$.
It is known that f is harmonic if and only if $\tau(f) = 0$ [6].
In this paper, we study pseudo Riemannian submersions $f : \bar{M} \longrightarrow M$ such that fibres $f^{-1}(q)$ over $q \in M$ be pseudo-Riemannian submanifolds admitting non-lightlike vector fields.

3 Bi-slant pseudo-Riemannian submersions

Definition 3.1. *Let $(\bar{M}^{2m}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and (M^n, g) be a pseudo-Riemannian manifold where $m > n$. A pseudo-Riemannian submersion $f: \bar{M} \longrightarrow M$ is called a bi-slant pseudo-Riemannian submersion if there exist two orthogonal distributions $\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2 \subseteq \ker f_*$ such that*

(i) $\ker f_* = \bar{\mathcal{D}}_1 \oplus \bar{\mathcal{D}}_2$;

(ii) for any non-zero vector field $\bar{X}_{1_p} \in \bar{\mathcal{D}}_{1_p}$, the angle θ_1 between $\bar{J}\bar{X}_{1_p}$ and the space $\bar{\mathcal{D}}_{1_p}$ is constant;

(iii) for any non-zero vector field $\bar{X}_{2_p} \in \bar{\mathcal{D}}_{2_p}$, the angle θ_2 between $\bar{J}\bar{X}_{2_p}$ and the space $\bar{\mathcal{D}}_{2_p}$ is constant.

These angles θ_1 and θ_2 are called slant angles of the submersion, which does not depend on the point p.

We observe that

(i) If $dim\bar{\mathcal{D}}_1 = 0$ and $\theta_2 = 0$, then f is an invariant pseudo-Riemannian submersion.

(ii) If $dim\bar{\mathcal{D}}_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then f is an anti-invariant pseudo-Riemannian submersion.

(iii) If $dim\bar{\mathcal{D}}_1 \neq 0 \neq dim\bar{\mathcal{D}}_2, \theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then f is a semi-invariant pseudo-Riemannian submersion.

(iv) If $dim\bar{\mathcal{D}}_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then f is a proper slant pseudo-Riemannian submersion.

(v) If $dim\bar{\mathcal{D}}_1 \neq 0 \neq dim\bar{\mathcal{D}}_2, \theta_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then f is a semi-slant pseudo-Riemannian submersion.

(vi) If $dim\bar{\mathcal{D}}_1 \neq 0 \neq dim\bar{\mathcal{D}}_2, \theta_1 = \frac{\pi}{2}, 0 < \theta_2 < \frac{\pi}{2}$, then f is a pseudo-slant pseudo-Riemannian submersion.

For any vector field $U \in \mathcal{V}$, we put

$$U = PU + QU \tag{3.1}$$

where $PU \in \bar{\mathcal{D}}_1, QU \in \bar{\mathcal{D}}_2$.

Also, for vector fields $U \in \bar{\mathcal{D}}_1, V \in \bar{\mathcal{D}}_2$, we set

$$\bar{J}U = \psi_1 U + \omega_1 U, \tag{3.2}$$

$$\bar{J}V = \psi_2 V + \omega_2 V \tag{3.3}$$

where $\psi_1 U, \psi_2 V \in \bar{\mathcal{D}}_1$ and $\omega_1 U, \omega_2 V \in \bar{\mathcal{D}}_2$ and for any $\bar{X} \in \mathcal{H}$, we put

$$\bar{J}\bar{X} = t\bar{X} + n\bar{X}, \tag{3.4}$$

where $t\bar{X} \in \mathcal{H}$ and $n\bar{X} \in \mathcal{V}$.

