A Coincidence Point Theorem in Partial Metric Space

M. C. Arya, N. Chandra and Mahesh C. Joshi

Department of Mathematics,
D. S. B. Campus, Kumaun University
Nainital, India

mcarya1986@gmail.com, cnaveen329@gmail.com, & mcjoshi69@gmail.com

Abstract

The aim of this paper is to obtain some coincidence point results on complete partial metric spaces under a generalized contractive condition. Our results extend some well known results on the existence of fixed/coincidence points for single valued maps in partial metric spaces.

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1 Introduction

Banach contraction mapping principle is the most celebrated result in fixed point theory and and its applications. Banach contraction principle guarantees that any contraction map on complete metric space has a unique fixed point. Kannan [3] was first, who gave a new contractive condition for which a map need not be continuous even if it has a fixed point on complete metric space. Also, Chatterjea [1] generalized the Banach's contractive condition and obtained a fixed point theorem. In this direction, Hardy and Rogers [2] obtained a fixed point theorem under the generalized condition which is the combination of Kannan, Chatterjea and Banach contractive type condition. Hardy and Rogers result is stated as follows.

Theorem 1.1. *Let T be a selfmap on a complete metric space (X, d) such that*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(y, Tx)] + c\, d(x, y)$$

for all $x, y \in X$, where $0 \leq 2a + 2b + c \leq k < 1$. Then T has a unique fixed point.

Matthews [4] introduced the notion of **partial metric space (PMS)** which is the part of study of denotational semantics of data for network. In the same paper, Matthews extended the Banach contraction mapping principle and proved the fixed point theorem in the setting of PMS. Here we recall the definitions and basic properties of PMS.

Definition 1.1. [4] Let X be a non empty set, then a partial metric on X is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(a) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$

(b) $p(x, x) \leq p(x, y)$

(c) $p(x, y) = p(y, x)$

(d) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

And the pair (X, p) is said to be partial metric space (PMS).

If p is a partial metric on X, then the functions $p^s, p^t : X \times X \to \mathbb{R}^+$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ and $p^t(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ are metrics on X.

Definition 1.2. ([4]) Let (X, p) be a partial metric space. Then,

(i) a sequence $\{x_n\}$ in (X, p) is said to be convergent to a point $x \in X$ if and only $p(x, x) = \lim_{n\to\infty} p(x_n, x)$,

(ii) a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists (and finite),

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Lemma 1.1. *([4]) Let (X, p) be a partial metric space. Then*

(i) a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, p^s),

(ii) (X, p) is complete if and only if the metric space (X, p^s) is complete,

(iii) a subset E of a (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Definition 1.3. [5] Let (X, p) be a partial metric space. Then,

(i) a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n) = 0$,

(ii) (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$, such that $p(x, x) = 0$.

Remark 1.1. Let (X, p) be a PMS, then $p(x, y) = 0 \Rightarrow x = y$, but converse is not true in general.

Now, we prove our main results using Hardy and Rogers [2] type condition in partial metric spaces.

2 Main Results

Theorem 2.1. *Let T be a self-map on a complete partial metric space X satisfying the condition*

$$p(Tx,Ty) \leq a[p(x,Tx)+p(y,Ty)]+b[p(x,Ty)+p(y,Tx)]+c\,p(x,y) \tag{2.1}$$

for all $x,y \in X$, where $0 \leq 2a+2b+c \leq k < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X which is defined as $x_n = Tx_{n-1}$ for $n = 1,2,3,\ldots$, then $x_n = T^n x_0$. Now, we have

$$\begin{aligned} p(x_n,x_{n+1}) &= p(Tx_{n-1},Tx_n) \\ &\leq a[p(x_{n-1},Tx_{n-1})+p(x_n,Tx_n)] \\ &\quad +b[p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})]+cp(x_{n-1},x_n) \\ &= a[p(x_{n-1},x_n)+p(x_n,x_{n+1})] \\ &\quad +b[p(x_{n-1},x_{n+1})+p(x_n,x_n)]+cp(x_{n-1},x_n) \\ &\leq a[p(x_{n-1},x_n)+p(x_n,x_{n+1})] \\ &\quad +b[p(x_{n-1},x_n)+p(x_n,x_{n+1})]+cp(x_{n-1},x_n) \\ &\leq (a+b+c)p(x_{n-1},x_n)+(a+b)p(x_n,x_{n+1}) \end{aligned}$$

$$\begin{aligned} \Rightarrow (1-a-b)p(x_n,x_{n+1}) &\leq (a+b+c)p(x_{n-1},x_n) \\ \Rightarrow p(x_n,x_{n+1}) &\leq \frac{a+b+c}{(1-a-b)}p(x_{n-1},x_n). \\ \text{Since } k<1, \quad p(x_n,x_{n+1}) &\leq kp(x_{n-1},x_n) \leq k^2 p(x_{n-2},x_{n-1}) \ldots \leq k^n p(x_0,x_1). \end{aligned}$$

