Coefficient Estimates For Certain Subclass of Bi-Univalent Functions Associated With Chebyshev Polynomial

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Abstract

In this paper we have defined new subclass $H_\Sigma(\lambda, t, \beta)$ of bi-univalent functions in D and established the estimates on the coefficients $| a_2 |$ and $| a_3 |$ by using Chebyshev polynomial

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1 Introduction

A functions of the form $f(z)$ normalized by the following Taylor Maclaurin series :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and belongs to class A.
Let $M(\lambda)$ denote the subclass of λ-convex functions in D defined as follows (see [7]):

$$M(\lambda) = \left\{ f \in A : \Re \left((1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0, \ \ \lambda \geq 0 \right\}.$$

Let S be the class of all functions in A which are univalent in D . By Koebe one quarter theorem [5] we can see that the image of D under every function f from the class S contains a disc or radius $\frac{1}{4}$. Thus every function $f \in S$ has an inverse f^{-1}, defined by
$f^{-1}(f(z)) = z \ (z \in D)$ and $f\left(f^{-1}(w)\right) = w$, $\left(| w |< r_0(f); r_0(f) \geq \frac{1}{4}\right)$
where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \ . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in D if both $f(z)$ and $f^{-1}(z)$ are univalent in D . Let Σ be the class of bi univalent functions in D investigated by Lewin (see[9]). Recently Joshi and others introduced and investigated the subclass of Σ associated with pseudo starlike functions. and obtain non sharp estimates on initial coefficients $\mid a_2 \mid$ and $\mid a_3 \mid$. (for details one could refer [3],[8],[11][12]).
Also in [2] Babalola proved that all pseudo-starlike functions are Bazilevic of type $\left(1-\dfrac{1}{\lambda}\right)$, order $\beta^{\frac{1}{\lambda}}$ and univalent in open disc D.

In literature there are only few works about the bounds on the general coefficient $\mid a_n \mid$ for $n \geq 4$. The problem of coefficient estimate for general coefficient $\mid a_n \mid$ $n \in \mathbb{N} \setminus \{1,2\}$ where $\mathbb{N}$ is set of natural numbers is still open.
In numerical analysis the chebyshev polynomial has great importance in theory and practical both.There are four kinds of chebyshev polynomials.The majority of research papers dealing with orthogonal polynomials of Chebyshev family, contains first and second kinds Chebyshev polynomials, which is denoted by $T_n(t)$and $U_n(t)$ see Doha[4] and Mason[10]. The Chebyshev polynomial of first and second kinds are well known and given as follows. In the case of real variable x on $(-1,1)$, they are defined by

$$T_n(t) = cosn\theta$$

$$U_n(t) = \frac{sin(n+1)\theta}{sin\theta}$$

where the subscript n denotes the polynomial degree and $t = cos\theta$.
Also if the functions f and g are analytic in D, then f is said to be subordinate to g and can be written as

$$f(z) \prec g(z) \qquad (z \in D).$$

If there exists a schwartz function $w(z)$ analytic in D, with $w(0) = 0$, $\mid w(z) \mid < 1$ such that $f(z) = g(w(z)) \qquad (z \in D)$.
Definition 1.1. A function f given by (1.1) is said to be in the class $H_\Sigma(\lambda, t, \beta)$, $\beta > 0$, $\lambda \geq 0$ and $t \in (0,1]$ if the following subordination hold :

$$(1.3) \quad f \in \Sigma\,, \quad \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^\beta \prec H(z,t) = \frac{1}{1-2tz+z^2} \quad (z \in D)$$

(1.4)

$$and \qquad \left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^\beta \prec H(w,t) = \frac{1}{1-2tw+w^2} \quad (w \in D),$$

where g is extension of f^{-1} to D given by (1.2)

Here note that if $t = cos\alpha \ \ \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then

$$\begin{aligned} H(z,t) &= \frac{1}{1-2tz+z^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n \qquad (z \in D). \end{aligned}$$

