Geometry of an infinite dimensional Lie group and applications

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Abstract

Statisticians while looking for general parametric-invariant procedures in statistical inference and asymptotics via) derivative and differential strings came across an infinite dimensional Lie group in 1990 as jet group which is currently called the phylon group $P(d)$.

This group has a very rich geometry and richer representation theory than the general linear group $GL(d)$. Just as general tensor fields are related to the tensor representations of $GL(d)$, all the string fields of statistics are realizable by representations of $P(d)$. Moreover these string fields and their generalisations occur as sections of certain vector bundles whose construction is well known in differential geometry.

There are applications of these representations of $P(d)$ in statistical asymptotics. The representation theory of $P(d)$ is new and difficult and is at begining state only posing many challenges.

§1 Introduction

Statisticians while studying statistical inference and asymptotics are concerned with parametric-invariant procedures. This necessitated them to develop the theory of generalized tensors 'or strings' systematically. In fact it was started by McCullah and Cox [1986][1]. While studying Bartlet adjustments decomposed it into 6 parts of tensors which are parameter-invariant. Starting with this paper statisticians developped statistical string theory during last 3 decades in the form of derivative strings, differnetial strings both structurally symmetric ones and general ones and in this process there evolved a certain infinite dimensional group in 1992 denoted by $\mathcal{P}(d)$ in the work of Murray 1990,1992 [2,3] for the first time.

It is well known to statisticians that differnential geometry provides a convenient language and tool for studying parametric-independent questions in inference. This was studied by Amari [4] extensively. For statistical parametric models differential geometry provides global structures like dual structures, affine structure etc on statistical manifold $\mathcal{P}$, that is, global geometries and their most general generalization on $\mathcal{P}$ [5] and neighborhood geometry of $\mathcal{P}$ such as divergence geometry [6] and finally differential form interpretation of invariants [7] under neighborhood geometry were investigated.

The infinite dimensional Lie group G that arose in statistical string theory was studied under the name of infinite phylon group (referring to family $\mathcal{P}$). It is closely related and contains the general linear group $GL(n)$ This group was also studied by Carey and Murray [3] under the name of infinite Jet group $J^{\infty}(d)$ from representation theoretic aspects since 1988.

On the otherhand the Japaneese group led by H. Omori etc., while studying the theory of infinite dimensional Lie groups that arise in classical, Hamiltonian and symplectic mechanics, quantum mechanics and more generally in integrable dynamical systems came to a class of infinite dimensional Frechet Lie groups which are regular [8],[9]. In that class Omori studied this particular infinite dimensional Lie group of statistics which he denoted by $GF(n)$ the group of formal coordinate transformations of $\mathbb{R}^n$ or $\mathbb{C}^n$. In otherwords this group $GF(n)$ is amenable to formal algebraic geometry i.e., the geometry of formal neighbourhoods of $\mathbb{R}^n$. In fact Omori [1980] used this group to study the classification of expansive singularities of algebraic varieties. From another classification aspect of natural vector bundles and natural differential operators this group was also studied by Palais-Terng 1977 [10] and Terng (1978 [11]). In this article we focus on the mathematical aspects of this infinite dimensional Lie group denoted by $\mathcal{P}(d)$ from the representation theory view point. At the same time we capture all the tensional string theory work of Barndorff-Nielsen etc., in a series of papers [12] as a theory of generalized tensors or arrays of functions attached to coordinate systems on statistical manifold $\mathcal{P}$ of dimension d (in coordinate version). This is done by means of a well known differential-geometric construction (or principle) of associating a vector bundle to a principal bundle [13],[14]. It should be pointed out this group $\mathcal{P}(d)$ of statistics has a very rich geometry and has richer representation theory than that of general linear group $GL(d)$. In fact all the classical tensor representations are included in those of $\mathcal{P}(d)$. The representation theory of $\mathcal{P}(d)$ is in general rather difficult. This is just the beginning only. There are many open problems and conjectures to be investigated.

Nevertheless, the statistical string theory studies of Barndorff-Nielsen and others gave new insights into differential geometry such as connection strings and intertwining of connection string and scalar strings and the representation theory of $\mathcal{P}(d)$ is quite new and revealing in many new aspects and is offering new challenges.

This statistical string theory has deep applications in statistical inference and statistical asymptotics via representations of $\mathcal{P}(d)$ and in general is computationally efficient in the generalized tensor formulation via transformation rules [12].

In §2 we introduced finite and infinite phylon groups in jet form and formal power series form and discussed its propoerties. §3 deals with classical and general tensors, generalized tensors and strings as representations of suitable group. §4 concerns with coordinate-free or invariant formulation of these generalized strings as geometric objects and all the upto-date information is given in a tabular form. In §5 the study of certain finite-dimensional special phylon representations was done. §6 treats the infinite dimensional phylon representations including the adjoint and co-adjoint representations of $P(d)$ and also the Kirillov orbit theory representations and also the twisted phylon representations. It was proved that an indecomposable representation of $P(d)$ need not be irreducible. At this time our knowledge about representations of $P(d)$ is very limited whereas it is complete for $GL(d)$. We used several test representative spaces to understand $P(d)$ and in this process we gave several open problems and the current status on this topic was brought out.

Finally in the appendix we applied the twisted phylon representation theory to the asymptotic behavior of the maximum likelihood estimator (MLE) (see [29] for other applications). We gave only limited references though the literature on string theory is vast.

From a more general aspect the analysis of measure parametric statistical models was investigated by Ay etc. in [26,27]. For a survey on statistical string theory the reader can see [30].

§2 The infinite phylon group $\mathcal{P}(d)$:

I. Local setup: We define the infinite phylon group $\mathcal{P}(d)$ in two equivalent ways.

(a) consider the set of all $\mathbb{R}^d$-valued infinite formal power series in d variables with no constant term. That is

$$\underline{f} = (f^1, f^2, \cdots, f^i, \cdots, f^d) \quad \text{where} \quad f^i = f^i_{j_1} x^{j_1} + \frac{1}{2!} f^i_{j_1 j_2} x^{j_1} x^{j_2} + \cdots \tag{2.1}$$

where $(f^i_{j_1})_{d\times d}$ is an invertible matrix or invertible linear term. All such $\underline{f}$s form a group under composition. That is, if $\underline{f}, \underline{g} \in P(d) \stackrel{\text{def}}{=} \mathbb{R}^d_{inv,0}[[x^1, x^2, \cdots, x^d]]$ then $\underline{f} \circ \underline{g}$ is obtained from (2.1) by replacing x^j by g^j as series with $g = (g^1, g^2, \cdots, g^j, \cdots, g^d)$. this is one definition (Barndorff-Nielsen etc., [15]).

(b) (i) We say two functions f and g defined in a nbd. of a point p are jet equivalent at p if they have the same Taylor series at p i.e., $T(f)_p = T(g)_p$.
(ii) f and g are germ equivalent at p if f and g agree on same nbd. of p.
Both of these are equivalence relations. The later gives a finer classification than the former one and the equivalence classes are called jet of f at p or $j^\infty_p(f)$ and the germ of f at p or $[f]_p$.
Consider a smooth function $f : \mathbb{R}^d \to \mathbb{R}^d$ with $f(0) = 0$ with invertible Jacobian i.e., such f is a local diffeomorphism from a nbd. 0 onto some nbd. of $\underline{0}$. Then the Taylor series $T(f)$ defines a formal power series.

By inverse function theorem f has a local inverse f^{-1} and $T(f^{-1}) = T(f)^{-1}$ and the chain rule gives $T(f \circ g) = T(f) \cdot T(g)$ where $f \circ g$ is composition and $\cdot$ is power series multiplication as defined in (a) for $P(d)$. So we have the Taylor map $T : J^\infty(d) \to P(d)$ where $J^\infty(d)$ denotes the group of local diffeomorphism of nbds. of 0 in $\mathbb{R}^d$ to itself fixing the origin modulo jet equivalence i.e.,

$$J^\infty(d) = \{J^\infty_o(f)\} \quad \text{under multiplication} \quad J^\infty_o(f) \cdot J^\infty_o(g) = J^\infty_o(f \circ g)$$

called the <u>infinite jet group</u> (Carey and Murray [3]). The above map T sending $J^\infty_o(f) \to T(f) \in \mathbb{R}^d_{inv,0}[[x^1, \cdots, x^d]]$ is an identification and T is onto by a Theorem of Borel [16]. These give the two ways ((a) and (b)) of understanding the infinite Phylon *gp* $P(d)$

For finite phylon groups $P_k(d)$ truncate the Taylor series at k-th stage or as k-th jet.

II. Manifold setup: Let M be a smooth d-dimensional manifold and $m \in M$. Let ω and ψ be local coordinate systems at m. They define an element of group $P(d)$ as follows:

For ω define new coordinate system $\hat{\omega}_m \stackrel{\text{def}}{=} \omega - \omega(m)$ so that $\hat{\omega}_m(m) = 0$. Define $D(\omega, \psi)(m) = J^\infty_0(\hat{\omega}_m \circ \hat{\psi}^{-1}_m) \in P(d)$. Note that $\hat{\omega}_m \circ \hat{\psi}^{-1}_m$ is a local diffeom on $\mathbb{R}^d$ at the origin as in our above local setup. More precisely, $D(\omega, \psi) : \cup_\omega \cap \cup_\psi \to P(d)$ and on three coordinate nbds. $\cup_\omega$, $\cup_\psi$ and $\cup_\zeta$ we have $\hat{\psi}_m \circ \hat{\omega}^{-1}_m = (\hat{\psi}_m \circ \hat{\zeta}^{-1}_m) \circ (\hat{\zeta}_m \circ \hat{\omega}^{-1}_m)$ and so taking J^∞_o operation, we get

$$D(\psi, \omega) = D(\psi, \zeta) D(\zeta, \omega) \tag{2.2}$$

which we call a cocycle condition on $\cup_\omega \cap \cup_\psi \cap \cup_\zeta$ of M.
III. Structure of $P(d)$: For any positive integer T one can consider the finite dimensional T-phylon groups $P_T(d)$ by taking T-jet equivalence of local diffeomorphisms or formal power series truncated after T^{th} term with same *gp* operations and we have projection maps $J^k : P(d) \to P_k(d)\ \forall\ k = 1, 2, \cdots$. which is also a group homomorphism onto with kernel a normal sub group of $P(d)$ denoted by $P^{(k)}(d)$ and the elements of $P^{(k)}(d)$ are of the form as formal power series

$$\delta^i_j x^j + \frac{1}{(k+1)!}\ f^i_{j_1,j_2,\cdots,j_{k+1}} x^{j_1} x^{j_2} \cdots x^{j_{k+1}} + \cdots$$

or equivalently it is a jet of a diffeom that agree with the identity upto order k.
Thus each $P_k(d)$ is a finite dimensional Lie group and note that $P_1(d)$ is simply $GL(d)$ the general linear group and $P^{(1)}(d)$ is the corresponding normal subgroup so that

$$P(d)/P^{(1)}(d) \cong P_1(d) = GL(d)$$

and we have the sequence of projection maps as

$$P(d) \to \cdots P_{k+1}(d) \to P_k(d) \to P_{k-1}(d) \to \cdots \to P_1(d)$$

In other words, the infinite phylon groups is the projective limit of the sequence $(P_k(d))$ of finite dimensional Lie groups. If $P(d) = \mathbb{R}^d_{inv,0}[[x^1, \cdots, x^d]]$ is given the pointwise convergence (by homogeneous degree components) then each map, j^k becomes a continuous map and this topology is infact the projective limit topology. With this topology the phylon group $P(d)$ becomes a <u>Frechet Lie group</u>.
Remarks: $P(d)$ as a manifold is a Frechet manifold.