Proposition 3.1. *Let $f : \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). If θ_1 and θ_2 are bi-slant angles of the submersion, then for $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in \mathcal{H}$,*

$$\bar{g}(\omega_1 PU + \omega_2 QU, V) = -\bar{g}(U, \omega_1 PV + \omega_2 QV), \tag{3.5}$$

$$\bar{g}(n\bar{X}, U) = -\bar{g}(\bar{X}\psi_i PU + \psi_2 QU), \tag{3.6}$$

$$\bar{g}(t\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, t\bar{Y}). \tag{3.7}$$

Proof. For $U, V \in \mathcal{V}$, equation (2.3) implies

$$\bar{g}(\bar{J}(PU + QU),\ V) = \bar{g}(U,\ \bar{J}(PV + QV)).$$

By using equations (3.2) and (3.3), above equation gives

$$\bar{g}(\psi_1 PU + \omega_1 PU + \psi_2 QU + \omega_2 QU,\ V) = \bar{g}(U,\ \psi_1 PV + \omega_1 PV + \psi_2 QV + \omega_2 QV),$$

which implies equation (3.5).
Similarly, we can obtain other equations. □

Theorem 3.1. *Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and (M, g) be a pseudo-Riemannian manifold. Then, a pseudo-Riemannian submersion $f : \bar{M} \longrightarrow M$ is a bi-slant pseudo-Riemannian submersion if and only if there exist $\lambda_1, \lambda_2 \in [0, 1]$ such that*

$$(P\omega_1)^2 = \lambda_1 \bar{J}^2, \tag{3.8}$$

and

$$(Q\omega_2)^2 = \lambda_2 \bar{J}^2. \tag{3.9}$$

Moreover, if θ_1 and θ_2 are bi-slant angles of the submersion, then $\lambda_1 = \cos^2\theta_1$ and $\lambda_2 = \cos^2\theta_2$.

Proof. Let $U \in \bar{\mathcal{D}}_1$. Then,

$$\cos\theta_1 = \frac{\bar{g}(\bar{J}U, P\omega_1 U)}{|\bar{J}U||P\omega_1 U|}. \tag{3.10}$$

Again,

$$\cos\theta_1 = \frac{|P\omega_1 U|}{|\bar{J}U|}. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\cos^2\theta_1 = \frac{-\bar{g}(U,\ (P\omega_1)^2 U)}{-\bar{g}(U,\ \bar{J}^2 U)}. \tag{3.12}$$

Now, equation (3.12) implies that $\cos^2\theta_1$ is constant if and only if $(P\omega_1)^2$ and $\bar{J}^2$ are conformally parallel. Thus, there is $\lambda_1 \in [0,\ \infty)$ such that $(P\omega_1)^2 = \lambda_1 \bar{J}^2$.
Again, from equation (3.12), $\lambda_1 = \cos^2\theta_1$. So, $\lambda_1 \in [0, 1]$.
Similarly, we can prove $(Q\omega_2)^2 = \lambda_2 \bar{J}^2$, where $\lambda_2 \in [0, 1]$. □

Proposition 3.2. *Let $f : \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemnnian manifold (M, g). If θ_1 and θ_2 are bi-slant angles of the submersion, then for $U_1, U_2 \in \bar{\mathcal{D}}_1$ and $V_1, V_2 \in \bar{\mathcal{D}}_2$, we have*

$$\bar{g}(P\omega_1 U_1, U_2) = -\bar{g}(U_1, P\omega_1 U_2), \tag{3.13}$$

$$\bar{g}(P\omega_1 U_1, P\omega_1 U_2) = \bar{g}(U_1, U_2)\cos^2\theta_1, \tag{3.14}$$

$$\bar{g}(\psi_1 U_1, \psi_1 U_2) = \bar{g}(\bar{J}U_1, \bar{J}U_2)\sin^2\theta_1 - \bar{g}(P\omega_1 U_1, Q\omega_1 U_2), \tag{3.15}$$

$$\bar{g}(P\omega_2 V_1, V_2) = -\bar{g}(V_1, P\omega_2 V_2), \tag{3.16}$$

$$\bar{g}(P\omega_2 V_1, P\omega_2 V_2) = \bar{g}(V_1, V_2)\cos^2\theta_2. \tag{3.17}$$

$$\bar{g}(\psi_2 V_1, \psi_2 V_2) = \bar{g}(\bar{J}V_1, \bar{J}V_2)\sin^2\theta_2 - \bar{g}(P\omega_2 V_1, Q\omega_2 V_2). \tag{3.18}$$

Proof. Let $U_1, U_2 \in \bar{\mathcal{D}}_1$. Then, by using equations (2.3) and (3.2), we have

$$\bar{g}(\psi_1 U_1 + \omega_1 U_1, U_2) = -\bar{g}(U_1, \psi_1 U_2 + \omega_1 U_2).$$

