A NON-METRIC $\Phi-$CONNECTION ON A RIEMANNIAN MANIFOLD

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Abstract

The present paper deals with the study of non-metric $\Phi-$connection on a Riemannian manifold. Semi-symmetric non-metric connection and Quarter-symmetric non-metric connection are two important examples of this new connection. The purpose of this paper is to study of some properties of Curvature tensor, Ricci tensor and Weyl projective curvature tensor with respect to non-metric $\Phi-$connection .

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1 Introduction

Dae Ho Jin [7] introduces a non-metric $\Phi-$symmetric connection on a semi Riemannian manifold. In 1924, Friedmann and Schouten [3, 10] introduced the idea of semi-symmetric connection on a differentiable manifold.

H. A. Hyden [5] introduced semi-symmetric metric connection on a Riemannian manifold and this was further develop by K. Yano [12], T. Imai [6] and Nakao [9]. Further the concept of semi-symmetric non-metric connection on a Riemannian manifold was introduced by Agashe and Chafle. S. Golab [4] introduced the notion of quarter-symmetric connection in a Riemannian manifold and further quarter-symmetric non-metric connection was studied by J. Sengupta and Biswas [11], U. C. De, K. Yano and T. Q. Binh [1]. Motivated from the above works, In this paper, we study a non-metric $\Phi-$connection on a Riemannian manifold.

The paper is organized as follows : In section 2, we give a brief introduction of a non-metric $\Phi-$connection on a Riemannian manifold. In section 3, we study the curvature tensor with respect to non-metric $\Phi-$connection. In section 4, we study projective curvature tensor of a Riemannian manifold with respect to non metric $\Phi-$connection .

2 Non-metric $\Phi-$connection

A linear connection $\overline{\nabla}$ on an $n-$dimensional Riemannian manifold (M, g) is called a non-metric $\Phi-$connection if it and its torsion tensor T satisfy

$$(\overline{\nabla}_X g)(Y,Z) = -u(Y)\Phi(X,Z) - u(Z)\Phi(X,Y), \tag{2.1}$$

and

$$\overline{T}(X,Y) = u(Y)JX - u(X)JY, \tag{2.2}$$

where Φ is a tensor field of type $(0,2)$, J is a tensor field of types $(1,1)$ and u is a 1-form associated with a smooth vector field U, given by

$$u(X) = g(X,U).$$

Throughout this paper, we denote by X, Y and Z the smooth vector fields on M.

A linear connection ∇ on a differentiable manifold M is said to be a semi-symmetric non metric connection if the torsion T of the connection ∇ satisfies

$$T(X,Y) = u(Y)X - u(X)Y, \tag{2.3}$$

and

$$(\nabla_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y), \tag{2.4}$$

where u is a 1-form.

Taking $\Phi = g$ and $J = I$ in (2.1), where I is the identity tensor field of type $(1,1)$, then non-metric $\Phi-$connection reduces to semi-symmetric non-metric connection and in the case $\Phi = g$ in the (2.1), it reduces to quarter-symmetric non-metric connection.

The subject of this paper is to study Riemannian manifold with non-metric $\Phi-$connection.

Remark 2.1. Denote ∇ by the Levi-Civita connection of (M,g) with respect to the metric g.

We define a linear connection $\overline{\nabla}$ on M given by

$$\overline{\nabla}_X Y = \nabla_X Y + u(Y)JX. \tag{2.5}$$

By direct calculation, we see that $\overline{\nabla}$ is a non-metric Φ-connection. Conversely, if $\overline{\nabla}$ is a non-metric $\Phi-$connection then we can write