We know Banach space modeled or Hilbert space modeled manifolds behave like finite dimensional manifolds where as Frechet space modelled manifolds behave differently. the main problem with them is (1) Inverse function theorem is not valid (ii) the uniqueness of solutions of O.D. Equations with initial condition also fails in Frechet manifolds. The geometric consequence of this is the exponential map may not eixst for Frechet Lie groups and even if it exists it need not be a local diffeomorphism from its Lie algebra to the group [17], [18]. Thus $P(d)$ is an infinite dimensional Frechet manifold. But $P(d)$ being the projective limit of finite dimensional Lie groups $P_k(d)$ and each $P_k(d)$ as Lie group enjoys exponential map: $LP_k(d) \to P_k(d)$ as local diffeom, $P(d)$ also enjoys exponential map: $LP(d) \to P(d)$ and is smooth and it is a regular Frechet Lie group (Omori [8]). Moreover exp: $LP^{(1)}(d) \to P^{(1)}(d)$ is a smooth bijection.
Note that $GL(d)$ is a sub group of $P(d)$ as we can regard $\forall\ X \in GL(d)$ as power series of the form $X^a_i x^i$ and also a quotient group of $P(d)$ and hence we have a bijection: $P^{(1)}(d)XGL(d) \to P(d)$ and we can put a group product on $P^{(1)}(d)XGL(d)$ making $P^{(1)}(d)XGL(d)$ isomorphic to $P(d)$. But this is a semi-direct product defined by $(f, X) \cdot (g, Y) \overset{\text{def}}{=} (f(XgX^{-1}), XY)$.
We study the representation theory of $P(d)$ under this product (cf. §5 in the following).

§3 Classical tensors, generalized tensors and strings as representations of a group

1. **Notations:** We give first some convenient notation to handle higher order derivatives of functions or coordinate change functions.
Let $\omega = (\omega^1, \cdots, \omega^d)$ and $\psi = (\psi^1, \psi^2, \cdots, \psi^d)$ be two local coordinate systems on a d-dimensional manifold M. Generically we write $\omega^i, \omega^j, \omega^k$ etc., and ψ^a, ψ^b, ψ^c etc., for them respectively. Let $\omega^i_{|a} = \frac{\partial \omega^i}{\partial \psi^a}$, $\omega^i_{|a,b} = \frac{\partial^2 \omega^i}{\partial \psi^a \partial \psi^b}$ etc., and more generally, the function $\omega^k_{|C} = \frac{\partial^t \omega^k}{\partial \psi^{c_1} \partial \psi^{c_2} \cdots \partial \psi^{c_t}}$ where $C = (c_1, c_2, \cdots, c_t)$ is a multi-index. Still more generally, let $K = (k_1, k_2, \cdots, k_u)$ be another multi-index then we put
$$\omega^K_{|C} = \sum_{C||u} \omega^{k_1}_{|C_1} \omega^{k_2}_{|C_2} \cdots \omega^{k_u}_{|C_u} \tag{3.1}$$
where $u = |K|$, length of K and $C||u$ means the sum is taken over all ordered partitions $(C_1, C_2, \cdots, C_u)$ of C into u subsets each having the same order as in C and with no gaps. That is, we are grouping the higher order derivatives in a definite arrangements.
We have $\omega^K_{|C} = 0$ if $|K| > |C|$. If K and C are empty then $\omega^K_{|C} = 1$, $\omega^\phi_{|C} = 0$ and $\omega^K_{|\phi} = 0$.
Note that $\omega^{i_1 i_2 \cdots i_n}_{|a_1 a_2 \cdots a_n} = \omega^{i_1}_{|a_1} \omega^{i_2}_{|a_2} \omega^{i_3}_{|a_3} \cdots \omega^{i_n}_{|a_n}$ and $\omega^{i_1 i_2}_{|a_1 a_2 a_3} = \omega^{i_1}_{|a_1} \omega^{i_2}_{|a_2 a_3} + \omega^{i_1}_{|a_1 a_3} \omega^{i_2}_{|a_2} + \omega^{i_2}_{|a_3} \omega^{i_1}_{|a_1 a_2}$

2. **Strings:** Barndorff-Nielsen [19] introduced strings or generalized tensors broadly as derivative string fields and differential string fields [20].

3. **Definition (a)** Given a point $m \in M$, a derivative string of tensorial degree (r, s) and length (T, U) at m assigns to each local coordinate system ω around m a set of real-valued arrays H^{IL}_{JK} indexed by multi-indices I, J, K, L with $|I| = r$, $|J| = s$, $|K| \le T$, $|L| \le U$ which transform under coordinate change from ω to ψ by
$$H^{AD}_{BC} = \psi^A_{|I} \omega^J_{|B} H^{IL}_{JK} \omega^K_{|C} \psi^D_{|L} \tag{3.2}$$
where $|A| = r$ and $|B| = s$ and the derivatives are valued at $\psi(m)$ or $\omega(m)$ as appropriate and extended summation convention followed. The space $\mathcal{S}^{rU}_{sT}(m)$ of such strings is a finite dimensional vector space. Taking union over m in M we get the space $\mathcal{S}^{rU}_{sT}(M)$ of such global strings on M and elements of $\mathcal{S}^{r0}_{sT}(M)$ and of $\mathcal{S}^{rU}_{s0}(M)$ are called (r, s)-costrings and (r, s)-contrastrings respectively. In H^{IL}_{JK}, the sets I and J are called *tensonal indices* and K and L are called *structural indices*. String H is called structurally symmetric if K and L are symmetric under respective permutations of their indices separately. Note that a derivative string field H of degree (r, s) and length (T, U) on M is a section $H : M \to \mathcal{S}^{rU}_{sT}(M) : m \to H_m$ in the vector space $\mathcal{S}^{rU}_{sT}(M)_m$ [19].
Definition (b): Definition of differential string ([20]). First we generalize the arrays $\omega^K_{|C}$ to arrays $[\psi, \omega]^{EK}_{IC}$ defined by
$$[\psi, \omega]^{EK}_{IC} = \sum_{C||2} \omega^K_{|C_1} \omega^L_{|C_2} (\psi^E_{|I})_{|L} \tag{3.3}$$

where $C||2$ denote sumover all order partition (C_1, C_2) of C into 2 subsets, either or which may be empty with same order as in C. A *differential string of degree* (r,s), *type* (p,q) *and length* (T,U) *at* $m \in M$ assigns to each local coordinate system ω around m a set of real valued arrays H^{ILM}_{JKN} indexed by multi-indices I, J, K, L, M, N with $|I| = r$, $|J| = s$, $|K| \leq T$, $|L| \leq U$, $|M| = p$, $|N| = q$ which transform under coordinate change from ω to ψ by

$$H^{ADE}_{BCF} = \psi^A_{|I}\omega^J_{|B}H^{ILM}_{JKN}[\omega,\psi]^{ND}_{FL}[\psi,\omega]^{EK}_{MC} \tag{3.4}$$

where $|A| = r$, $|B| = s$, $|E| = p$ and $|F| = q$.
Let $\mathcal{D}^{rUp}_{sTq}(M)$ denote the space of all such global differential string fields on M and each such differential string field $H = \left(H^{ILM}_{JKN}\right)$ is a section $H : M \to \mathcal{D}^{rUp}_{sTq}$.
Note that both these definitions are coordinate-based on M.

4. **Classical tensor fields:** classically in coordinates $\omega = (\omega^1, \cdots, \omega^d)$ on M a(1,2)-tensor field T is a collection of d^3 functions T^i_{jk} $i, j, k = 1$ to d which under coordinate change from ω to $\psi = (\psi^1, \cdots, \psi^d)$ transforms by

$$T^a_{bc} = T(\psi) = (T^i_{jk})\psi^a_{|i}\omega^j_{|b}\omega^k_{|c} \tag{3.5}$$

with $T^i_{jk} = T(\omega)$
The change of coordinate matrix $(\omega^i_{|a} = \frac{\partial \omega^i}{\partial \psi^a};\ i = 1$ to $d,\ a = 1$ to $d)$ defines a function: $\cup_\omega \cap \cup_\psi \to GL(d)$ denoted by $d(\omega,\psi)$ where $\cup_\omega \cap \cup_\psi$ is an open subset of M.
Note that the group $GL(d)$ acts in $\mathbb{R}^d$ by matrix multiplication and on its dual $\mathbb{R}^{d*}$ by its transposed inverse. More generally, consider the tensor product vector space $V^{r,s} = (\otimes^r \mathbb{R}^d) \otimes (\otimes^s \mathbb{R}^{d*})$ and its elements are tensors T with components $T^{i_1 i_2 \cdots i_r}_{j_1, j_2 \cdots j_s}$ and the group $GL(d)$ acts on $V^{(r,s)}$ given by for $X \in GL(d)$ and $T \in V^{(r,s)}$, $XT \in V^{(r,s)}$ with components given by

$$(XT)^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots b_s} = T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} X^{a_1}_{i_1} X^{a_2}_{i_2} \cdots X^{a_r}_{i_r} \hat{X}^{j_1}_{b_1} \hat{X}^{j_2}_{b_2} \cdots \hat{X}^{j_s}_{b_s} \tag{3.6}$$

where $X = (X^a_i) \in GL(d)$ and $\hat{X} = (X^t)^{-1}$.
This gives a representation of $GL(d)$ in the vectors space $V^{(r,s)}$, namely, $\tau_{rs} : GL(d) \to GL(V^{(r,s)})$ ..

5. **Remark:** The above (1,2)-tensor T can be understood as tensor representation $\tau_{12} : GL(d) \to GL(V^{(1,2)})$ such that $T(\omega) = T^i_{jk}$ and $T(\psi) = T^a_{bc}$ and the transformation law is given by $T(\psi)(m) = (\tau_{12} \circ d(\psi,\omega)(m))(T(\omega)(m)) \forall\ m \in M$ or globally as $T(\psi) = (\tau_{12} \circ d(\psi,\omega))T(\omega)$.

   In general, a tensor field T of type (r,s) on M transforms as

$$T(\psi) = (\tau_{rs} \circ d(\psi,\omega))T(\omega)\,. \tag{3.7}$$

We say that T is a tensor field transforming in the representation $V^{(r,s)}$ under general linear group $GL(d)$. To define a global tensor T on M it must satisfy the compatibility condition for any 3 coordinate charts on M say ω, ψ and ζ, namely

$$\tau_{r,s} \circ d(\psi,\omega) = (\tau_{rs} \circ d(\psi,\zeta))(\tau_{rs} \circ d(\zeta,\omega)) \tag{3.8}$$

called the *cocycle condition.*

6. **Remarks 1):** Then the collection $\{T(\omega)|\forall$ coordinate system ω on $M\}$ satisfying (3.7) is consistent and defines a global tensor field T on M.
2) By chain rule

$$\frac{\partial\psi^a}{\partial\omega^i} = \frac{\partial\psi^a}{\partial\zeta^r}\frac{\partial\zeta^r}{\partial\omega^i} \text{ or } \psi^a_{|i} = \psi^a_{|r}\zeta^r_{|i} \text{ or } d(\psi,\omega) = d(\psi,\zeta)d(\zeta,\omega) \tag{3.9}$$

Since $\tau_{rs} : GL(d) \to GL(V^{(r,s)})$ is a group homomorphism, (3.9) gives (3.8).
3) In this set up all the classical tensor fields can be interpretation as $GL(d)$-representations $V^{(r,s)}$ as above, and the properties of these tensors can be studied from the corresponding representation theory of $GL(d)$ such as decomposability of a tensor etc. We extend this to the phylon group $P(d)$ to define what are called phylon fields.
Recall $P(d)$ is the set of diffeomorphism of $\mathbb{R}^d$ which fix the origin, upto ∞-jet equivalence and it can also be regarded as the set $\mathbb{R}^d_{inv,0}[[x^1, x^2, \cdots, x^d]]$ of all $\mathbb{R}^d$-valued (infinite) formal power series in d variables $x^1, \cdots, x^d$ with zero constant term and with invertible linear term.

7. **Definition a):** Let ρ be a representation of the phylon group $P(d)$ on vector space V. We call ρ is an *algebraic representation* if in any basis of V the matrices $\rho(J^\infty_0 f)$ representating the elements $j^\infty_0 f$ of $P(d)$ have entries that are polynomial functions of the derivative (Taylor coefficients of f) $f^i_{j_1 j_2 \cdots j_k}$ and $\det(f^i_{j_1})^{-1}$
b) A finite dimensional algebraic representation ρ of $P(d)$ on V is called a *special phylon representation of* $P(d)$.