Now we show that $\{x_n\}_{n=1}^{n=\infty}$ is a Cauchy sequence in X. Let $m,n>0$ with $m>n$, then

$$\begin{aligned} p(x_n,x_m) &\leq p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\ldots+p(x_{n+m-1},x_m) \\ &\quad -p(x_{n+1},x_{n+1})-p(x_{n+2},x_{n+2})-\ldots-p(x_{n+m-1},x_{n+m-1}) \\ &\leq k^n p(x_0,x_1)+k^{n+1}p(x_0,x_1)+\ldots+k^{n+m-1}p(x_0,x_1) \\ &\leq k^n[p(x_0,x_1)+kp(x_0,x_1)+\ldots+k^{m-1}p(x_0,x_1)] \\ &\leq k^n\frac{1-k^{m-1}}{1-k}p(x_0,x_1). \end{aligned}$$

Taking $n,m \to \infty$ we get $p(x_n,x_m) \to 0$, hence $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in X. Thus by lemma this sequence will also Cauchy in (X,p^s). As X is complete therefore the sequence $\{x_n\}$ will converge to $x \in X$, i.e, $x_n \to x$. Thus,

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n,\to\infty} p(x_n,x) = p(x,x) = 0.$$

Existence of fixed point: Now, we show that x is fixed point of T. As,

$$\begin{aligned}
p(x,Tx) &\leq p(x,x_{n+1}) + p(Tx_n,Tx) - p(x_{n+1},x_{n+1})\\
p(x,Tx) &\leq p(x,x_{n+1}) + a[p(x_n,x_{n+1}) + p(x,Tx)]\\
&+ b[p(x_n,Tx) + p(x,x_{n+1})] + cp(x_n,x) - p(x_{n+1},x_{n+1})\\
p(x,Tx) &\leq p(x,x_{n+1}) + a[p(x_n,x) + p(x,x_{n+1}) - p(x,x) + p(x,Tx)]\\
&\quad +b[p(x_n,x) + p(x,Tx) - p(x,x) + p(x,x_{n+1})]\\
&\quad +cp(x_n,x) - p(x_{n+1},x_{n+1})\\
&\leq (1+a+b)p(x,x_{n+1}) + (a+b+c)p(x_n,x) + (a+b)p(x,Tx)\\
\Rightarrow \quad (1-a-b)p(x,Tx) &\leq (1+a+b)p(x,x_{n+1}) + (a+b+c)p(x_n,x)\\
\Rightarrow \quad p(x,Tx) &\leq \frac{(1+a+b)}{(1-a-b)}p(x,x_{n+1}) + \frac{(a+b+c)}{(1-a-b)}p(x_n,x).
\end{aligned}$$

Letting $n \to \infty$, then we have

$$p(x,Tx) \leq 0 \Rightarrow p(x,Tx) = 0 \Rightarrow Tx = x.$$

Now, if y is another fixed point of T, i.e., $p(y,Ty) = 0$. Then,

$$\begin{aligned}
p(x,y) &= p(Tx,Ty)\\
&\leq a[p(x,Tx) + (y,Ty)] + b[p(x,Ty) + p(y,Tx)] + cp(x,y)\\
&\leq a[p(x,x) + (y,y)] + b[p(x,y) + p(y,x)] + cp(x,y)\\
&\leq b[p(x,y) + p(y,x)] + cp(x,y) \leq (2b+c)p(x,y)\\
\Rightarrow \quad (1-2b-c)p(x,y) &\leq 0 \Rightarrow p(x,y) = 0 \text{ i.e. } x = y,
\end{aligned}$$

which completes the theorem. □

Taking particular values of a, b and c in Theorem 2.1, we obtained the following results as corollaries due Kannan [3], Chatterjea [1] and Banach contraction principle in partial metric spaces.

Corollary 2.1. *Let T be a self-map of a complete partial metric space X such that*

$$p(Tx,Ty) \leq a[p(x,Tx) + p(y,Ty)]$$

for each $x, y \in X$, where $0 \leq a < \frac{1}{2}$. Then T has a unique fixed point on X.

Corollary 2.2. *Let T be a self-map of a complete partial metric space X such that*

$$p(Tx,Ty) \leq b[p(x,Ty) + p(y,Tx)]$$

for each $x, y \in X$, where $0 \leq b < \frac{1}{2}$. Then T has a unique fixed point on X.

Corollary 2.3. *Let T be a self-map of a complete partial metric space X such that*

$$p(Tx,Ty) \leq kp(x,y)$$

for each $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point on X.

Coincidence Point Result for Two Mapping

Theorem 2.2. *Let T and f be a self-maps on a complete partial metric space X such that*

$$p(Tx,Ty) \leq a[p(fx,Tx)+p(fy,Ty)]+b[p(fx,Ty)+p(fy,Tx)]+cp(fx,fy) \tag{2.2}$$

for all $x, y \in X$, where $0 \leq 2a+2b+c \leq k < 1$. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then T and f have a coincidence fixed point.