In view of equation (3.1), above equation implies (3.13).
Now, replacing U_2 by $P\omega_1 U_2$ in equation (3.13), we get

$$\bar{g}(P\omega_1 U_1, P\omega_1 U_2) = -\bar{g}(U, P\omega_1 P\omega_1 U_2).$$

In view of equation (3.8), above equation gives

$$\bar{g}(P\omega_1 U_1, P\omega_1 U_2) = \bar{g}(\bar{J}U_1, \bar{J}U_2)\cos^2\theta_1,$$

which is equation (3.14).
Similarly, we can obtain other equations. □

Theorem 3.2. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, $(\psi_1\bar{\mathcal{D}}_1)^\perp$ and $(\psi_2\bar{\mathcal{D}}_2)^\perp$ are invariant with respect to $\bar{J}$.*

Proof. Let $U, V \in \bar{\mathcal{D}}_2$. Then, $\bar{J}V = \psi_2 V + P\omega_2 V + Q\omega_2 V$. We have

$$\begin{aligned} \bar{g}(\bar{J}U, Q\omega_2 V) &= -\bar{g}(U, \bar{J}Q\omega_2 V) \\ &= -\bar{g}(U, \psi_2 Q\omega_2 V + \omega_2 V) \\ &= -\bar{g}(U, \psi_2 Q\omega_2 V + P\omega_2 Q\omega_2 V + Q\omega_2 Q\omega_2 V) \\ &= 0. \end{aligned}$$

Also, for any $V \in \bar{\mathcal{D}}_2$,

$$\begin{aligned} \bar{g}(\bar{J}U, V) &= -\bar{g}(U, \bar{J}V) \\ &= -\bar{g}(U, \psi_2 V + P\omega_1 V + Q\omega_2 V) \\ &= 0. \end{aligned}$$

Hence, $\bar{J}U \in (\psi_2\bar{\mathcal{D}}_2)^\perp$ and $\bar{J}(\psi_2\bar{\mathcal{D}}_2)^\perp \subseteq (\psi_2\bar{\mathcal{D}}_2)^\perp$.
Similarly, we can show that $\bar{J}(\psi_1\bar{\mathcal{D}}_1)^\perp \subseteq (\psi_1\bar{\mathcal{D}}_1)^\perp$. □

Lemma 3.1. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in \mathcal{H}$, we have*

$$
\begin{aligned}
(3.19)\qquad & h\bar{\nabla}_{PU}(\psi_1 PV) + h\bar{\nabla}_{PU}(\omega_1 PV) + v\bar{\nabla}_{pU}(\psi_{PV}) + v\bar{\nabla}_{PU}(\omega_1 PV)\\
&+h\bar{\nabla}_{QU}(\psi_1 PV) + h\bar{\nabla}_{QU}(\omega_1 PV) + v\bar{\nabla}_{QU}(\psi_1 PV) + v\bar{\nabla}_{QU}(\omega_1 PV)\\
&+h\bar{\nabla}_{PU}(\psi_2 QV) + v\bar{\nabla}_{PU}(\psi_2 QV) + h\bar{\nabla}_{pU}(\omega_2 QV) + v\bar{\nabla}_{PU}(\omega_2 QV)\\
&+h\bar{\nabla}_{QU}(\psi_2 QV) + v\bar{\nabla}_{QU}(\psi_2 QU) + h\bar{\nabla}_{QU}(\omega_2 QV) + v\bar{\nabla}_{QU}(\omega_2 QV)\\
&= (\bar{\nabla}_U \bar{J})V + t(h\bar{\nabla}_U V) + n(h\bar{\nabla}_U V) + \psi_1 P(v\bar{\nabla}_U V)\\
&\quad +\omega_1 P(v\bar{\nabla}_U V) + \psi_2 Q(v\bar{\nabla}_U V) + \omega_2 Q(v\bar{\nabla}_U V);
\end{aligned}
$$