8. **Remark:** From the general theory of algebraic groups and their representations we have if ρ_1, ρ_2 are special phylon representations so are their direct sum $\rho_1 \oplus \rho_2$, tensor product $\rho_1 \otimes \rho_2$ and dual ρ_1^* and if W is a ρ-invariant subspace of V then $\rho_{|W}$ is also phylon and also the induced representation of $P(d)$ on V/W is phylon. In analogy to tensor fields of type (r,s) we have.

9. **Definition:** Let $\rho : P(d) \to GL(V)$ be a phylon representation. Then a phylon field P of type ρ is a collection of maps $P(\omega)$ one for each coordinate system ω on M namely $P(\omega) : \cup_\omega \subset M \to V$ satisfying the transformation rule

$$P(\psi) = (\rho \circ D(\psi,\omega))P(\omega) \quad \text{(analogue of (3.7) for } GL(d)) \tag{3.10}$$

and these maps $\{P(\omega)\}$s are compatible as the $D(\psi,\omega)$s satisfy the cocycle condition (done in §2 as (2.2))

10. **Remarks 1):** we interpret later a phylon field P of type or simply a phylon as a section of certain bundle in a coordinate-free manner. Note the above definition is coordinate-based generalizing strings or tensors to a collection of maps $\{P(\omega)\}_\omega$.
2) Phyla can be alternatively defined using D-matrices $\{D(\psi,\omega)\}_{\omega,\psi}$ of functions [15].
3) Since the coefficients in the transformation formula (3.2) for derivative strings and in (3.4) for differential strings are polynormal in $\omega^i_{|A}$ and $\psi^a_{|I}$ of the corresponding tensors, they give the coordinate description of algebraic representations of the phylon group $P(d)$.
4) Comparison of representations of groups $GL(d)$ and $P(d)$: Many of the problems classical tensor fields such as how they decompose, how to multiply them tensorially and their existence etc., reduces to studying the representation theory of the general linear group $GL(d)$. we have complete information on the representation theory of $GL(d)$, namely (i) we classify *all* the finite dimensional *indecomposable* representations ρ of $GL(d)$. (ρ indecomparable means it is not a direct sum of two representations).
(ii) to know how a tensor product $\rho_1 \otimes \rho_2$ of two indecomposable representations decomposes into a sum of indecomposable representations. For $GL(d)$ representation theory we can answer all these questions using the combinatorics of Young tableaux, namely for $GL(d)$, every indecomposable representation is irreducible also (i.e. they have no proper nontrivial invariant subspaces) and there is a *discrete collection of indecomposables* for $GL(d)$ labeled by Young tableaux and there are combinatoric rules for decomposing a tensor product $\rho_1 \otimes \rho_2$ into indecomposables. Similarly the theory of strings, new tensors and phylon fields reduces to the representation theory of the phylon group $P(d)$. Moreover since $GL(d) \subset P(d)$ as a subgroup, the classical tensor theory is part of string theory.

The representation theory of $P(d)$ is more complicated because (1) here indecomposable representations of $P(d)$ need not be irreduciable (ii) compared to $GL(d)$, $P(d)$ has a very large unipotent subgroup $P^{(1)}(d)$ = kernel of $P(d) \to P_1(d) = GL(d)$. So $P^{(1)}(d)$ acts in the representations like infinite upper triangular block matrices with identity along the diagonal. This means with analogy to the theory of algebraic groups that there are *continuous families* of indecomposable representations of $P(d)$.

Nevertheless we get a class of finite dimensional representations of $P(d)$ by projecting to $GL(d)$. We study later some finite representation of $P(d)$ called class of special phylon representations and their structure which can be computed in the special case of rank 2 on $P_2(d)$ and also some examples of infinite dimensional phylon representations of $P(d)$ (cf. §5 and §6 below).
5) So we can interpret the study of new tensors and strings and even the study of old tensors as simply the study of the representation theory of the infinite phylon group $P(d)$. That is, *all* strings and new tensors are tensor fields of type W for some representation W of $P(d)$ i.e. $\rho : P(d) \to GL(W)$ group homomorphism for some vector space W transforming by $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to P(d)$ as defined before.
6) Let M be a d-dimensional manifold. Then a *general tensor field-like object* Γ can be defined on M as follows: Instead of group $P(d)$, start with a group G and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be given maps satisfying the cocycle condition $g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ and also given an action of G on some vector space W. Then

with this data we can define a $(G, \{g_{\alpha\beta}\}, W)$-tensor field Γ as a collection of maps $\{T_\alpha : U_\alpha \to W | T_\alpha = g_{\alpha\beta} T_\beta\}$. *Most of the string fields of statistics occur like this only and the study of tensor like objects can be reduced to the study of representations of group G.*

7) We can define tensor field-like objects, more general than phylon fields as follows: Let $M = \mathbb{R}^d$. Two functions f and g on M are germ equivalent at $x \in M$ if f and g agree on some nbd. of x; denoted by $[f]_x$ the germ of f at x. Let $\mathcal{G}_0(d)$ be the set of all germs at $0 \in \mathbb{R}^d$ of origin-preserving diffeomorphisms of $\mathbb{R}^d$ to itself. That is, $\mathcal{G}_0(d) = \{[f]_0 | f$ diffeomorphisms of $\mathbb{R}^d$ to itself and $f(0) = 0\}$ then define $[f]_x \cdot [g]_x] = [f \circ g]_x$. Thus $\mathcal{G}_0(d)$ becomes a group. Define $\chi : \mathcal{G}_0(d) \to P(d)$ by the infinite jet (or Taylor) map sending $[f]_x \to J_0^\infty(f)$ which is well defined and it is a group homomorphism *onto*.

As in above remark 6) we can construct more general tensor fields Γ as $(\mathcal{G}_0(d), \{g_{\alpha\beta}\}, W)$-tensor field where W is a representation of $\mathcal{G}_0(d)$ and the properties of Γ can be studied from the representations of group $\mathcal{G}_0(d)$. By a theorem of Terng ([11], [1978]) every (continuous) finite dimensional representation of $\mathcal{G}_0(d)$ (as well as that of $P(d)$) factors through some finite phylon group $P_T(d)$. (Theorem 1.3 of Terng [11]). By looking at spaces W on which $\mathcal{G}_0(d)$ acts we may get some information on the set of indecomposable representations of $P(d)$, as well as invariant subspaces of W [21].

§4 Coordinate-free (or invariant) formulation of geometric string objects

In differential geometry there is general construction of a vector bundle E from a principal bundle P where its structure group G acts on a vector space F so that F is the fiber of E. That $E = P \times F(G)$. There is a natural principal bundle of frames on a manifold and we can construct an associated vector bundle from it.

1 Definition: A principal bundle consists of three manifolds F, P, G, denoted by $F(P, G)$, F total space, P base manifold, G is a Lie group called the structure group and G acts freely on F on the right and there is a smooth projection $\pi : F \to P$ such that $\pi^{-1}(p)$ called the fiber over p is diffeomorphic to G-orbits in F and π admits local sections so that $\pi^{-1}(U) \cong U \times G$ (local triviality).

Examples: 1) $F = P \times G$. Then $P \times G(P, G)$ is a trivial principal bundle.

2) Let E be a vector bundle over P of rank r. Then $F(E) = \bigcup_{p \in P} F(E_p)$ where E_p is fiber over p which is r-dimensional vector space and $F(E_p)$ is the set of all frames ($f : \mathbb{R}^r \to E_p$ linear isomerphism) of E_p. Then $F(E)(P, G = GL(r))$ is a principal bundle. In particular, P is manifold of dimension n. Then the tangent bundle $T(P)$ is a manifold and dimension $T_p(P) = n$ and $T(P)$ is vector bundle over P of rank n. Then $F(P) = F(T(P)) = \bigcup_{p \in P} F(T_p P)$ is a principal bundle over P with structure group $G = GL(n)$ i.e., $F(P)$ is a manifold on which $GL(n)$ acts freely and $\pi : F(P) \to P$ projection with fibers $\pi^{-1}(p) \cong GL(n)$. These are finite frame bundles.

2. Infinite frame bundle:

Now we consider for a statistical d-manifold $P(\subset \mathcal{P} \subset \mathcal{M})$ [6] and discuss its *infinite frame bundle* $\mathcal{F}^\infty(P)$. Let $p \in P$ and φ be a coordinate system about p. Then expand any function f about p in a Taylor series in coordinates φ. Let φ and ψ be two coordinate systems about p. We say φ and ψ are infinite jet equivalent if any function f expanded w.r.t. φ and ψ have same Taylor coefficients. Then the equivalence class $[\varphi]_p$ is called *an infinite frame at p*. Similarly r-frames at p for $1 \leq r < \infty$ can be defined for P. Denote by $\mathcal{F}^{\infty}(P)$ (respectively $\mathcal{F}^{r}(P)$ the collection of all infinite frames (respectively r-frames) at $p, \forall\, p \in P$. By above construction we get a principal bundle over P namely $\mathcal{F}^{\infty}(P)(P, \mathcal{P}(d))$ (respectively $\mathcal{F}^{r}(P)(P, \mathcal{P}_r(d))$ for $1 \leq r < \infty$). Note for $r = 1$, $\mathcal{F}^{1}(P)(P, \mathcal{P}_1(d) = GL(d))$ is the standard frame bundle of P as in differential geometry. Thus corresponding to various phylon groups $P_r(d)$ $(r = 1, 2, \cdots, \infty)$ (i.e $P_{\infty}(d) = P(d)$) we have the corresponding principal bundle of frames of P and in particular, the structure group of $\mathcal{F}^{\infty}(P)$ is the infinite phylon group $P(d)$.

3 Remark: Each $\mathcal{F}^{r}(P)$ is finite dimensional for $1 \leq r < \infty$ whereas for $r = \infty$, $\mathcal{F}^{\infty}(P)$ is infinite dimensional and the structure group $P(d)$ is also an infinite dimensional regular Frechet Lie group.

4. Associated vector bundle: We can associate a vector bundle E to a principal bundle $F(P, G)$ if a G-action is given on a vector space F ([13], [14] for general situations) as follows: consider the product $F \times V$. Define $(f_1, v_1) \sim (f_2 v_2)$ if $\exists g \in G$ such that $(f_1, v_1) \cdot g = (f_2, v_2)$, where $(f_1, v_1) \cdot g = (f_1 g, g_1^{-1} v_1)$ (respectively for representation ρ of G)). Let $E = \{[f, v] | (f, v) \in F \times V\}$. Then E is a $GL(r)$-vector *bundle over* P with fiber r-dimensional vector space V denoted by $E = F \times V(G, \rho)$ called the *associated* vector bundle of the prinicpal bundle $F(P, G)$ via the representation $\rho : G \to GL(V)$.

5. Special Cases: (a) $F^{r}(P)$ r-frame bundle over P $(1 \leq r \leq \infty)$. Let $\rho : P_r(d) \to GL(V)$ be a representation of phylon group $(1 \leq r \leq \infty)$. Then we get the associated vector bundle $E = F^{r}(P) \times V(P_r(d), \rho)$ over P with fiber V. We assume V has appropriate smoothness as necessary.

(b) $F(P)$ bundle of frames of manifold P, V vector space, and $T^{(r,s)}(V) = (\otimes^{r} V) \otimes (\otimes^{s} V^{*})$ is the tensor bundle of type (r, s). Take $V = \mathbb{R}^n$. We saw $GL(n)$ acts on $T^{(r,s)}(\mathbb{R}^n)$ by (3.6) giving a representation $\rho : GL(n) \to GL(T^{(r,s)}(\mathbb{R}^n))$. By above associated bundle construction we get the vector bundle $E = F(P) \times T^{(r,s)}(\mathbb{R}^n)(GL(n), \rho)$. Then the tensor field T with components $T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ satisfying the transformation law (3.6) precisely gives a global section of E.

(c) The Fisher information metric can be interpreted a section of the tensor bundle $T^{(1,2)}(P)$.

Now we briefly interpret various tensor fields or string fields or phylon fields as sections of vector bundles via) actions (or representations) on appropriate vector space. We give this in tabular form for d-dimensional manifold P in Appendix B.