Thus

$$H(z,t) = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \dots \qquad (z \in D).$$

then we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \qquad (z \in D, t \in (-1,1)).$$

where $U_{n-1} = \dfrac{sin(n\ arccos\ t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ are the Chebyshev polynomial of the second kind and we known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \tag{1.5}$$

$$U_1(t) = 2t \tag{1.6}$$

$$U_2(t) = 4t^2 - 1 \tag{1.7}$$

$$U_3(t) = 8t^3 - 4t \tag{1.8}$$

$$\vdots$$

The generating function of the Chebyshev polynomial of the first kind $T_n(t),\ t \in [-1,1]$ have the form

$$\sum_{n=1}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \qquad (z \in D).$$

However, the relationship between Chebyshev polynomial of first kind and second kind can be given as follows

$$\begin{aligned} \frac{dT_n(t)}{dt} &= nU_{n-1}(t) \\ T_n(t) &= U_n(t) - tU_{n-1}(t) \\ 2T_n(t) &= U_n(t) - U_{n-2}(t) \end{aligned}$$

Motivated by the earlier work of Dziok *et al.* [6], Sahsene Altinkaya and Sibel Yalcin [1] we use Chebyshev polynomial expansions to provide estimates for coefficients $\mid a_2 \mid$ and $\mid a_3 \mid$ of bi-univalent functions in $H_\Sigma(\lambda, t, \beta)$.

MAIN RESULTS

2 Coefficient Estimates for the function class $H_\Sigma(\lambda, t, \beta)$.

Theorem 2.1. Let f given by (1.1) be in the class $H_\Sigma(\lambda, t, \beta)$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\beta(1+\lambda)[(1+\lambda)(\beta - 2t^2(\beta+1) + 4t^2]}} \tag{2.1}$$

and

$$|a_3| \leq \frac{4t^2}{\beta^2(1+\lambda)^2} + \frac{t}{\beta(1+2\lambda)}. \tag{2.2}$$

Proof : Let $f \in H_\Sigma(\lambda, t, \beta)$. From (1.3)and (1.4)we have

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^\beta = 1 + U_1(t)w(z) + U_2(t)w^2(z) + \ldots \quad , \tag{2.3}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^\beta = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \ldots \quad , \tag{2.4}$$

for some analytic functions w, v such that$w(0) = v(0) = 0$ and $|w(z)| < 1$, $|v(w)| < 1$ $\forall$ $z \in D$
from equalities (2.3) and (2.4) we have

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^\beta = 1 + U_1(t)c_1z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \ldots \quad , \tag{2.5}$$

and
(2.6)

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^\beta = 1 + U_1(t)d_1w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + \ldots \quad .$$

It is well known that if $|w(z)| = |c_1z + c_2z^2 + c_3z^3 + \ldots| < 1$ and $|v(w)| = |d_1w + d_2w^2 + d_3w^3 + \ldots| < 1$, $z, w \in D$, then

$$| c_j |\leq 1 \qquad \forall \;\; j \in \mathbb{N}.$$

From equations (2.5) and (2.6) we can write

$$\beta(1+\lambda)a_2 = U_1(t)c_1 \;\; , \tag{2.7}$$

$$\beta(2+4\lambda)a_3 + \left(\frac{\beta(\beta-1)}{2}(1+\lambda)^2 - \beta(3\lambda+1)\right)a_2^2 = U_1(t)c_2 + U_2(t)c_1^2 \;\; , \tag{2.8}$$

$$-\beta(1+\lambda)a_2 = U_1(t)d_1 \;\; , \tag{2.9}$$

$$\left[\beta(3+5\lambda) + \frac{\beta(\beta-1)}{2}(1+\lambda)^2\right]a_2^2 - \beta(2+4\lambda)a_3 = U_1(t)d_2 + U_2(t)d_1^2 \;\; . \tag{2.10}$$

From equations (2.7) and (2.9) we get

$$c_1 = -d_1 \tag{2.11}$$

and

$$2\beta^2(1+\lambda)^2 a_2^2 = U_1^2(t)(c_1^2 + d_1^2) \;\; . \tag{2.12}$$

Taking addition of (2.8) and (2.10) we get

$$\left[\beta(\beta-1)(1+\lambda)^2 + \beta(2+2\lambda)\right]a_2^2 = U_1(t)(c_2+d_2) + U_2(t)\left(c_1^2+d_1^2\right), \tag{2.13}$$