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X,Y). \tag{2.6}$$

Substituting (2.6) into (2.1) and using the fact that ∇ is a metric. We have

$$g(\psi(X,Y),Z) + g(\psi(X,Z),Y) = u(Y)\Phi(X,Z) + u(Z)\Phi(X,Y). \tag{2.7}$$

Also from (2.6) and the fact that ∇ is torsion-free, it follows that

$$\overline{T}(X,Y) = \psi(X,Y) - \psi(Y,X). \tag{2.8}$$

Thus using (2.2), we obtain

$$\psi(X,Y)-\psi(Y,X)=u(Y)JX-u(X)JY. \tag{2.9}$$

Exchanging X and Y in (2.7), we have

$$g(\psi(Y,X),Z)+g(\psi(Y,Z),X)=u(X)\Phi(Y,Z)+u(Z)\Phi(Y,X). \tag{2.10}$$

Subtracting (2.10) from (2.7) and using (2.8), we obtain

$$g(\psi(X,Z),Y)-g(\psi(Y,Z),X)=u(Z)[\Phi(X,Y)-\Phi(Y,X)]. \tag{2.11}$$

Again from (2.9), we get

$$g(\psi(X,Y),Z)-g(\psi(Y,X),Z)=u(Y)\Phi(X,Z)-u(X)\Phi(Y,Z), \text{ and}$$

$$g(\psi(X,Z),Y)-g(\psi(Z,X),Y)=u(Z)\Phi(X,Y)-u(X)\Phi(Z,Y).$$

Adding these two equations and using (2.7), we have

$$g(\psi(Y,X),Z)+g(\psi(Z,X),Y)=u(X)[\Phi(Y,Z)+\Phi(Z,Y)]. \tag{2.12}$$

Interchanging X and Z, we have

$$g(\psi(Y,Z),X)+g(\psi(X,Z),Y)=u(Z)[\Phi(Y,X)+\Phi(X,Y)]. \tag{2.13}$$

Adding (2.11) and (2.13), we have

$$g(\psi(X,Z),Y)=u(Z)\Phi(X,Y),$$
$$\psi(X,Z)=u(Z)JX,$$

So,

$$\psi(X,Y)=u(Y)JX. \tag{2.14}$$

Thus the non-metric $\Phi-$connection $\overline{\nabla}$ satisfies (2.5). It shows that for a linear connection $\overline{\nabla}$ on a Riemannian manifold (M,g), $\overline{\nabla}$ is a non-metric $\Phi-$connection if and only if it satisfies (2.5).

Further for a 1-form π on M, we have

$$\begin{aligned}\overline{\nabla}_X(\pi(Y)) &= (\overline{\nabla}_X\pi)Y+\pi(\overline{\nabla}_XY),\\ &= (\overline{\nabla}_X\pi)Y+\pi(\nabla_XY+u(Y)JX),\\ &= (\overline{\nabla}_X\pi)Y+\pi(\nabla_XY)+u(Y)\pi(JX),\end{aligned}$$

This implies,

$$(\overline{\nabla}_X\pi)Y=(\nabla_X\pi)Y-u(Y)\pi(JX). \tag{2.15}$$

for vector fields X, Y on M.

Covariant differentiation of the torsion tensor $\overline{T}$ is given by

$$(2.16)\qquad \begin{aligned}(\overline{\nabla}_X\overline{T})(Y,Z) &= \overline{\nabla}_X(\overline{T}(Y,Z)) - \overline{T}(\overline{\nabla}_X Y,Z) - \overline{T}(Y,\overline{\nabla}_X Z),\\ &= ((\overline{\nabla}_X u)Z)JY - ((\overline{\nabla}_X u)Y)JZ\\ &\quad +u(Z)(\overline{\nabla}_X(JY) - J(\overline{\nabla}_X Y))\\ &\quad -u(Y)(\overline{\nabla}_X(JZ) - J(\overline{\nabla}_X Z)).\end{aligned}$$

Further we define,

$$(2.17)\qquad \widetilde{T}(X,Y,Z) = g(\overline{T}(X,Y),Z).$$

From (2.2) and (2.17), we have

$$(2.18)\qquad \begin{aligned}&\widetilde{T}(X,Y,Z) + \widetilde{T}(Y,Z,X) + \widetilde{T}(Z,X,Y)\\ = {}& u(X)[\Phi(Z,Y) - \Phi(Y,Z)]\\ &+u(Y)[\Phi(X,Z) - \Phi(Z,X)]\\ &+u(Z)[\Phi(Y,X) - \Phi(X,Y)].\end{aligned}$$

In particular, if Φ is symmetric. Then (2.18) reduces to

$$(2.19)\qquad \widetilde{T}(X,Y,Z) + \widetilde{T}(Y,Z,X) + \widetilde{T}(Z,X,Y) = 0.$$

and if Φ is skew-symmetric then (2.18) reduces to

(2.20)
$$\widetilde{T}(X,Y,Z) + \widetilde{T}(Y,Z,X) + \widetilde{T}(Z,X,Y) = 2[u(X)\Phi(Z,Y) + u(Y)\Phi(X,Z) + u(Z)\Phi(Y,X)]$$