§5 Structure of special phylon representations of $P(d)$

1. Another interpretation of phylon group:

Let P be a d-dimensional manifold, $m \in P$ such that $\omega(m) = \psi(m) = 0$, ψ, ω are two coordinate systems around $m \in P$. Then the multi-array $\{[\psi^{c}_{|k_1}], [\psi^{c}_{|k_1 k_2}], \cdots, [\psi^{c}_{|k_1 k_2 \cdots k_T}]\}$ evaluated at 0 where $1 \leq T \leq \infty$; is essentially the set of coefficients in the T^{th}-order (infinite order) Taylor series of the coordinate change function $\psi \circ \omega^{-1}$ from some open set of $\mathbb{R}^d$ to $\mathbb{R}^d$.

a) <u>T-finite case:</u> Define the phylon group of order T of $\mathbb{R}^d$ as

$P_T(d) = \{$multi-array $\{a^{c}_{k_1}, a^{c}_{k_1 k_2}, \cdots, a^{c}_{k_1 k_2 \cdots k_T}\} |$ with $a^{c}_{k_1 k_2 \cdots k_t}$ symmetric in $k_1, k_2, \cdots, k_t$

and with (a^c_k) forming a non-singular matrix and $c = 1, 2, \cdots, d\}$. Under identification of multi-arrays in $P_T(d)$ with T-th order $\mathbb{R}^d$-valued Taylor series

$$f^c(x) = a^c_{k_1} x^{k_1} + \frac{1}{2!} a^c_{k_1 k_2} x^{k_1} x^{k_2} + \cdots + \frac{1}{T!} a^c_{k_1 \cdots k_T} x^{k_1} x^{k_2} \cdots x^{k_T} \tag{5.1}$$

the group operation in $P_T(d)$ corresponds to composition of functions. In coordinate-free language, $P_T(d)$ is the group of T-jets at 0 of local diffeomorphisms of $(\mathbb{R}^d, 0)$ with itself. The group operation being composition (two functions have the same T-jet at a point x if they have the same T-th order Taylor series around x). Thus for finite T, $P_T(d)$ is the group of $\mathbb{R}^d$-valued polynomial functions on $\mathbb{R}^d$ of degree at most T with zero constant term and invertible linear term. Thus $P_T(d) = \{(A_1, A_2, \cdots, A_T) | A_i \in \odot^i (\mathbb{R}^d)^* \otimes \mathbb{R}^d, A_1 \in GL(d)\}$ where $\odot$ denotes the symmetric tensor product.
b) T-infinite: The above interpretations carry on for T infinite by replacing T-jet with infinite jet finite multiarray with infinite multiarray and finite Taylor series with infinite formal power series on $\mathbb{R}^d$. Define the infinite phylon group $P_\infty(d) = P(d)$ as

$$P(d) = \{(A_1, A_2, \cdots, A_T, \cdots, \infty) |\ A_i \in \odot^i (\mathbb{R}^d)^* \otimes \mathbb{R}^d, A_1 \in GL(d)\}\mathbb{R}^d. \tag{5.2}$$

and has the coordinate free interpretation as given before in §2.
2 Remark: Transformation rules for derivative and differential strings (3.2) and (3.4) can be interpretated as giving representations of $P(d)$ as follows: Note that the transformation laws for derivative and differential strings involve higher derivatives $\omega^k_{|C}$ and $\psi^c_{|K}$ of coordinate changes and give coordinate descriptions of representations of $P(d)$. Also these transformation laws are polynomials in $\omega^c_{|C}$ and $\det(\omega^k_{|C})^{-1}$; and hence these representations of $P(d)$ are algebraic [22]. Any such general algebraic representation of group $P(d)$ defines geometrical objects called *phyla* and these phyla are representated by arrays $H^{A_1 A_2 \cdots A_r}_{B_1 B_2 \cdots B_s}$ which transform under coordinate change from ω to ψ by

$$H^{A_1 A_2 \cdots A_r}_{B_1 B_2 \cdots B_s} = H^{I_1 I_2 \cdots I_r}_{J_1 J_2 \cdots J_s} D[\omega, \psi]^{A_1 A_2 \cdots A_r J_1 \cdots J_s}_{B_1 B_2 \cdots B_s I_1 \cdots I_r} \tag{5.3}$$

where $D[\omega, \psi]$ is the block matrix in which the elements of the blocks are polys in $\omega^i_{|A}$ and $\psi^a_{|I}$.

In fact we can order these arrays $H^{A_1, \cdots A_r}_{B_1 \cdots B_s}$ in such a way that the matrix $D[\omega, \psi]$ is an upper triangular block matrix. We call the function which takes a pair (ω, ψ) of local coordinate systems of P to the matrix $D[\omega, \psi]$ satisfying (i) each $D[\omega, \psi]$ is a non-singular upper triangular block matrix in which elements of the blocks are polys. in $\omega^i_{|A}$ and $\psi^a_{|I}$ and (ii) (cocycle condition) $D[\omega, \psi] = D[\omega, \chi] C[\chi, \psi]$.

We saw in §4 the strings or above phyla H can be interpreted as section Γ of the associated vector bundle E of the infinite frame bundle $F(P)$ by a representation of $P(d)$. Thus the study of the algebraic properties of these phyla or strings or tensors H reduces to the study of representations of $P(d)$.
3. Definition: Let V be a representation of $P_T(d)$ i.e. $\chi : P_T(d) \to GL(V)$ is group homomorphism; χ is called *algebraic* if in any basis for V the matrices representing elements

$[f] \in P_T(d)$ have entries that are polynomial functions of $f^i_{|j_1 j_2 \cdots j_k}$ and $\det(f^i_{j_1})^{-1}$. We say $\chi : P_T(d) \to GL(V)$ is finite dimensional or special phylon representation if it is algebraic in above sense. Here f has the formal power series $f = f^i_{j_1} x^{j_1} + \frac{1}{2!} f^i_{j_1 j_2} x^{j_1} x^{j_2} + \cdots$ and V is a complex vector space.

4. Remark: These phylon representations respect direct sum, tensor product, subrepresentations and quotient representations are also phylon. We have the following result from Terng [11].

5. Theorem (Terng): A representation $\chi : P_T(d) \to GL(V)$ is finite dimensional phylon (i.e. algebraic) representation iff $\exists$ a decomposition of V as $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ with V_i as a $GL(d)$-irreducibles such that for each element $fX \in P_T(d)$ with $f \in P^{(1)}_T(d)$ and $X \in GL(d)$

(i) $\chi(X) \in GL(V)$ is diagonal block matrix $\begin{pmatrix} \chi_1(X) & 0 & \cdots & 0 & \\ 0 & \chi_2(X) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \chi_k(X) & \end{pmatrix}$

(ii) $\chi(f) = \begin{pmatrix} 1 & \chi_{12}(f) & \cdots & \chi_{1k}(f) \\ 0 & 1 & \cdots & \chi_{2k}(f) \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \cdots 1 \end{pmatrix}$ (upper triangular block matrix for $f \in P^{(1)}_T(d)$)

(iii) $\chi(fX) = \chi(f)\chi(X)$ where $\chi_i(X) : V_i \to V_i$ linear operator and $\chi_{ij}(f) : V_j \to V_i$ linear map satisfying $\chi_{ii}(I) = I$, $\chi_{ij}(I) = 0$ $(i \neq j)$.

6. Remarks a): $\chi : P_T(d) \to GL(V)$ representation $\Rightarrow \chi(fX\ gY) = \chi(fX)\chi(gY)$ and hence $\chi_i(X)\chi_i(Y) = \chi_i(XY)$ i.e. $\chi_i : GL(d) \to GL(V_i)$ are representations and also

$$\chi_{ij}(XgX^{-1}) = \chi_i(X)\chi_{ij}(g)\chi_j(X)^{-1} \tag{5.4}$$

and χ_{ij}'s satisfy the relation.

(iv) $\chi_{ij}(fg) = \chi_{ij}(g) + \chi_{i,i+1}(f)\chi_{i+1,j}(g) + \chi_{i,i+2}(f)\chi_{i+2,j}(g) + \cdots + \chi_{i,j-1}(f)\chi_{j-1,j}(g) + \chi_{ij}(f)$ $(i < j)$ (5.5)

(Note that $\chi_{ij} : P^{(1)}_T(d) \to L(V_j, V_i) \cong V^*_j \otimes V_i$ and since $GL(d)$ acts on both sides and since (5.3) holds we have χ_{ij} are $GL(d)$-equivariant maps).

b) Conversely given χ_i and χ_{ij} satisfying the above three relations they define a phylon representation χ of $P_T(d)$ on V.

c) The *ordering* of these subspaces V_i can be so done that $\chi(f)$, $f \in P^{(1)}_T(d)$ are upper triangular block matrices with identity I on the diagonal.

Then χ is called a phylon representation of type $(V_1, V_2, \cdots, V_k)$ and k is called the *rank* of the representation. (this is equivalent to rearranging the components $H^{A_1 \cdots A_r}_{B_1 \cdots B_s}$ of the phylon H in the coordinate representation above).

d) Since $\chi(fX)$ is upper triangular block matrix $\forall\ fX \subset P_T(d)$ and $\chi_i : GL(d) \to GL(V_i)$ are representations and $\chi_{ij} : P^{(1)}_T(d) \to V^*_j \otimes V_i$ is an $GL(d)$-equivariant map we can arrange the arrays $H^{A_1 \cdots A_r}_{B_1 \cdots B_s}$ of phylon H into arrays representing $(P_1, P_2, \cdots, P_i, \cdots P_k)$ with $P_i \in V_i$. Then We can project phylon H onto other phyla which it contains say $(P_1, P_2, \cdots P_j)$ $j < k$, called the tails of the phylon H. That is, phyla of length ν

and block structure $(m_1, m_2, \cdots, m_\nu)$ projects to phyla of length μ and block structure $(m_1, m_2, \cdots, m_\mu)$ for $1 \leq \mu < \nu$. More precisely if phylon $P = (P_1, P_2, \cdots P_k)$ then its coordinate representatives are transformed by D-matrices

$$\begin{aligned} P(\psi) &= (P_1(\psi), \cdots, P_k(\psi)) \\ &= (P_1(\omega), P_2(\omega), \cdots, P_k(\omega)) \begin{pmatrix} D_{11}(\omega,\psi) & D_{12}(\omega,\psi) & \cdots & \cdots & D_{1k}(\omega,\psi) \\ 0 & D_{22}(\omega,\psi) & \cdots & \cdots & D_{2k}(\omega,\psi) \\ 0 & 0 & D_{33}(\omega,\psi) & \cdots & D_{3k}(\omega,\psi) \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & D_{kk}(\omega,\psi) \end{pmatrix} \end{aligned} \quad (5.6)$$

and so

$$\begin{aligned} &(P_1(\psi), P_2(\psi), \cdots, P_j(\psi)) \\ &= (P_1(\omega), P_2(\omega), \cdots, P_j(\omega)) \begin{pmatrix} D_{11}(\omega,\psi), & \cdots & \cdots & \cdots & D_{1j}(\omega,\psi) \\ 0 & D_{22}(\omega,\psi) & \cdots & \cdots & D_{2j}(\omega,\psi) \\ 0 & 0 & 0 & 0 & D_{jj}(\omega,\psi) \end{pmatrix} \end{aligned} \quad (5.7)$$

for $1 \leq j < k$ are also phyla of length j and type $(m_1, m_2, \cdots, m_j)$, called the tails of the phylon P.
e) Terng proved Theorem 3.8 of [11] in much more general setup above characterizing all phylons of $P_T(d)$ while studying natural vector bundles.
We illustrate Terng's theorem by an example
7. Example: $G = P_T(d)$ and the representation space V is the space of power series of degree $\leq T$ with no constant term denoted by $C_T(d)$ with $P_T(d)$-action given by $J_0^T(\varphi) \cdot J_0^T(f) = J_0^T(f \circ \varphi^{-1})$ with $J_0^T(\varphi) \in P_T(d)$ and $J_0^T(f) \in C_T(d)$.
Then $C_T = \bigoplus^T S^k(\mathbb{R}^d)^*; S^k(\mathbb{R}^d)^*$ is the space of polynomials of homogeneous degree k where $S^k(\mathbb{R}^d)^*$ is $GL(d)$- irreducible representation of $P_T(d)$ with $GL(d) \subset P_T(d)$. With respect to this decomposition every element of $P_T(d)$ can be represented by a block matrix say upper triangular and the entry in the i-th row and j-th column is a linear map $\in L(S^j(\mathbb{R}^{d^*}), S^i(\mathbb{R}^{d^*}))$ and the elements of $GL(d)$ are represented by block diagonal matrices and those of $P_T^{(1)}(d)$ are represented by upper triangular block matrices having identity matrices along the diagonal.