$$
\begin{aligned}
(3.20)\qquad & h\bar{\nabla}_{\bar{X}}(t\bar{Y}) + v\bar{\nabla}_{\bar{X}}(t\bar{Y}) + h\bar{\nabla}_{\bar{X}}(n\bar{Y}) + v\bar{\nabla}_{\bar{X}}(n\bar{Y})\\
&= (\bar{\nabla}_{\bar{X}}\bar{J})\bar{Y} + t(h\bar{\nabla}_{\bar{X}})\bar{Y} + n(h\bar{\nabla}_{\bar{X}})\bar{Y} + \psi_1 P(\mathcal{A}_{\bar{X}}\bar{Y})\\
&\quad +\omega_1 P(\mathcal{A}_{\bar{X}}\bar{Y}) + \psi_2 Q(\mathcal{A}_{\bar{X}}\bar{Y}) + \omega_2 Q(\mathcal{A}_{\bar{X}}\bar{Y});
\end{aligned}
$$

$$
\begin{aligned}
(3.21)\qquad & h\bar{\nabla}_{\bar{X}}(\psi_1 PU) + h\bar{\nabla}_{\bar{X}}(\omega_1 PU) + v\bar{\nabla}_{\bar{X}}(\psi_1 PU) + v\bar{\nabla}_{\bar{X}}(\omega_1 PU)\\
&+h\bar{\nabla}_{\bar{X}}(\psi_2 QV) + \mathcal{A}_{\bar{X}}(\psi_2 QU) + \mathcal{A}_{\bar{X}}(\omega_2 QU) + v\bar{\nabla}_{\bar{X}}(\omega_2 QU)\\
&= (\bar{\nabla}_{\bar{X}}\bar{J})U + t(\mathcal{A}_{\bar{X}}U) + n(\mathcal{A}_{\bar{X}}U) + \psi_1 P(v(\bar{\nabla}_{\bar{X}}U))\\
&\quad +\omega_1 P(v\bar{\nabla}_{\bar{X}}U) + \psi_2 Q(v\bar{\nabla}_{\bar{X}}U) + \omega_2 Q(V\bar{\nabla}_{\bar{X}}U);
\end{aligned}
$$

$$
\begin{aligned}
(3.22)\qquad & h\bar{\nabla}_U(t\bar{X}) + \mathcal{T}_U(t\bar{X}) + \mathcal{T}_U(n\bar{X}) + v\bar{\nabla}_U(n\bar{X})\\
&= (\bar{\nabla}_U\bar{J})\bar{X} + t(h\bar{\nabla}_U\bar{X}) + n(h\bar{\nabla}_U\bar{X}) + \psi_1 P(\mathcal{T}_U\bar{X})\\
&\quad +\omega_1 P(\mathcal{T}_U\bar{X}) + \psi_2 Q(T_U\bar{X}) + \omega_2 Q(v\bar{\nabla}_U\bar{X}).
\end{aligned}
$$

Proof. For $U, V \in \mathcal{V}$, we have

$$\bar{\nabla}_U(\bar{J}V) \;=\; (\bar{\nabla}_U\bar{J})V + \bar{J}(\bar{\nabla}_U V),$$

which gives equation (3.19). Similarly, we can obtain other equations. □

By using similar steps as in lemma 3.1, we have

Lemma 3.2. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Kähler manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, for any $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in \mathcal{H}$, we have*

$$
\begin{aligned}
(3.23)\qquad & h\bar{\nabla}_{PU}(\psi_1 PV) + h\bar{\nabla}_{PU}(\omega_1 PV) + h\bar{\nabla}_{QU}(\psi_1 PV) + h\bar{\nabla}_{QU}(\omega_1 PV)\\
&+h\bar{\nabla}_{PU}(\psi_1 QV) + h\bar{\nabla}_{PU}(\omega_2 QV) + h\bar{\nabla}_{QU}(\psi_2 QV) + h\bar{\nabla}_{QU}(\omega_2 QV)\\
&= t(h\bar{\nabla}_U V + \psi_1 P(\bar{\nabla}_U V) + \psi_2 Q(v\bar{\nabla}_U V);
\end{aligned}
$$