3 The Curvature Tensor of a Riemannian Manifold With Respect to Non-Metric $\Phi-$Connection $\overline{\nabla}$.

Corresponding to the definition of curvature tensor of a Riemannian manifold M with respect to Riemannian connection ∇ we define the curvature tensor of M with respect to non-metric Φ-connection $\overline{\nabla}$ by

$$(3.1)\qquad \overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z.$$

From (2.5) and (3.1), we have

$$
\begin{aligned}
\overline{R}(X,Y)Z &= \overline{\nabla}_X(\nabla_Y Z + u(Z)JY) - \overline{\nabla}_Y(\nabla_X Z + u(Z)JX) \\
&\quad -\nabla_{[X,Y]}Z - u(Z)J[X,Y], \\
&= \nabla_X(\nabla_Y Z + u(Z)JY) + u(\nabla_Y Z + u(Z)JY)JX \\
&\quad -\{\nabla_Y(\nabla_X Z + u(Z)JX) + u(\nabla_X Z + u(Z)JX)JY\} \\
&\quad -\nabla_{[X,Y]}Z - u(Z)J[X,Y], \\
&= R(X,Y)Z + \alpha(X,Z)JY - \alpha(Y,Z)JX \\
&\quad +u(Z)\{\nabla_X(JY) - \nabla_Y(JX) - J[X,Y]\},
\end{aligned} \tag{3.2}
$$

where $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature tensor of the Riemannian manifold M with respect to the Riemannian connection ∇ and α is a tensor of type $(0,2)$ on M given by

$$
\begin{aligned}
\alpha(X,Y) &= (\nabla_X u)Y - u(Y)u(JX), \\
&= (\overline{\nabla}_X u)Y.
\end{aligned} \tag{3.3}
$$

From (3.3) we have

$$
\begin{aligned}
\alpha(X,Y) - \alpha(Y,X) &= (\overline{\nabla}_X u)Y - (\overline{\nabla}_Y u)X, \\
&= \overline{\nabla}_X u(Y) - u(\overline{\nabla}_X Y) \\
&\quad -\overline{\nabla}_Y u(X) + u(\overline{\nabla}_Y X), \\
&= X\omega(Y) - Y\omega(X) - u[X,Y] \\
&\quad -u(Y)u(JX) + u(X)u(JY), \\
&= d\omega(X,Y) - u(Y)u(JX) + u(X)u(JY).
\end{aligned} \tag{3.4}
$$

If $1-$form u is closed, then

$$
\alpha(X,Y) - \alpha(Y,X) = u(X)u(JY) - u(Y)u(JX). \tag{3.5}
$$

If we take the exterior product or wedge product, $u \, \Lambda \, u(J) = 0$ in the above equation, we have

$$
\alpha(X,Y) = \alpha(Y,X), \tag{3.6}
$$

i.e. α is symmetric if $u \, \Lambda \, u(J) = 0$.

Let

$$
{}'R(X,Y,Z,W) = g(R(X,Y)Z,W), \tag{3.7}
$$

and

$$
{}'\overline{R}(X,Y,Z,W) = g(\overline{R}(X,Y)Z,W), \tag{3.8}
$$

for vector fields X, Y, Z and W on M.
From (3.2) and (3.6), we have

$$\begin{aligned} {}'\overline{R}(X,Y,Z,W) &= {}'R(X,Y,Z,W) + \alpha(X,Z)\Phi(Y,W) \\ &\quad -\alpha(Y,Z)\Phi(X,W) + u(Z)\{g(\nabla_X(JY),W) \\ &\quad -g(\nabla_Y(JX),W) - g(J[X,Y],W)\}. \end{aligned} \tag{3.9}$$