Now we close this section with finding all phyla of $P_2(d)$ of rank 2 by using Terng's theorem.
8. Action of $P_2(d)$ on $C_2(d)$: Consider $h = (h^1, h^2, \cdots, h^d) : \mathbb{R}^d \to \mathbb{R}^d$ with $h^i(x) = h^i_j x^j + h^i_{jk} x^j x^k$ with (h^i_j) invertible matrix ($i = 1$ to d) representing an element $j_o^2(h)$ in $P_2(d)$. Similarly elements of $C_2(d)$ are represented by $g : \mathbb{R}^d \to \mathbb{R}$ with $g(x) = g_i x^i + g_{il} x^i x^l$. The action of h^{-1} takes g to $j_o^2(g \circ h)$. since $(g \circ h)(x) = g_i h^i_j x^j + (g_i h_{jk} + h^i_j g_{il} h_k) x^j x^k +$ higher order terms and so

$$j_o^2(g \circ h)(x) = g_i h^i_j x^j (g_i h^i_{jk} + h^i_j g_{il} h^l_k) x^j x^k \quad (5.8)$$

Take $V_1 = S^2(\mathbb{R}^d)^*$ homogeneous quadratic function on $\mathbb{R}^d$, $V_2 = \mathbb{R}^{d^*}$ linear functions on $\mathbb{R}^d$ so that $C_2(d) = S^2(\mathbb{R}^d)^* \otimes \mathbb{R}^d$ Then the action of $P_2(d)$ on $C_2(d)$ gives the representation

$\chi : P_2(d) \to GL(C_2(d))$ for which

$$\chi(h) = \begin{pmatrix} h^i_j h^l_k & h^i_{jk} \\ 0 & h^i_j \end{pmatrix}. \tag{5.9}$$

9. Finding phylon representations of $P_2(d)$ in $C_2(d)$:
Now take a general splitting of V as $V = V_1 \oplus V_2$ where V_1, V_2 are $GL(d)$-irreducibles. By Terng scheme above to define a representation of $P_2(d)$ we have to choose the two representations $\chi_i : Gl(d) \to GL(V_i)$ $(i = 1, 2)$ and a map $\chi_{12} : P_2^{(1)}(d) \to V_2^* \oplus V_1$ such that $\chi_{12}(XgX^{-1}) = \chi_1(X)\chi_{12}(g)\chi_2(X^{-1})$ and

$$\chi_{12}(fg) = \chi_{12}(f) + \chi_{12}(g). \tag{5.10}$$

then for $f, g \in P_2^{(1)}(d)$ given by $f^i(x) = f^i_j x^j + f^i_{jk} x^j x^k$; $g^i(x) = g^i_j x^j + g^i_{jk} x^j x^k$ $(fg)^i = f^i_j x^j + (f^i_{jk} + g^i_{jk}) x^j x^k$. Since $P_2^{(1)}(d) \cong S^2(\mathbb{R}^d)^* \otimes \mathbb{R}^d$ is group isomorphism, we have by (5.10) $\chi_{12} : LS^2(\mathbb{R}^d)^* \otimes \mathbb{R}^d \to V_2^* \otimes V_1$ is an additive continuous map which is $GL(d)$-equivariant and so it is linear.
Thus the problem reduced to the study of these linear maps χ_{12}. For the case of splitting $V_1 = S^2(\mathbb{R}^d)^*$ and $V_2 = \mathbb{R}^{d*}$, and the corresponding representation of $P_2(d)$ of rank 2, note that $\chi_{12} : S^2(\mathbb{R}^d)^* \oplus \mathbb{R}^d \otimes S^2(\mathbb{R}^d)^2$ is the standard identification.

Our interest is in the isomorphism classes of such representations: χ, χ' phylon representations of $P_2(d)$ are isomorphic iff $\exists\ \Phi : V \to V'$ isomorphism s.t.

$$\chi(fX)\Phi(\nu) = \Phi(\chi(fX)(\nu)) \tag{5.11}$$

for all ν in V.
For $V = V_1 \oplus V_2$ then Φ can be taken into 2 by 2 block form. We take all representations χ of $P_2(d)$ as complex. By (5.11) Φ intertwines the $GL(d)$-actions obtained from χ and χ' and then Schur's lemma gives

$$\Phi = \begin{pmatrix} \alpha I_1 & 0 \\ 0 & \beta I_2 \end{pmatrix} \qquad \text{with } \alpha, \beta \in \mathbb{C}^*. \tag{5.12}$$

I_1, I_2 identity matrices of V_1 and V_2.
On the other hand, given a representation χ on $V_1 \oplus V_2$ and given arbitrary nonzero $\alpha, \beta \in \mathbb{C}^*$, (5.12) defines a Φ which intertwines χ and χ' where $\chi'_i = \chi_i$ $(i = 1, 2)$ and $\chi'_{12} = (\frac{\alpha}{\beta}\chi_{12})$.
Thus multiplying χ_{12} by a nonzero scalar gives rise to an isomorphic representation. Thus we reduced the problem to: the isomorphism classes of phylon representations of $P_2(d)$ of type (V_1, V_2) are in 1-1 correspondence with $GL(d)$-equivariant maps $\chi_{12} : S^2(\mathbb{R}^d)^* \otimes \mathbb{R}^d \to V_2^* \otimes V_1$ taken upto scaling in $\mathbb{C}^*$.
We close this section with finding all the phylon representations for $P_2(d)$ with $k = 2$ ($\cdot$ rank).
Let $X(V_1, V_2)$ denote the set of all such $GL(d)$-equivariant maps. Then the isomorphism classes of phylon representations of $P_2(d)$ are in 1-1 correspondence with the orbit space $X(V_1, V_2)/\mathbb{C}^*$ under $\mathbb{C}^*$-action.
If $S^2\mathbb{R}(d)^* \otimes \mathbb{R}^d$ is sum of two non-isomorphic irreducibles W_1 and W_2 and $V_2^* \otimes V_1$ is a

sum of r irreducibles $U_1, U_2, \cdots, U_r$ say. Then by Schur's lemma the only possible $GL(d)$-equivariant maps χ_{12} are those which map each W_i iesomorphically to U_j. That is, if there are p_1 U_js isomorphic to W_1 and $p_2 U_j$'s isomorphic to W_2 with $r = p_1 + p_2$ then $X(V_1, V_2) = \mathbb{C}^{p_1+p_2}$ and under $\mathbb{C}^*$-action the orbits are $\{0\}$ orbit or a complex line without origin. So $X(V_1, V_2)/\mathbb{C}^*$ is 0 orbit or complex projective space $\mathbb{P}C^{p_1+p_2-1}$.
The trivial representations of $P_2(d)$ corresponds to $\{0\}$ orbit. If $V_1 = \mathbb{R}^{d^*}$ and $V_2 = S^2\mathbb{R}^{d^*}$ then $X(V_1, V_2) = \mathbb{C}^2$ and the isomorphic classes of phylon representations correspond to points of 2-sphere $\mathbb{P}^1(\mathbb{C})$. This completes the example.
10. Remarks: (i) Terng classified all the phylon representations of $P_T(d)$ in terms of orbits under group action on the Lie algebra cohomology space [11] which is a systematic mathematical scheme but not explicitly easy to calculate.
In that sense phylon representations of $P_2(d)$ in $C_2(d)$ are geometrically realizable.
(ii) By carefully studying the features of tensors or strings occurring in stochastic calculus and asymptotic statistical inference and generalizing them may give a subclass of phylon representations amenable to computation.

§6 Infinite dimensional phylon representations of $P(d)$

1. Definition: Even though a general Frechet space F is a complete locally convex, metrizable topological vector space, we consider a particular one F which is a direct product $\prod_{i=1}^{\infty} V_i$ of a sequence of finite dimensional vector spaces with the component wise convergence topology. In this sense $P(d) = \mathbb{R}^d_{inv,0}[[x^1, x^2, \cdots, x^d)]$ becomes a *Frechat space.*
2. First note that at scalar function level the space of formal power series $\mathbb{R}[x^1, \cdots, x^d]]$ can be identified with $J_0^{\infty}(\mathbb{R}^d, \mathbb{R})$ the space of all infinite jets at the origin of smooth real valued, function on $\mathbb{R}^d$. Also we have the Taylor map $T : C^{\infty}(\mathbb{R}^d, \mathbb{R}) \to \mathbb{R}[[x^1 \cdot ; ; x^d]]$ and this is onto and ker T = ideal I of smooth function whose all partial derivatives at origin vanish and hence

$$J_0^{\infty}(\mathbb{R}^d, \mathbb{R}) \cong C^{\infty}(\mathbb{R}^d, \mathbb{R})/I \cong \mathbb{R}[[x^1, x^2, \cdots, x^d]]. \tag{6.1}$$

We can extend this identification to $\mathbb{R}^d$-valued smooth functions component wise using (6.1) and so we get $P(d)$ as two equivalent definitions we gave before and it has the Frechet manifold structure.
3. Remark: There is a natural duality between $\mathbb{R}[[x^1, \cdots, x^d]]$ formal poly ring and polynomial ring $\mathbb{R}[x^1, \cdots, x^d]$ of finite degree polys [23]. via distributions on $\mathbb{R}^d$.
4. We saw in the study of $P(d)$ in §2 in $P(d)$ there is a sequence $P^{(k)}(d)$ $(k = 1, 2, \cdots)$ of normal sub groups $(P^{(k)}(d) = \ker j^k : P(d) \to P_k(d))$ namely $P^{(k)}(d)$ consists of jets of diffeomorphism that agree to the identity upto order k (or in formal power series sense, $f^i(x) = \delta^i_j x^j + \frac{1}{(k+1)!} f^i_{j_1}, \cdots j_{k+1} x^{j_1} x^{j_2} \cdots x^{j_k} + \cdots$) such that $P(d)/P^{(k)}(d) \cong P_k(d)$ =group of k-jets of local diffeomorphism of $\mathbb{R}^d$ fixing the origin. these $P_k(d)$s are finite dimensional Lie groups $\forall\ k = 1, 2, \cdots$.
If $P(d)$ is given the pointwise convergence topology (Frechet) so that each $j^k : P(d) \to P_k(d)$ is continuous. In fact this is the projective limit topology on $P(d)$ making each j^k continuous map and hence $P(d)$ is an infinite dimensional Frechet Lie group with the underlying manifold a Frechet manifold.
5. Remark: If M is a Frechet manifold then (1) Inversion function theorem (ii) Uniqueness of solutions of OD Eqns. fail. Hence in $P(d)$ or more generally in any Frechet Lie group

G exponential map: $LG \to G$ may not exist (open problem) here LG denotes the Lie algebra of G. Omori [8] proved $P(d)$ is a regular Frechet Lie group (that is, an analogue of primitive or indefinite integral called C^1-hair which is a limit of sequences of areas defined by a sequence of step functions exists).
Then the Lie algebra $LP(d) = T_e(P(d))$ = the space of jets at 0 of vector field X on $\mathbb{R}^d$ with

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \tag{6.2}$$

where $X^i \in \mathbb{R}[[x^1, \cdots, x^d]]$ with constant term zero i.e., X is zero at the origin ($LP(d), [\,]$) is the Lie algebra of $P(d)$).
6. Remark: origin is a singularity and Omori classified these expansive singularities using Lie algebra techniques and using the regular Frechet Lie group $P(d)$ structure [8].