$$\begin{aligned} &v\bar{\nabla}_{PU}(\psi_1 PV) + v\bar{\nabla}_{PU}(\omega_1 PV) + v\bar{\nabla}_{QU}(\psi_1 PV) + v\bar{\nabla}_{QU}(\omega_1 PV) \\ &+v\bar{\nabla}_{PU}(\psi_2 QV) + v\bar{\nabla}_{PU}(\omega_2 QV) + v\bar{\nabla}_{QU}(\psi_2 QV) + v\bar{\nabla}_{QU}(\omega_2 QV) \\ &= n(h\bar{\nabla}_U V) + \omega_1 P(v\bar{\nabla}_U) + \omega_2 Q(v\bar{\nabla}_U V). \end{aligned} \tag{3.24}$$

$$h\bar{\nabla}_{\bar{X}}(t\bar{Y}) + h\bar{\nabla}_{\bar{X}}(n\bar{Y}) = t(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_1 P(\mathcal{A}_{\bar{X}}\bar{Y}) + \psi_2 Q(\mathcal{A}_{\bar{X}}\bar{Y}); \tag{3.25}$$

$$v\bar{\nabla}_{\bar{X}}(t\bar{Y}) + v\bar{\nabla}_{\bar{X}}(\bar{Y}) = n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 P(\mathcal{A}_{\bar{X}}\bar{Y}) + \omega_2 Q(\mathcal{A}_{\bar{X}}\bar{Y}). \tag{3.26}$$

$$\begin{aligned} &h\bar{\nabla}_{\bar{X}}(\psi_1 PU) + h\bar{\nabla}_{\bar{X}}(\omega_1 PU) + h\bar{\nabla}_{\bar{X}}(\psi_2 QV) + \mathcal{A}_{\bar{X}}(\omega_2 QU) \\ &= t(\mathcal{A}_{\bar{X}}U) + \psi_1 P(v\bar{\nabla}_{\bar{X}}U) + \psi_2 Q(v\bar{\nabla}_{\bar{X}}U); \end{aligned} \tag{3.27}$$

$$\begin{aligned} &v\bar{\nabla}_{\bar{X}}(\psi_1 PU) + v\bar{\nabla}_{\bar{X}}(\omega_1 PU) + \mathcal{A}_{\bar{X}}(\psi_2 QU) + v\bar{\nabla}_{\bar{X}}(\omega_2 QU) \\ &= n(\mathcal{A}_{\bar{X}}U) + \omega_1 P(v\bar{\nabla}_{bX}U) + \omega_2 Q(v\bar{\nabla}_{\bar{X}}U). \end{aligned} \tag{3.28}$$

$$h\bar{\nabla}_U(t\bar{X}) + \mathcal{T}_U(n\bar{X}) = t(h\bar{\nabla}_U\bar{X}) + \psi_1 P(\mathcal{T}_U\bar{X}) + \psi_2 Q(\mathcal{T}_U\bar{X}); \tag{3.29}$$

$$\mathcal{T}_U(t\bar{X}) + v\bar{\nabla}_U(n\bar{X}) = n(h\bar{\nabla}_U\bar{X}) + \omega_1 P(\mathcal{T}_U\bar{X}) + \omega_2 Q(v\bar{\nabla}_U\bar{X}). \tag{3.30}$$

Theorem 3.3. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Kähler manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the fibres of f are totally geodesic if and only if $\bar{\nabla}_U(\bar{J}V) = \bar{J}(v\bar{\nabla}_U V)$, for any vector fields $U, V \in \mathcal{V}$.*

Proof. For vector fields $U, V \in \mathcal{V}$, we have

$$\bar{\nabla}_U(\bar{J}V) = \bar{J}(\mathcal{T}_U V) + \bar{J}(v\bar{\nabla}_U V). \tag{3.31}$$

The fibres of f are totally geodesic if and only if $\mathcal{T} = 0$. So, the proof follows from (3.31). □

Theorem 3.4. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation if and only if*

$$\begin{aligned} &\bar{g}(t(h\bar{\nabla}_{\bar{X}}\bar{Y})) + \psi_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_1 PU + \psi_2 QU) \\ &+\bar{g}(n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_1 PU + \omega_2 QU) = 0, \end{aligned} \tag{3.32}$$

for $\bar{X}, \bar{Y} \in \mathcal{H}$ and $U, V \in \mathcal{V}$.