From (3.1) and (3.6), we get

$$ {}'\overline{R}(X,Y,Z,W) = -{}'\overline{R}(Y,X,Z,W). \tag{3.10}$$

Using (3.2) and the first Bianchi identity with respect to Riemannian connection ∇, we have

$$\begin{aligned} &\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y \\ = \;& \{\alpha(Y,X) - \alpha(X,Y)\}JZ \\ &+\{\alpha(Z,Y) - \alpha(Y,Z)\}JX \\ &+\{\alpha(X,Z) - \alpha(Z,X)\}JY \\ &+u(Z)\{\nabla_X(JY) - \nabla_Y(JX) - J[X,Y]\} \\ &+u(Y)\{\nabla_Z(JX) - \nabla_X(JZ) - J[Z,X]\} \\ &+u(X)\{\nabla_Y(JZ) - \nabla_Z(JY) - J[Y,Z]\}. \end{aligned} \tag{3.11}$$

We call (3.11) as the first Bianchi identity with respect to non-metric $\Phi-$connection connection $\overline{\nabla}$.
Using (3.9), we get

$$\begin{aligned} &{}'\overline{R}(X,Y,Z,W) + {}'\overline{R}(X,Y,W,Z) \\ = \;& \alpha(X,Z)\Phi(Y,W) - \alpha(Y,Z)\Phi(X,W) \\ &+\alpha(X,W)\Phi(Y,Z) - \alpha(Y,W)\Phi(X,Z) \\ &+u(Z)\{g(\nabla_X(JY),W) - g(\nabla_Y(JX),W) \\ &-g(J[X,Y],W)\} + u(W)\{g(\nabla_X(JY),Z) \\ &-g(\nabla_Y(JX),Z) - g(J[X,Y],Z)\}. \end{aligned} \tag{3.12}$$

And

$$\begin{aligned} &{}'\overline{R}(X,Y,Z,W) - {}'\overline{R}(Z,W,X,Y) \\ = \;& \{\alpha(X,Z) - \alpha(Z,X)\}\Phi(Y,W) \\ &+\alpha(W,X)\Phi(Y,Z) - \alpha(Y,Z)\Phi(X,W) \\ &+u(Z)\{g(\nabla_X(JY),W) - g(\nabla_Y(JX),W) \\ &-g(J[X,Y],W)\} - u(X)\{g(\nabla_Z(JW),Y) \\ &-g(\nabla_W(JZ),Y) - g(J[Z,W],Y)\}. \end{aligned} \tag{3.13}$$

Corresponding to the definition of Ricci tensor S of Riemannian manifold M with respect to Riemannian connection ∇, we define Ricci tensor $\overline{S}$ of Riemannian manifold M with respect to non-metric Φ- connection $\overline{\nabla}$ by

$$\begin{aligned} \overline{S}(Y,Z) &= \Sigma_{i=1}^{n\prime} \overline{R}(e_i,Y,Z,e_i), \\ &= S(Y,Z) + \Sigma_{i=1}^{n\prime}\{(\nabla_{e_i}u)Z - u(Z)u(Je_i)\}g(JY,e_i) \\ &\quad -\alpha(Y,Z)trace(J) + u(Z)\{(divJ)Y - trace(\nabla_Y J)\} \end{aligned} \tag{3.14}$$

where $e_i, 1 \leq i \leq n$ are orthonormal vector fields on M.

From (3.9) and (3.14) and taking $J = I$, we have

$$\overline{S}(Y,Z) = S(Y,Z) - (n-1)\alpha(Y,Z), \tag{3.15}$$

where $S(Y,Z) = \Sigma_{i=1}^{n\prime} R(e_i,Y,Z,e_i)$ is Ricci tensor of Riemannian manifold M with respect to Riemannian connection ∇.

A relation between Ricci tensor $\overline{S}$ with respect to non-metric Φ- connection $\overline{\nabla}$ and Ricci tensor S with respect to the Riemannian connection ∇, taking $J = I$ is given by (3.15).

Further if $\overline{S}(Y,Z) = 0$ on M then (3.15) implies α is symmetric.

From (3.4) and (3.15) and taking $J = I$, we have

$$\overline{S}(X,Y) - \overline{S}(Y,X) = -(n-1)d\omega(X,Y). \tag{3.16}$$

Hence Ricci tensor $\overline{S}$ with respect to non-metric Φ-connection $\overline{\nabla}$ is symmetric if and only if $1-$form u is closed and $J = I$, hence if and only if α is symmetric.