Since $P(d)$ is the projective limit of f.d. Lie groups $P_k(d)$ which all have the exponential map, $P(d)$ also get an exponential map which is smooth. On the other hand, $\exp : LP_k^{(1)}(d) \to P_k^{(1)}(d)$ is a diffeom [24].
Claim: $\exp LP^{(r)}(d) \to P^{(r)}(d)$ $(r \geq 1)$ is a smooth bijection. Infact one can show this by proving exponent on $LP^{(r)}(d)$ is restriction of exponential on $LP(d)$ which is smooth and its image is precisely $P^{(r)}(d)$.
7. Relation between representations of $P(d)$ and of its Lie algebra $LP(d)$:
Let $\rho : P(d) \to GL(V)$ be a representation of $P(d)$ on a Frechet space V and that the map $P(d) \times V \to V$ is smooth, then on differentiation ρ gives a representation $\tilde{\rho} : LP(d) \to gl(V)$ and the map: $LP(d) \times V \to V$ is smooth.

Let W be a closed subspace of finite co-dimension in V. Then we have the following:

Proposition 8: Let W be a closed subspace of finite codimension in a vector space V on which the phylon group $P(d)$ acts smoothly. Then (i) If $LP^{(k)}(d)$ stabilizes W then group $P^{(k)}(d)$ also stabilizes W. (ii) if $LP^{(k)}(d)(V) \subset W$ then the induced action of group $P^{(k)}(d)$ on V/W is trivial. $(k = 1, 2, \cdots)$
Proof: For $X \in LP^{(k)}(d)$ consider the curve $g_t = \exp(tX)$ and apply it to $v \in V$ and then differentiation gives

$$\frac{d}{dt}(g_t \cdot v) = X(g_t \cdot v) \tag{6.3}$$

Let $\pi : V \to V/W$ be the projection map. Then

$$\frac{d}{dt}\pi(g_t \cdot v) = \pi(X(g_t \cdot v)) \tag{6.4}$$

Then if either $LP^{(k)}(d)$ stabilizes W or $LP^{(k)}(d)(V) \subset W$ give $\pi(g_t \cdot v) = 0$. Since $g_o = I$, uniqueness of solutions of first order differential equations with values in V/W implies then $\pi(g_t \cdot v) = \pi(v)$ for all t. q.e.d
9. Infinite diemsnsional representation of $P(d)$: We studied the structure of finite dimensional (algebraic) representations of the phylon group $P(d)$ by projecting $P(d)$ onto $P_1(d) = GL(d)$. Now we define infinite dimensional representations of $P(d)$. First we give a definition following Terng [11]. Note that there exists a 1-dimensional sub group H of dilations of $\mathbb{R}^d$, namely $H = \{\lambda Id | \lambda \in \mathbb{R}^*\} \subset GL(d) \subset P(d)$. So if $P(d)$ acts linearly on a

vector space V then H also acts on V.
Definition 10: We say an element $v \in V$ is *homogeneous of degree* n if $\lambda \cdot 1 \cdot v = \lambda^n v$. All such vs form a subspace V_n of V *called the homogeneous subspace of degree* n.

From Terng's characterization result of §5 (theorem 5) we define
Definition 11: An infinite dimensional phylon representation on a Frechet space V is a group homomorphism:

$$P(d) \to GL(V) \tag{6.5}$$

such that (i) $V = \prod_{i=1}^{\infty} V_{n_i}$ (ii) V_{n_i} is the subspace of all elements of homogeneous degree n_i of V and $n_1 > n_2 > n_3 > \cdots$. (iii) for $i = 1, 2, \cdots$, each V_{n_i} is a finite dimensional $GL(d)$-module (iv) the map $P(d) \times V \to V$ is smooth as a map between Frechet manifolds. (6.6)
12. Remarks (i): The decomposition in (i) $V = \prod_{i=1}^{\infty} V_{n_i}$ is called *the homogeneous decomposition of* V.
(ii) More generally, any continuous action of $P(d)$ on a Frechet space V is called a phylon representation if V decomposes into a direct product of finite dimensional homogeneous subspaces whose degrees are bounded above.
13. Example of an infinite dimensional phylon representation: Let $P(d)$ be the infinite dimensional phylon group and $LP(d)$ is its Lie algebra. Let $P(d)$ act on $LP(d)$ by conjugation $(g, X) \to gXg^{-1}$, $g \in P(d)$, $X \in LP(d)$. In this case we have the homogeneous decomposition

$$LP(d) = \prod_{k=1}^{\infty} [R^d \otimes S^k(\mathbb{R}^d)^*] \tag{6.7}$$

and the homogeneous degrees are $0, -1, -2, \cdots$ with $LP(d)_{-k} = \mathbb{R}^d \otimes S^{k+1}(\mathbb{R}^d)^*$ for all k. The vector space structure of the subalgebra $LP^{(k)}(d)$ of $LP(d)$ is

$$LP^{(k)}(d) = \prod_{j \geq k} LP(d)_{-j} \tag{6.8}$$

Thus the adjoint representation of $P(d)$ in $LP(d)$ is an infinite phylon representation.
Theorem 14: Every phylon representation of $P(d)$ is a projective limit of finite dimensional representations of the phylon group.
Proof: Consider the dilation $\lambda \cdot 1 \in P(d)$. As λ varies over $\mathbb{R}^*$, we get a curve in phylon group $P(d)$ and its tangent vector at $\lambda = 1$ is the vector field

$$Z = \sum_i x^i \frac{\partial}{\partial x^i} \tag{6.9}$$

in the Lie algebra $LP(d)$.

Let v be an element of homogeneous degree n in a phylon representation (as in definition). Then $\lambda \cdot v = \lambda^n v$ (6.10), $\cdot$ is action on left and salar multiplication on the right side. Differentiating (6.10) at $\lambda = 1$ gives $Zv = nv$ (6.11)
In particular, if X is an element of $LP(d)$ of homogeneous degree $-k$, then we have $[Z, X] = -kX$ (6.12). Thus v has homogeneous degree n and X has homogeneous degree $-k$ and X act on v. Then $Z(Xv) = [Z, X]v + X(Zv) = -kXv + nXv = (n-k)Xv, \forall\, v \in V_n$ and so we have

$$\forall\ v \in V_n,\ \forall\ X \in LP(d)_{-k},\ Xv \in V_{n-k} \text{ i.e. } LP(d)_{-k}V_n \subset V_{n-k} \tag{6.12}$$

Then from (6.8) using the vector space structure of $LP^{(k)}(d)$ and (6.12) we get

$$LP^{(k)}(d)\left(\prod_{n<m} V_n\right) = \left(\prod_{j\geq k} LP(d)_{-j}\right)\left(\prod_{n<m} V_n\right) \subset \prod_{n<m-k} V_n \tag{6.13}$$

By proposition 8 (i) part on group action

$$P^{(k)}(d)\left(\prod_{n<m} Vn\right) \subset \prod_{n<m} V_n \tag{6.14}$$

Since $n_1 > n_2 > \cdots$ and using (6.13) and propostion 8 (ii) part with $W = \prod_{n\leq n_1-k} V_n$ we see that $P_k(d)$ acts on the quotient space

$$W_k = V \Big/ \prod_{n\leq n_1-k} V_n \tag{6.15}$$

Since V is the projective limit of these W_ks and these W_ks are the representations of the finite phylon groups $P_k(d)$, the given phylon representation is a projective limit of f.d. phylon representations of $P(d)$ q.e.d.

15. Twisted phylon representations:

Let V be a phylon representation with decomposition

$$V = \prod_{i=1}^{\infty} V_{n_i}, \quad V_{n_i} \text{ homogeneous subspace of degree } n_i \tag{6.16}$$

Consider the space $V[[t]]$ of all formal asymptotic power series in

$$t : v = v_0 + v_1 t + v_2 t^2 + \cdots \tag{6.17}$$

where each $v_i \in V$. From (6.16), $V[[t]]$ is bigraded as

$$\prod_{i\geq 1}\prod_{m\geq 0} V_{n_i}\ t^m \tag{6.18}$$

Hence $\forall\ v \in V[t]$ is of the form

$$v = \sum_{i\geq 1}\sum_{m\geq 0} v_i t^m \tag{6.19}$$

with $v_i \in V_{n_i}$.

Given an element $w = v_i t^a$ with $v_i \in V_{n_i}$, we call n_i *the homogeneous degree of* w and a *the asymptotic degree of* w.

We define the twisted action of $P(d)$ on $V[[t]]$ by

$$g * v = t^{-1} g t \cdot v \tag{6.20}$$

For example, let $X \in LP(d)_{-k}$ and $v \in V_{n_i}$. Then the twisted Lie algebra action is given by

$$X * vt^a = t^{-1}Xt \cdot vt^a = t^{-1}Xvt^{a+n_i} = (Xv)t^{a+k} \tag{6.21}$$

so that the homogeneous degree decreases and the asymptotic degree increases and hence

$$LP(d)_{-k}V_{n_i}t^m \subset V_{n_i-k}t^{m+k} \tag{6.22}$$

This tells how the twisted action $*$ works.

Now if we filter $V[[t]]$ by the subspaces $V(a) = V[[t]]t^a$ (by asymptotic degree filtration) so that

$$V[[t]] = V(0) \supset V(1) \supset V(2) \supset \cdots \tag{6.23}$$

then

$$LP(d)_{-k}V(a) \subset V(a+k). \tag{6.24}$$

Hence by proposition 8 (i) and (ii) parts we get an action of $P(d)$ on the quotient spaces

$$V(a)/V(a+k) \tag{6.25}$$

and the $LP^{(k)}(d)$ acts trivially on $V(a)/V(a+k)$. Hence the action of $P(d)$ on $V(a)/V(a+k)$ factors through an action of $P_k(d)$ (6.25a).

16. Remark: This has a deep application to the MLE in statistical inference relating the asymptotic order and parametric order in asymptotic expansions (see Appendix A).

17. The Coadjoint action of $P(d)$ on $LP(d)^*$: We saw the regular Frechet Lie group $P(d)$ has its Lie algebra $LP(d)$ as a direct product, its dual $LP(d)^*$ is a direct sum and hence it is not a Banach space and by a theorem that the dual F^* of a Frechet space F such that F^* is not a Banach space is never a Frechet space ([17]), $LP(d)^*$ is not a Frechet space and hence $P(d)$ has no action on $LP(d)^*$ and hence the coadjoint action $\rho : P(d) \times LP(d)^* \to LP(d)^*$ is *not a* phylon representation as we defined. That is, $LP(d)^*$ is not **a good space** for the action of $P(d)$ by coadjoint action on $LP(d)^*$.

18. Remark: Interpreting infinite dimensional strings when interpreted as phylons, the contravariant and covariant infinite strings behave quite differently.

19. Remark: Kirilov [25]'s orbit theory for infinite Lie groups provides a method for constructing a class of representations.