Proof. Let $\bar{X}, \bar{Y} \in \mathcal{H}$ and $U \in \mathcal{V}$. Then, we have

$$\begin{aligned}\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ U) &= \bar{g}(\bar{J}(\bar{\nabla}_{\bar{X}}\bar{Y}),\ \bar{J}U)\\ &= \bar{g}(t(h\bar{\nabla}_{\bar{X}}\bar{Y}) + n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \bar{J}(P(v\bar{\nabla}_{\bar{X}}\bar{Y})) + Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\\ &\qquad \psi_1 PU + \omega_1 PU + \psi_2 QU + \omega_2 QU),\end{aligned}$$

which implies

$$\begin{aligned}\bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y},\ U) &= \bar{g}(t(h\bar{\nabla}_{\bar{X}}\bar{Y})) + \psi_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_1 PU + \psi_2 QU)\\ &\quad + \bar{g}(\psi_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \psi_1 PU + \psi_2 QU)\\ &\quad + \bar{g}(n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_1 PU + \omega_2 QU)\\ &\quad + \bar{g}(\omega_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_1 PU + \omega_2 QU).\end{aligned} \tag{3.33}$$

Now, $\mathcal{H}$ defines a totally geodesic foliation if and only if $\bar{\nabla}_{\bar{X}}\bar{Y} \in \mathcal{H}$. So, the proof follows from above equation. □

Corollary 3.1. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the following statements are equivalent:*

(a) The horizontal distribution $\mathcal{H}$ defines a totally geodesic foliation,

(b) $\bar{g}(t(h\bar{\nabla}_{\bar{X}}\bar{Y})) + \psi_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \psi_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y},\ \psi_1 PU + \psi_2 QU)$
$+ \bar{g}(n(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ \omega_1 PU + \omega_2 QU) = 0,$

(c) $\bar{g}(h\bar{\nabla}_{\bar{X}}\bar{Y},\ t\psi_1 PU) + \psi_1 P\omega_1 PU + \psi_2 Q\omega_1 PU + t\psi_2 QU$
$+ \psi_1 P\omega_2 QU + \psi_2 Q\omega_2 QU) + \bar{g}(v\bar{\nabla}_{\bar{X}}\bar{Y}),$
$n\psi_1 PU + \omega_1 P\omega_1 PU + \omega_2 Q\omega_1 PU$
$+ n\psi_2 QU + \omega_1 P\omega_2 QU + \omega_2 Q\omega_2 QU) = 0,$

(d) $\bar{g}(nt(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 Pn(h\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_2 Qn(h\bar{\nabla}_{\bar{X}}\bar{Y})$
$+ n\psi_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 P\omega_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_2 Q\omega_1 P(v\bar{\nabla}_{\bar{X}}\bar{Y})$
$+ n\psi_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}) + \omega_1 P\omega_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y})$
$+ \omega_2 Q\omega_2 Q(v\bar{\nabla}_{\bar{X}}\bar{Y}),\ U) = 0,$

for all $\bar{X}, \bar{Y} \in \mathcal{H}$ and $U \in \mathcal{V}$.

Proof. In theorem (3.4), we have proved $(a) \Leftrightarrow (b)$. Similarly, we can prove $(b) \Leftrightarrow (c)$, $(c) \Leftrightarrow (d)$ and $(d) \Leftrightarrow (a)$. □

Theorem 3.5. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}&\bar{g}(t(h\bar{\nabla}_U V) + \psi_1 P(v\bar{\nabla}_U V) + \psi_2 Q(v\bar{\nabla}_U V),\ t\bar{X})\\ &+\bar{g}(n(h\bar{\nabla}_U V) + \omega_1 P(v\bar{\nabla}_U V) + \omega_2 Qv(\bar{\nabla}_U V),\ n\bar{X}) = 0,\end{aligned} \tag{3.34}$$

for all $\bar{X} \in \mathcal{H}$ and $U, V \in \mathcal{V}$.