Proof. Let $x_0 \in X$ and Choose a point x_1 in X such that $Tx_0 = fx_1, \ldots, Tx_n = fx_{n+1}$, we get

$$\begin{aligned}
p(fx_n, fx_{n+1}) &= p(Tx_{n-1}, Tx_n) \\
&\leq a[p(fx_{n-1}, Tx_{n-1}) + p(fx_n, Tx_n)] \\
&\quad +b[pf(x_{n-1}, Tx_n) + p(fx_n, Tx_{n-1})] + cp(fx_{n-1}, fx_n) \\
&= a[p(fx_{n-1}, fx_n) + p(fx_n, fx_{n+1})] \\
&\quad +b[p(fx_{n-1}, fx_{n+1}) + p(fx_n, fx_n)] + cp(fx_{n-1}, fx_n) \\
&\leq (a+b+c)p(fx_{n-1}f, x_n) + (a+b)p(fx_n, fx_{n+1}) \\
\Rightarrow (1-a-b)p(fx_n, fx_{n+1}) &\leq (a+b+c)p(fx_{n-1}, fx_n) \\
\Rightarrow p(fx_n, fx_{n+1}) &\leq \frac{a+b+c}{(1-a-b)} p(fx_{n-1}, fx_n). \\
\text{Since } k<1,\ p(fx_n, fx_{n+1}) &\leq kp(fx_{n-1}, fx_n) \leq k^2 p(fx_{n-2}, fx_{n-1}) \ldots \leq k^n p(fx_0, fx_1).
\end{aligned}$$

Now, we show that $\{fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, so we have

$$\begin{aligned}
p(fx_n, fx_m) &\leq p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2}) + \ldots + p(fx_{n+m-1}, fx_m) \\
&\quad -p(fx_{n+1}, fx_{n+1}) - p(fx_{n+2}, fx_{n+2}) - \ldots - p(fx_{n+m-1}, fx_{n+m-1}) \\
&\leq k^n p(fx_0, fx_1) + k^{n+1} p(fx_0, fx_1) + \ldots k^{n+m-1} p(fx_0, fx_1) \\
&\leq k^n [p(fx_0, fx_1) + kp(fx_0, fx_1) + \ldots + k^{m-1} p(fx_0, fx_1)] \\
&\leq k^n \frac{1-k^{m-1}}{1-k} p(fx_0, fx_1).
\end{aligned}$$

Letting $n, m \to \infty$, we get $p(fx_n, fx_m) \to 0$, hence $\{fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Thus by lemma this sequence will be Cauchy in (X, p^s) also. As X is complete therefore the sequence $\{x_n\}$ will converge to x in X, i.e., $x_n \to x \Rightarrow fx_n \to fx$. Therefore,

$$\lim_{n,m\to\infty} p(fx_n, fx_m) = \lim_{n\to\infty} p(fx_n, fx) = p(fx, fx) = 0.$$

Existence of Coincidence fixed point : For if,

$$\begin{aligned}
p(fx,Tx) &\leq p(fx,fx_{n+1})+p(Tx_n,Tx)-p(fx_{n+1},fx_{n+1}) \\
p(fx,Tx) &\leq p(fx,fx_{n+1})+a[p(fx_n,fx_{n+1})+p(fx,Tx)] \\
&\quad +b[p(fx_n,Tx)+p(fx,fx_{n+1})]+cp(fx_n,fx)-p(fx_{n+1},fx_{n+1} \\
p(fx,Tx) &\leq p(fx,fx_{n+1})+a[p(fx_n,fx)+p(fx,fx_{n+1})-p(fx,fx)+p(fx,Tx)] \\
&\quad +b[p(x_n,x)+p(x,Tx)-p(x,x)+p(fx,fx_{n+1})] \\
&\quad +cp(fx_n,fx)-p(fx_{n+1},fx_{n+1} \\
&\leq (1+a+b)p(fx,fx_{n+1})+(a+b+c)p(fx_n,fx)+(a+b)p(fx,Tx) \\
\Rightarrow (1-a-b)p(fx,Tx) &\leq (1+a+b)p(fx,fx_{n+1})+(a+b+c)p(fx_n,fx) \\
\Rightarrow p(fx,Tx) &\leq \frac{(1+a+b)}{(1-a-b)}p(fx,fx_{n+1})+\frac{(a+b+c)}{(1-a-b)}p(fx_n,fx).
\end{aligned}$$

Taking $n \to \infty$, we get $p(fx,Tx) \leq 0 \Rightarrow p(fx,Tx)=0 \Rightarrow Tx=fx$. □

Remark 2.1. Taking $f = I$ and T is the single valued map in Theorem 2.2, we get Theorem 2.1.

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