Let ξ be an element of $LP(d)^*$. We want to find the $P(d)$-orbit of ξ under coadjoint action. Let $<,>$ be the pairing map of V and V^*. Then $P(d) \times LP(d)^* \to LP(d)^*$ sending $(g, \xi) \to g^{-1}\xi g$ where $< g^{-1}\xi g, X >=< \xi, g^{-1}Xg >$ for all $X \in LP(d)$. We have projections

$$j^k : LP(d) \to LP_k(d) \tag{6.26}$$

and hence we have dual inclusions

$$j^{k*} : LP_k(d)^* \hookrightarrow LP(d)^* \tag{6.27}$$

Since the dual of the Lie algebra is a a direct sum (i.e. $LP(d)^* = \oplus_{i\geq 1}[\mathbb{R}^d \otimes S^i(\mathbb{R}^d)^*]$) if $\xi \neq 0$ then

$$\xi = (\xi_1, \xi_2, \cdots, \xi_k, 0, 0 \cdots) \ \text{ for some } k \tag{6.28}$$

with $\xi_i \in \mathbb{R}^d \otimes S^i(\mathbb{R}^d)^*$ and $\xi_k \neq 0$ and so $\xi = j^{k*}(\xi')$ where

$$\xi' = (\xi_1, \xi_2, \cdots, \xi_i, ; \cdots \xi_k) \in LP_k(d)^* \tag{6.29}$$

with k being the smallest integer such that (6.29) is true; k is called *the order of* ξ. Then for $g \in P(d)$ and $X \in LP(d)$ we have

$$\begin{aligned} < g^{-1}\xi g, X > &=< g^{-1}j^{k*}(\xi')g, X) =< \xi', j^k(g^{-1}Xg) > \\ &==< \xi', j^k(g)^{-1}j^k(X)j^k(g) > \\ &=< j^{k*}(j^k(g)^{-1}\xi' j^k(g)), X > . \end{aligned} \tag{6.30}$$

Hence we get the identity

$$g^{-1}\xi g = j^{k*}(j^k(g)^{-1}\xi' j^k(g)) \tag{6.31}$$

and hence the stabilizer of ξ, $\Omega_\xi = \{h \in P(d)|h^{-1}\xi h = \xi\}$ contains $P^{(k)}(d) = \ker$ of j^k and use (6.31) and $P_k(d) \cong P(d)/P^{(k)}(d)$ and so the group $P_k(d)$ acts transitively on $P(d)$-orbit of ξ and this orbit is finite dimensional. Movreover the coadjoint orbit of ξ is the image under $j^{k*} : LP_k(d)^* \hookrightarrow LP(d)^*$ of the coadjoint orbit of ξ' (under $P_k(d)$-coadjoint action) . Hence the representations of $P(d)$ obtained from the coadjoint orbits are actually representations of the finite dimensional phylon groups $P_k(d)$. Hence we proved the following.

Proposition 20: The orbit theory obtained from coadjoint orbits gives only finite dimensional phylon representations. i.e. Kirilov orbit theory gives no infinite dimensional phylon representations of $P(d)$.

21: We close this article with examples of $P(d)$-action on some other important spaces.

(1) Construct a $P(d)$- tensor field given by the projection $j^1 : P(d) \to GL(d) = P_1(d)$ as $[f] \to f^i_j v^j = f.v$, $v \in \mathbb{R}^d$.

This gives the classical tensor analysis i.e. $P(d)$-action restricted to 1-jet which gives linear action on $\mathbb{R}^d$.

(2) $\mathcal{C}$=space of all power series with no constant terms as

$$\varphi = \varphi_j z^j + \frac{1}{2!}\varphi_{jk} z^j z^k + \cdots \tag{6.32}$$

Let $P(d)$ act on $\mathcal{C}$ by composition as $f.\varphi = \varphi \circ f^{-1}$ $\mathcal{C}$ can also be understood as the vector space of functions defined about zero in $\mathbb{R}^d$ and vanishing at 0, upto jet equivalence.
Similarly $P_r(d)$ acts on $\mathcal{C}^r$ space of power series upto order r giving $P_r(d)$-tensor fields or representations of $P_r(d)$ on the vector space $\mathcal{C}^r$.
We also get Tensor fields of type $\mathcal{C}$ and its dual $\mathcal{C}^*$ as phylons from phylon representations of $P(d)$ on $\mathcal{C}$ or $\mathcal{C}^*$, which are structurally symmetric string fields.

General Problem 1: Determine all the indecomposable representations of $P(d)$.

(3) Let $\mathcal{S}$ denote the space of infinite contravariant strings with basis the set of all monomials in the coordinate vector fields $\frac{\partial}{\partial z^a}$ and a general element of $\mathcal{S}$ is of the form

$$\Gamma = \sum \Gamma^{a_1 a_2 \cdots a_d} \frac{\partial}{\partial z^{a_1}} \frac{\partial}{\partial z^{a_2}} \cdots \frac{\partial}{\partial z^{a_d}}$$

then $P(d)$ acts on $\mathcal{S}$: $(f, \Gamma) \to \Gamma \cdot f$ obeying some rules of differentiation giving $\mathcal{S}$-representations of $P(d)$.

(4) On the dual space $\mathcal{S}^*$, the space of infinite covariant strings or the space of certain linear differential operators D on germs of vector fields obeying certain differentiations rules, with costring coefficients

$$D_{i_1\cdots i_k} = D\left(\frac{\partial}{\partial z^{i_1}}, \frac{\partial}{\partial z^{i_2}}, \cdots, \frac{\partial}{\partial z^{i_k}}\right) \text{ for} k = 1, 2, \cdots$$

The vector space $\mathcal{S}^*$ contains the space $\mathcal{C}$ as a subspace under the identification $f \leftrightarrow D_f$ with $D_f(X_1, X_k) = X_1(X_2(\cdots(X_k)))f, X_i$: vector fields. Since the strings are structurally symmetric co-strings $P(d)$-respects symmetry, and so $\mathcal{C}$ is a $P(d)$-invariant subspace of $\mathcal{S}^*$.
Problem 2: Is $\mathcal{S}^*$ indecomposable under $P(d)$ action?
Problem 3: Are there other $P(d)$-invariant subspaces in $\mathcal{S}^*$ other than $\mathcal{C}$?
Proposition 22: Structurally symmetric strings are indecomposable, in otherwords, the space $\mathcal{C}$ is an indecomposable representation of $P(d)$.
Proof: Suppose $\mathcal{C}$ is the direct sum of two $P(d)$-invariant subspaces. Then one of these must contain an element with non-vanishing derivative or their sum is inside the subspace of $\mathcal{C}$ of functions with vanishing derivatives at the origin and hence is not all of $\mathcal{C}$. But then this subspace which contains a function with nonvanishing derivative at 0 contains all functions with non-vanishing derivatives at 0 by $P(d)$-invariance and hence is all of $\mathcal{C}$. So the space $\mathcal{C}$ is indecomposable. q.e.d
23 Remark: Let $I^k(\mathcal{C})$ be the subspace of functions in $\mathcal{C}$ vanishing to order k at 0 for $k \geq 1$.
Then we have a nested sequence of subspaces of $\mathcal{C} : \mathcal{C} = I^1(\mathcal{C}) \supset I^2(\mathcal{C}) \supset \cdots$
Problem 4: There are no other invariant subspaces in $\mathcal{C}$ under $P(d)$-action?
24. Remarks 1): Infact any $P(d)$-invariant subspace of $\mathcal{C}$ is either $\mathcal{C}$ or is inside $I^2(\mathcal{C})$ by proposition 22. Using techniques of germ equivalence at 0 and catastrophe theory [21] it may help for this problem.
Problem 5. Are there other $P(d)$-invariant subspaces of $\mathcal{C}^*$ other than $\mathcal{C}$ and $I^k(\mathcal{C})$s?
25. Before closing it must be mentioned that Terng studies in a different context the phylon group $P(d)$ ([10][11]).
26. Definition: A natural vector bundle v over n-manifolds assigns to each n-dimensional smooth manifold M a smooth fiber bundle over M with total space $F(M)$ s.t. if $\varphi : M \to N$ is an embedding then there is a bundle map $F(\varphi) : F(M) \to F(N)$ over φ on base manifolds such that F is continuous in some sense. Terng showed that natural vector bundles with d-dimensional fiber are given by representations of $P(d)$. i.e. finite dimensional phyla are precisely the elements of algebraic natural vector bundles i.e. natural vector bundles for which this representation is algebraic. For finite dimensional natural vector bundles the continuity of F is automatic and that the phyla of dimension d have coordinate forms which change by a representation of finite phylon group $P_T(d)$ with $T \leq 2d + 1$.
27. Remark: In fact Terng [11] studied phylon representations in her study of natural vector bundles and natural differential operators between them. She reduced the study of these problems to algebraic problems by showing that (i) natural vector bundles of order T over d-dimensional manifolds correspond to $P_T(d)$-modules (ii) natural differential operators of order k correspond to $P_{T+k}(d)$-equivariant maps between such modules. She also gave a general classification theorem for $P_T(d)$-modules in terms of orbits of a group action on the Lie algebra cohomology space. Our classification of phylon representations of $P_2(T)$ of rank 2 is a simple illustration of this deep theorem.
28. Remark: The phylon representations we have studied has deeper applications in statistical inference such as parameter-invariance of (i) Likelihood ratio test (ii) score test for

hypothesis and (iii) non-invariance of Wald [31] test. Also the asymptotic behavior of the maximum likelihood estimator (MLE) under parametric change and also the interrelation between the orders used there as (a) order of term in Taylor expansion w.r.t some parameters (b) order in the asymptotic expansion in terms of the size of the sample which are related by jet of the coordinate change function of certain degree.
These applications to statistics we discuss with more details in another article (cf. Appendix A).

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Appendix A

Applications to Statistical Asymptotics:

We are interested in Taylor expanding measures and so we assume the sample space Ω is a smooth manifold of dimension r or an open subset of $\mathbb{R}^r$ so that the measure μ can be taken as smooth function times Lebesgure measure.

(A.1) Definition: A density on a f.d. vector space is a map $\omega : F(V) \to \mathbb{R}$ s.t. for two bases $v = (v^1, v^2, \cdots, v^r)$ and $w = (w^1, w^2, \cdots, w^r)$, there exists $X \in GL(r)$ satisfying

$$\omega(v) = \omega(Xw) = |\det(X)|\omega(w) \tag{A.2}$$

Then the set $\Delta(V)$ of all densities on V is a 1-dimensional vector space.

(b) Consider $\Delta(\Omega) = \bigcup_{x\in\Omega} \Delta T_x(\Omega)$ which is a vector bundle of rank 1. Then a smooth density on Ω is a smooth choice of density on each tangent space and so is a smooth section of $\Delta(\Omega)$ i.e. an element of $\Gamma(\Delta(\Omega))$.

For Ω=open subset $U \subset \mathbb{R}^r$, these are of the form

$$f(x)dx^1dx^2 \cdots dx^r \tag{A.3}$$

Locally every density on manifold Ω is of this form. We can consider the infinite jet of section in $\Gamma(\Delta(\Omega))$ i.e. $J_x^\infty \Delta(\Omega))$ denotes the vector space of infinite jets of section of $\Delta(\Omega)$ at x.

Let P be a family of (mutually absolutely continuous probability measures on the sample space Ω [7]. Denote by $f(w, p)$ the value at w of the Radon-Nikodym derivative of the prob. measure p w.r.t. some reference measure (see [7]). Then the max· Likelihood estimator (MLE) based on random samples of size N is the function $M^N : \Omega^N \to P$ defined by $M^N(w_1, \cdots, \omega_N)$ being the element $p \in P$ (assumed unique) which maximizes $\prod_{i=1}^N f(w_i, p)$. If p is any point of P then the push-out $M_*^N(p)$ is a measure on P. We are interested in its asymptotic behavior as $N \to \infty$. More generally it is of interest to study the measure

$$(N^{1/2}\varphi)_* M_*^N(p) \tag{A.4}$$

where

$$\varphi : P \to T_p(P) \tag{A.5}$$

is a map sending p to 0 which is a local diffeomorphism.

This measure (A.4) is asymptotically a normal distribution on $T_p(P)$ with mean 0 and variance dual of Fisher metric [28] Cox and Hinkley 1974. We want to understand the way in which the asymptotics of $(N^{1/2}\varphi)_* M_*^N(p)$ depends on the choice of φ.

For this we assume that $M_*^N(p)$ is a density on P that is a special push-out measure and consider its infinite jet

$$J_p^\infty(M_*^N(p)) \in J_p^\infty(\Delta(P)) \tag{A.6}$$

Then we choose a set of coordinates at p or its infinite jet

$$J_p^\infty(\varphi) \in J_p^\infty(P, \mathbb{R}^d)_p^o \tag{A.7}$$

Then define the element

$$J_0^\infty \left(\left(N^{1/2}\varphi \right)_* M_*^N(p) \right) \in J_0^\infty(\Delta(\mathbb{R}^d) \tag{A.8}$$

Thus $(N^{1/2}\varphi)_* M_*^N(p)$ is sequence of measures on $\mathbb{R}^d$ denoted by $\mu(N)$.
We can associate to this sequence $\mu(N)$ of measures a formal asymptotic series as

$$\mu_\varphi = \mu_0 + \frac{1}{N^{1/2}}\ \mu_1 + \frac{1}{(N^{1/2})^2}\ \mu_2 + \cdots \tag{A.9}$$

where the μ_r are signed measures on $\mathbb{R}^d$.