Proof. Let $\bar{X} \in \mathcal{H}$ and $U, V \in \mathcal{V}$. Then, we have

$$\begin{aligned}\bar{g}(\bar{\nabla}_U V, \bar{X}) &= \bar{g}(\bar{J}\bar{\nabla}_U V,\ \bar{J}\bar{X}) \\ &= \bar{g}(t(h\bar{\nabla}_U V) + n(h\bar{\nabla}_U V) + \psi_1 P(V\bar{\nabla}_U V) \\ &\quad +\omega_1 P(v\bar{\nabla}_U V) + \psi_2 Q(v\bar{\nabla}_U V) \\ &\quad +\omega_2 Q(v\bar{\nabla}_U V),\ t\bar{X} + n\bar{X}),\end{aligned}$$

which gives

$$\begin{aligned}\bar{g}(\bar{\nabla}_U V, \bar{X}) &= \bar{g}(t(h\bar{\nabla}_U V) + \psi_1 P(v\bar{\nabla}_U V) + \psi_2 Q(v\bar{\nabla}_U V),\ t\bar{X}) \\ &\quad +\bar{g}(n(h\bar{\nabla}_U V) + \omega_1 P(v\bar{\nabla}_U V) + \omega_2 Qv(\bar{\nabla}_U V),\ n\bar{X}).\end{aligned} \tag{3.35}$$

The distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if $\bar{\nabla}_U V \in \mathcal{V}$. This completes the proof. □

Theorem 3.6. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite Kähler manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the vertical distribution $\mathcal{V}$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}&\bar{g}(h\bar{\nabla}_U(\psi_1 PV + \omega_1 PV + \psi_2 QV + \omega_2 QV),\ t\bar{X}) \\ &+\bar{g}(v\bar{\nabla}_U(\psi_1 PV + \omega_1 PV + \psi_2 QV + \omega_2 QV),\ n\bar{X}) = 0,\end{aligned} \tag{3.36}$$

for $U, V \in \mathcal{V}$ and $\bar{X} \in \mathcal{H}$.

Proof. Let $U, V \in \mathcal{V}$ and $\bar{X} \in \mathcal{H}$. We have

$$\begin{aligned}\bar{g}(\bar{\nabla}_U V,\ \bar{X}) &= \bar{g}(\bar{\nabla}_U(\bar{J}V),\ t\bar{X} + n\bar{X}), \\ &= \bar{g}(\bar{\nabla}_U(\psi_1 PV + \omega_1 PV + \psi_2 QV + \omega_2 QV),\ t\bar{X} + n\bar{X}),\end{aligned}$$

which implies

$$\begin{aligned}\bar{g}(\bar{\nabla}_U V,\ \bar{X}) &= \bar{g}(h\bar{\nabla}_U(\psi_1 PV + \omega_1 PV + \psi_2 QV + \omega_2 QV),\ t\bar{X}) \\ &\quad +\bar{g}(v\bar{\nabla}_U(\psi_1 PV + \omega_1 PV + \psi_2 QV + \omega_2 QV),\ n\bar{X}).\end{aligned} \tag{3.37}$$

$\bar{\nabla}_U V \in \mathcal{V}$ if and only if right side of above equation vanishes.
Hence, the proof follows from equation (3.37). □

Now, using similar steps as in theorem 22 and theorem 24 of [20], we have

Theorem 3.7. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the submersion f is an affine map on $\mathcal{H}$.*

Theorem 3.8. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the submersion f is an affine map if and only if $h(\bar{\nabla}_E hF) + \mathcal{A}_{hE} vF + \mathcal{T}_{vE} vF$ is f-related to $\nabla_X Y$, for any $E, F \in \Gamma(T\bar{M})$.*

Theorem 3.9. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ onto a pseudo-Riemannian manifold (M, g). Then, the submersion map f is totally geodesic if and only if*

$$\mathcal{T}_U V + \mathcal{A}_{\bar{X}} V + h\bar{\nabla}_U \bar{Y} = 0, \tag{3.38}$$

for any $U, V \in \mathcal{V}$ and $\bar{X}, \bar{Y} \in \mathcal{H}$.