Using equation (2.12) in (2.13) we get

$$a_2^2\left[\beta(1+\lambda)\left(\beta(1+\lambda) - (\lambda-1)\right) - \frac{U_2(t)}{U_1^2(t)}2\beta^2(1+\lambda)^2\right] = U_1(t)(c_2+d_2). \tag{2.14}$$

From (1.6),(1.7) and (2.14) we have

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\beta(1+\lambda)[(1+\lambda)(\beta - 2t^2(\beta+1) + 4t^2]}} \tag{2.15}$$

Now for bound on $| a_3 |$, by subtracting (2.10)from (2.8)we get

$$2\beta(2+4\lambda)a_3 - \beta(4+8\lambda)a_2^2 = U_1(t)(c_2-d_2) + U_2(t)(c_1^2 - d_1^2) \tag{2.16}$$

Now using (2.11) and (2.12) in (2.16) we get

$$a_3 = \frac{U_1^2(t)}{2\beta^2(1+\lambda)^2}(c_1^2+d_1^2) + \frac{U_1(t)}{2\beta(2+4\lambda)}(c_2-d_2) \tag{2.17}$$

Now from (1.6) we get

$$|a_3| \leq \frac{4t^2}{\beta^2(1+\lambda)^2} + \frac{t}{\beta(1+2\lambda)}. \tag{2.18}$$

3 Fekete-Szegö inequalities for the class $H_\Sigma(\lambda, t, \beta)$

Theorem 3.1. If f given by (1.1) is in the class $H_\Sigma(\lambda, t, \beta)$ and $\delta \in R$, then

$$| a_3 - \delta a_2^2 | \leq \begin{cases} \dfrac{t}{\beta(1+2\lambda)}; \\ \qquad\qquad \text{for } | \delta - 1 | \leq \dfrac{1}{4(1+2\lambda)} \left| \dfrac{\beta(1+\lambda)^2}{2t^2} - \left(\beta(1+\lambda)^2 + (\lambda^2-1)\right) \right| \\ \dfrac{8t^3 \mid (1-\delta) \mid}{|2t^2(\beta^2(1+\lambda)^2 - \beta(\lambda^2-1)) - (4t^2-1)(\beta^2(1+\lambda)^2)|}; \\ \qquad\qquad \text{for } | \delta - 1 | \geq \dfrac{1}{4(1+2\lambda)} \left| \dfrac{\beta(1+\lambda)^2}{2t^2} - \left(\beta(1+\lambda)^2 + (\lambda^2-1)\right) \right| \end{cases}$$

Proof. From equations (2.17) and (2.14)

$$\begin{aligned} a_3 - \delta a_2^2 &= (1-\delta) \left[\frac{U_1^3(t)(c_2+d_2)}{U_1^2(t)[\beta(\beta-1)(1+\lambda)^2 + 2\beta(1+\lambda)] - U_2(t)2\beta^2(1+\lambda)^2} \right] + \frac{U_1(t)(c_2-d_2)}{2\beta(2+4\lambda)} \\ &= U_1(t) \left[\left(l(\delta) + \frac{1}{4\beta(1+2\lambda)} \right) c_2 + \left(l(\delta) - \frac{1}{4\beta(1+2\lambda)} \right) d_2 \right] \end{aligned}$$

where

$$l(\delta) = \frac{U_1^2(t)(1-\delta)}{U_1^2(t)[\beta(1+\lambda)(\beta(1+\lambda) - (\lambda-1))] - U_2(t)2\beta^2(1+\lambda)^2}.$$

Then from (1.6) and (1.7) we can write

$$| a_3 - \delta a_2^2 | \leq \begin{cases} \dfrac{t}{\beta(1+2\lambda)} & \text{for } 0 \leq | l(\delta) | \leq \dfrac{1}{4\beta(1+2\lambda)}; \\ 4t \mid l(\delta) \mid & \text{for } | l(\delta) | \geq \dfrac{1}{4\beta(1+2\lambda)}. \end{cases}$$

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