Taking $J = I$ in (2.16) and using (3.3) , we have

$$(\overline{\nabla}_X \overline{T})(Y,Z) = \alpha(X,Z)Y - \alpha(X,Y)Z. \tag{3.17}$$

In particular if Ricci tensor with respect to non-metric Φ- connection $\overline{\nabla}$ vanishes, then from (3.17) we have

$$(\overline{\nabla}_X \overline{T})(Y,Z) + (\overline{\nabla}_Y \overline{T})(Z,X) + (\overline{\nabla}_Z \overline{T})(X,Y) = 0. \tag{3.18}$$

Corresponding to the definition of scalar curvature of Riemannian manifold M with respect to Riemannian connection ∇, we define scalar curvature of M with respect to non-metric Φ-connection $\overline{\nabla}$ by

$$\overline{r} = \Sigma_{i=1}^{n} \overline{S}(e_i,e_i). \tag{3.19}$$

From (3.15) and (3.19), if $J = I$ we obtain a relation between the scalar curvature with respect to Riemannian connection ∇ and the non-metric Φ-connection $\overline{\nabla}$ by

$$\overline{r} = r - (n-1)trace(A), \tag{3.20}$$

where $r = \Sigma_{i=1}^{n} S(e_i,e_i)$ is the scalar curvature of Riemannian manifold M with respect to Riemannian connection ∇ and A is a tensor of type $(1,1)$ defined on M by

$$\alpha(X,Y) = g(AX,Y).$$

4 Projective Curvature Tensor of a Riemannian Manifold With Respect to Φ -Symmetric Non-Metric Connection $\overline{\nabla}$.

Weyl projective curvature tensor of a Riemannian manifold with respect to Riemannian connection ∇ is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\{S(Y,Z)X - S(X,Z)Y\}. \tag{4.1}$$

Corresponding to this definition, we define projective curvature tensor of a Riemannian manifold with respect to non-metric Φ-connection $\overline{\nabla}$ by

(4.2)
$$\begin{aligned}
\overline{P}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{(n-1)}\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\} \\
&= P(X,Y)Z + .\alpha(X,Z)JY - \alpha(Y,Z)JX \\
&\quad + u(Z)\{\nabla_X(JY) - \nabla_Y(JX) - J[X,Y]\} \\
&\quad - \frac{1}{n-1}[g(JY, gradu(Z))X - g(JX, gradu(Z))Y \\
&\quad - (\Sigma_{i=1}^n\{u(\nabla_{e_i}Z) + u(Z)u(Je_i)\})(g(JY,e_i)X - g(JX,e_i)Y) \\
&\quad - trace(J)\{\alpha(Y,Z)X - \alpha(X,Z)Y\} + u(Z)\{(divJ)YX \\
&\quad - (divJ)XY\} - trace(\nabla_Y J)X + trace(\nabla_X J)Y].
\end{aligned}$$

Hence we can state the following theorems

Theorem 4.1. *If M is a Riemannian manifold admitting non-metric Φ-connection $\overline{\nabla}$, then the Weyl projective curvature tensor with respect to non-metric Φ-connection $\overline{\nabla}$ must satisfy the following equation*

$$\begin{aligned}
\overline{P}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{(n-1)}\{\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\} \\
&= P(X,Y)Z + .\alpha(X,Z)JY - \alpha(Y,Z)JX + u(Z)\{\nabla_X(JY) - \nabla_Y(JX) - J[X,Y]\} \\
&\quad - \frac{1}{n-1}[g(JY, gradu(Z))X - g(JX, gradu(Z))Y \\
&\quad - (\Sigma_{i=1}^n\{u(\nabla_{e_i}Z) + u(Z)u(Je_i)\})(g(JY,e_i)X - g(JX,e_i)Y) \\
&\quad - trace(J)\{\alpha(Y,Z)X - \alpha(X,Z)Y\} + u(Z)\{(divJ)YX - (divJ)XY\} \\
&\quad - trace(\nabla_Y J)X + trace(\nabla_X J)Y].
\end{aligned}$$

In particular, taking $J = I$ in (3.2) *and using* (3.15), (4.1) *and* (4.2), *we have*

$$\overline{P}(X,Y)Z = P(X,Y)Z, \tag{4.3}$$

From (4.3) we have, the Weyl projective curvature tensor with respect to non-metric Φ-connection $\overline{\nabla}$ satisfies the following properties

$$\overline{P}(X,Y)Z = -\overline{P}(Y,X)Z,$$

and

$$\overline{P}(X,Y)Z + \overline{P}(Y,Z)X + \overline{P}(Z,X)Y = 0, \tag{4.4}$$

for every vector fields X, Y and Z on M.