Proof. Let $E = \bar{X} + U,\ F = \bar{Y} + V \in \Gamma(T\bar{M})$.
In view of equation (2.24), using similar steps as in proof of theorem (3.7), we have

$$\begin{aligned}(\bar{\nabla} f_*)(E,\ F) &= (\bar{\nabla} f_*)(U, V) + (\bar{\nabla} f_*)(\bar{X}, V) + (\bar{\nabla} f_*)(U, \bar{Y}) \\ &= -f_*(h(\bar{\nabla}_U V + \bar{\nabla}_{\bar{X}} V + \bar{\nabla}_U \bar{Y})),\end{aligned}$$

which gives

$$(\bar{\nabla} f_*)(E,\ F) = -f_*(\mathcal{T}_U V + \mathcal{A}_{\bar{X}} V + h\bar{\nabla}_U \bar{Y}). \tag{3.39}$$

As the submersion map f is totally geodesic if and only if $\bar{\nabla} f_* = 0$, the proof follows from (3.39). □

Theorem 3.10. *Let $f\colon \bar{M} \longrightarrow M$ be a bi-slant pseudo-Riemannian submersion from indefinite almost Hermitian manifold $(\bar{M}^{2m_1}, \bar{J}, \bar{g})$ onto pseudo-Riemannian manifold (M^{m_2}, g). If the fibres $f^{-1}(q)$ of f over $q \in M$ are totally geodesic, then f is a harmonic map.*

Proof. The tension field $\tau(f)$ of the map $f\colon \bar{M} \longrightarrow M$ is defined as

$$\tau(f) = trace(\bar{\nabla} f_*). \tag{3.40}$$

Let $\{e_1, e_2, \cdots, e_{2m_1-m_2}, e_{2m_1-m_2+1} = \bar{e}_1, \bar{e}_2, \cdots \bar{e}_{m_2}\}$ be an orthonormal basis of $\Gamma(T\bar{M})$, where $\{e_1, e_2, \cdots, e_{2m_1-m_2}\}$ is an orthonormal basis of $\mathcal{V}$ and $\{\bar{e}_1, \bar{e}_2, \cdots \bar{e}_{m_2}\}$ is an orthonormal basis of $\mathcal{H}$. Then, we have

$$\tau(f) = \sum_{i=1}^{2m_1-m_2} \bar{g}(e_i,\ e_i)(\bar{\nabla} f_*)(e_i,\ e_i) + \sum_{j=1}^{m_2} \bar{g}(\bar{e}_j,\ \bar{e}_j)(\bar{\nabla} f_*)(\bar{e}_j,\ \bar{e}_j). \tag{3.41}$$

For any vertical vector fields $U, V \in \mathcal{V}$, using equation (2.12), we have

$$\begin{aligned}(\bar{\nabla} f_*)(U, V) &= (\nabla^f_U (f_* V)) \circ f - f_*(\bar{\nabla}_U V) \\ &= -f_*(h\bar{\nabla}_U V) \\ &= -f_*(\mathcal{T}_U V),\end{aligned} \tag{3.42}$$

where ∇^f is the pullback connection of ∇ with respect to f. For any horizontal vector fields $\bar{X}, \bar{Y} \in \mathcal{H}$, which are f-related to $X, Y \in \Gamma(TM)$ respectively, lemma 2.1 and theorem 3.7 imply

$$\begin{aligned}(\bar{\nabla} f_*)(\bar{X}, \bar{Y}) &= (\nabla^f_{\bar{X}} (f_* \bar{Y})) \circ f - f_*(\bar{\nabla}_{\bar{X}} \bar{Y}) \\ &= (\nabla_{f_* \bar{X}} (f_* \bar{Y})) \circ f - f_*(h\bar{\nabla}_{\bar{X}} \bar{Y}) \\ &= 0.\end{aligned} \tag{3.43}$$

In view of equations (3.41), (3.42), (3.43) and theorem 3.8, we get

$$\tau(f) \quad = \quad -\sum_{i=1}^{4m+3-n} \bar{g}(e_i,\ e_i) f_*(\mathcal{T}_{e_i} e_i). \tag{3.44}$$

Now, if the fibres $f^{-1}(q)$ of f over $q \in M$ are totally geodesic, then $\mathcal{T} = 0$. So the proof of the theorem follows from equation (3.44). □

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