In particular, if M be a Riemannian manifold satisfying

$$\overline{R}(X,Y)Z = 0, \tag{4.5}$$

which implies

$$\overline{S}(Y,Z) = 0, \tag{4.6}$$

on M.

From (4.2), (4.3), (4.5) and (4.6), we have

$$P(X,Y)Z = 0 \qquad \text{on } M.$$

Necessary and sufficient condition for a manifold with a symmetric linear connection to be projectively flat is that the projective curvature tensor with respect to it vanishes identically on a manifold [2].

Taking $J = I$ in (3.3) and using (3.15) and (4.6), we have

$$(\nabla_X u)Y = \frac{1}{(n-1)}S(X,Y) + u(X)u(Y). \tag{4.7}$$

Using (4.7), we have

$$\begin{aligned} -u(R(X,Y)Z) &= (\nabla_X\nabla_Y u - \nabla_Y\nabla_X u - \nabla_{[X,Y]}u)Z \\ &= \frac{1}{(n-1)}\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &\quad +u(Y)S(X,Z) - u(X)S(Y,Z)\}. \end{aligned} \tag{4.8}$$

Further from (4.1), we have

$$u(R(X,Y)Z) = \frac{1}{(n-1)}\{u(X)S(Y,Z) - u(Y)S(X,Z)\}. \tag{4.9}$$

Adding (4.8) and (4.9), we get

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0.$$

Hence we have following theorem,

Theorem 4.2. *If M is a Riemannian manifold with vanishing curvature tensor with respect to non-metric Φ-connection then M is projectively flat and*

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0,$$

if $J = I$.

It is well known that a Riemannian manifold is of constant curvature if and only if it is projectively flat and a Riemannian manifold of constant curvature is conformally flat [2].

Hence from theorem 2, we have

Theorem 4.3. *If M is a Riemannian manifold with vanishing curvature tensor with respect to non-metric Φ-connection then M is a space of constant curvature and hence is conformally flat.*

A Riemannian manifold M is a group manifold [8] with respect to non-metric Φ-connection if

(4.10)
$$\begin{aligned} \overline{R}(X,Y)Z &= 0, \\ &\text{and} \\ (\overline{\nabla}_X \overline{T})(Y,Z) &= 0 \text{ on } M. \end{aligned}$$

If either α is symmetric or vanishing Ricci tensor with respect to non-metric Φ-connection $\overline{\nabla}$ then taking $J = I$ in (3.2) and using (3.17), we have

(4.11)
$$\overline{R}(X,Y)Z = R(X,Y)Z + (\overline{\nabla}_Z \overline{T})(Y,X).$$

Hence we have

Theorem 4.4. *Let M be a Riemannian manifold with respect to non-metric Φ-connection. If either the 1- form u is closed and $J = I$ or Ricci tensor with respect to non-metric Φ-connection vanishes, then*

(4.12)
$$\overline{R}(X,Y)Z = R(X,Y)Z + (\overline{\nabla}_Z \overline{T})(Y,X) \qquad \text{on } M.$$

In particular, If M is a Riemannian manifold with vanishing Ricci tensor with respect to non-metric Φ-connection $\overline{\nabla}$, then from (4.2), (4.3) and (4.11), we have

(4.13)
$$\begin{aligned} P(X,Y)Z &= \overline{R}(X,Y)Z, \\ &= R(X,Y)Z + (\overline{\nabla}_Z \overline{T})(Y,X) \qquad \text{on } M. \end{aligned}$$

Since a flat manifold is projectively flat, from (4.13), we have $\overline{R}(X,Y)Z = 0$ and $(\overline{\nabla}_Z \overline{T})(X,Y) = 0$ on M.

Hence from (4.10), we have

Theorem 4.5. *If a Riemannian manifold M with vanishing Ricci tensor with respect to non-metric Φ-connection is flat, then M is a group manifold with respect to non-metric Φ-connection.*

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