If we change the choice of coordinates from φ to χ we have then $\chi = g\cdot\varphi$ for $g \in P(d)$ the infinite phylon group. The corresponding change in the asymptotic series can be computed from

$$\left(N^{1/2}\chi\right)_* M_*(p) = \left(N^{1/2} g\cdot\varphi\right)_* M_*(p) = \left(N^{1/2} g\ \frac{1}{N^{1/2}}\right)_* \left(N^{1/2}(\varphi)_* M_*(p)\right) \tag{A.10}$$

Thus if μ is the asymptotic series of measures $\mu(N)$ of (A.9) induced by coordinates φ then the asymptotic series induced by coordinates χ is given by the sequence

$$\mu_\chi(N) = \left(N^{1/2}\ g\ \frac{1}{N^{1/2}}\right)_* \mu_\varphi(N) \tag{A.11}$$

Let $\mathcal{M}$ denote the space $J_0^\infty(\Delta(\mathbb{R}^d))$.
On passing to jets we obtain an action of $P(d)$ on $\mathcal{M}$ and we call this action of $P(d)$ the twisted phylon action (inview of tgt^{-1}-action of §6).
(Note that there is also a natural action of $g \in P(d)$ on the measures μ_i). Let, $\varphi : (P,p) \to (\mathbb{R}^d, 0)$ be local diffeom sending p to 0 and $\chi : (P,p) \to (\mathbb{R}^d, 0)$ local diffeom which are two coordinate systems on P. Then $\chi = g\cdot\varphi$, $g \in P(d)$ and $g = \dot{J}_0^\infty(\psi\circ\phi^{-1})$ and $P(d)$ acts on $\mathcal{M} = J_0^\infty(\Delta(\mathbb{R}^d))$ i.e. $P(d)\times\mathcal{M} \to \mathcal{M}$ is such that

$$J_0^\infty\left(\mu_\chi(N)\right) = g\cdot J_0^\infty(\mu_\varphi(N)) = \left(\sqrt{N}\, g\ \frac{1}{\sqrt{N}}\right)_* J_0^\infty\left(\mu_\varphi(N)\right) \tag{A.12}$$

where $(\sqrt{N}\, g\, \frac{1}{\sqrt{N}})_*$ denotes the composition of three elements of $P(d)$ as $\sqrt{N}\cdot 1$ and $\frac{1}{\sqrt{N}}1$ are dilations of $\mathbb{R}^d \in GL(d) \subset P(d)$ and $g \in P(d)$.
(A.13) Interpretation of $\mathcal{M} = J_0^\infty(\Delta(\mathbb{R}^d))$ the space of infinite jets of measures on $\mathbb{R}^d$: We can interpret elements of $\mathcal{M}$ as infinite formal power series multiplied by the Lebesqure measure and so they can be decomposed into sum of homogeneous terms of degree k for each $k \geq 0$. Hence the subspaces C_k consisting of homogeneous functions of degree k for each $k \geq 0$ are the irreducible $GL(d)$-factors of $\mathcal{M}$. Recall that for $g \in P(d)$, $f \in C_k$, $g.f = f\circ g^{-1}$ and that the action of a dilation $\lambda\cdot 1$ on the Lebesqure measure is multiplied by λ^d and so $g\cdot(f\mu) \in C_{d-k}$ where $g =$ dialation $\lambda\cdot 1$.

Recall the twisted phylon action of $P(d)$ on $\mathcal{M}$ and the discussion of the twisted phylon action of $P(d)$ on the space of formal asymptotic series $V[[t]]$ of §6 paragraph 15 using $t = \frac{1}{\sqrt{N}}$ and filtration by asymptotic degree gives $\mathcal{M}(0) \supset \mathcal{M}(1) \supset \cdots \supset \mathcal{M}(k) \supset \cdots$ and hence from that result (6.25a) of §6, the $P(d)$-action on $\mathcal{M}(0)/\mathcal{M}(k)$ factors through an

action of $P_k(d)$.
In general $LP(d)_{-k} * \mathcal{M}(m) \subset \mathcal{M}(m+k)$ as

$$X^* c_l t^m = t^{-1} X t(c_l t^m) = (X c_l) t^{k-(d-l)} t^{d-l} t^m \in LP(d)_{-k} c_l \in C_l \tag{A.14}$$

for $X \in LP(d)_{-k}$, and $c_\in C_l$.
Now take $m = 0$.
So $LP(d)_k * \mathcal{M}(0) \subset \mathcal{M}(k)$ by (A.14) and so $P(d)$-action on the quotient space $\mathcal{M}(0)/\mathcal{M}(k)$ factors through $P_k(d)$ as $P^{(k)}(d)$ acts trivially on $W_k = \mathcal{M}(0)/\mathcal{M}(k)$ by (6.25a).
This means if we want to understand the behavior of the asymptotic expansion of the sequence of measures $\mu_\phi(N)$ say upto order $\leq k$ in $t = \frac{1}{\sqrt{N}}$ (i.e. with asymptotic order k) under the change of parameter φ to χ i.e. under the action of phylon group $P(d)$ i.e. $\chi = g \cdot \varphi$, $\exists g \in P(d)$ (infact $g = J_0^\infty(f = \chi \circ \varphi^{-1})$) then the twisted phylon action results in $\mu_\chi(N) = (N^{1/2} g \frac{1}{N^{1/2}})_* \mu_\phi(N)$ the transformation formula for the corresponding coordinate sequences of measures on $\mathbb{R}^d$. By (6.25a) we need only to consider the Taylor expansion of $\psi = \chi \circ \varphi^{-1}$ upto order k i.e. $J_0^k(\psi) = g$ in the φ-coordinates on $\mathbb{R}^d$ at origin that is of parameteric order k. In otherwords the $P(d)$-phylon field $\{\mu_\varphi(N)\}_\varphi$ on P has a $P_k(d)$-reduction.
This gives a relation between the asymptotic order and the parametric order of asymptotic series expansion of parametric distributions with smooth density functions in signed measures.

Appendix B
The statisticians discovered several strings in statistical inference as arrays and gave their transformation formula under coordinate change. These formulae can be interpreted as global sections of certain vector bundles and infact they must be understood in this natural setting. We tabulated these statistical strings as sections of associated vector bundles (ASVs) of some frame bundles:

(1)	(2)	(3)	(4)	(5)
S.No.	Manifold	Tensor/String	Action/reprn.	Concerned bdle
1(a)	$P=\mathbb{R}^d$	$T=T^i_{jk}$	$GL(d)$-action on $V^{(1,2)}=\mathbb{R}^d\otimes\otimes^2\mathbb{R}^{d^*}$	$T^{(1,2)}(P)$
(b)	$P=\mathbb{R}^d$	$T=T^r_s=T^{i_1i_2\cdots i_r}_{j_1j_2,\cdots j_s}$	$GL(d)$-action on $V^{(r,s)}=\otimes^r\mathbb{R}^d\otimes\otimes^s\mathbb{R}^{d^*}s$	$T^{(r,s)}(P)$
2(a)	$P=\mathbb{R}^d$	$H=H_C\in\mathcal{S}^{00}_{0T}(P)$ spl. derivative string of length T	$P_T(d)$-action	$H^T(P)$:bdle of of T-frames on P
2(b)	$P=\mathbb{R}^d$	$H=\left(H^{IL}_{JK}\right)\in\mathcal{S}^{rU}_{sT}(P)$ general derivative string (Table 3.2)	$P_\lambda(d)$-action $\lambda=\max(T,U)$	$H^\lambda(P)$: bdle of λ-frames on P
2(c)	$P:d$-mfld	$H=$ same as above	same as above	same as above
3.	P d-mfld BJK [12]	$H=\left(H^{ILM}_{JKN}\right)\in\mathcal{D}^{rUp}_{sTq}(P)$ diffl. String, $T<\infty$ T.L (3.4)	$P_\lambda(d)$-action	
4(a)	P d-mfld BB[30]	$H=\left(H^{AD}_{BC}\right)\in\mathcal{S}^{rU}_{sT}(P)$ str.symm.tensors,$<\infty$	same as in 2(c)	
4(b)	P d-mfld (JUPP)	$H=\left(H^{IL}_{JK}\right)\in\mathcal{S}^{rU}_{sT}(P)$ genl.str.(non-symm.) $1\le T<\infty$, $U<\infty$	same as in 2(c)	
4(c)	P-d mfld.	$H=\left(H^{ILM}_{JKN}\right)$ general diffl.str., $T<\infty$	same as in 3	same as in 3
5	P d-mfdl.	$H=(A_1,A_2,\cdots,\infty)$ $T=\infty$ Case $A_i\in\mathbb{R}[[x^1,\cdots,x^d]]$		

(6)	(7)
Fibre of ve. bdle.	Section realization of ASVs
$V^{(1,2)}$ $V^{(r,s)}$	$\{T(\omega)\}_\omega$ s.t. $T(\omega)=\rho(\ )T(\varphi)$ same as above
$L(\otimes^T\mathbb{R}^d,\mathbb{R})$ $F=\mathcal{L}_{T,0}(\mathbb{R}^d,\mathbb{R})\otimes\mathcal{L}_{U,0}(\mathbb{R}^d,\mathbb{R})^*$ $\otimes^r\mathbb{R}^d\otimes\otimes^s\mathbb{R}^{d*}$ same as above	$H=(H_C)s.t\ H_C=H_K\omega^K_{\vert C}$ $H=(H^{IL}_{jk})$ as section of ASV H as section of $\mathcal{S}^{rU}_{sT}(P)=\otimes^rTP\otimes\otimes^sTP^*$ $\otimes\mathcal{S}^{00}_{0T}(P)\otimes\mathcal{S}^{00}_{0U}(P)^*$
$F=\otimes^r\mathbb{R}^d\otimes\otimes^s\mathbb{R}^{d^*}$ $\otimes\mathcal{L}_T(\mathbb{R}^d,\mathbb{R}^{dp})\otimes\mathcal{L}_T(\mathbb{R}^d,\mathbb{R}^{dq})^*$ with Cauchy product or convolutive multiplication	H as section of $H^\lambda(P)\ \mathcal{D}^{rU[}_{sTq}(P)=$ $\otimes^rTP\otimes\otimes^sTP\otimes$ $\otimes\mathcal{D}^{00p}_{0T0}(P)\otimes\mathcal{D}^{0Uq}_{000}(P)^*$
same as in 2(c) case except $\mathcal{S}^{00}_{0T}(P)$ and $\mathcal{S}^{00}_{0U}(P)$ have jet bdle interpretation by symmetry. same as above	H as a section of ASV $\otimes^rTP\otimes^s T^*P\otimes J^{T,0}(P,\mathbb{R})\otimes J^{U,0}(P,\mathbb{R})^*$ where $J^{T,0}(P,\mathbb{R})$ =space of T-jet of $\mathcal{C}^\infty_0(M,\mathbb{R})$ with 0-truncation and $f(x)=0$. F-complicated fiber H as section of $\mathcal{S}^{rU}_{sT}(P)\cong\otimes^4TP\otimes^s T^*P\otimes\bar{J}^{T,0}(P,\mathbb{R})\otimes\bar{J}^{U,0}(P,\mathbb{R})^*$ where $\bar{J}^{T,0}(P,\mathbb{R})$ is space of 0-truncated semi-holonomic jets of fns. in $\mathcal{C}^\infty(P,\mathbb{R})$.
6 and 7	
4(c) H as section of $\mathcal{D}^{rUp}_{sTq}(P)\cong\otimes^rTP\otimes^s T^*P\otimes\bar{J}^T(\otimes^pTP)\otimes\bar{J}^T(\otimes^qT^*P)$, $\bar{J}$ as in 4(b)	
5) $\mathcal{F}^\infty(P)$ pro. bdle of infinite, frames in P; $P(d)$: str.gp, $\rho: P(d)\to GL(V)$ reprn.	a) V:f.d. ve.sp. b) V: infinite ve.sp. (Frechet) In case (a) H is realized as a spl. phylon reprn. as a section of $E=\mathcal{F}^\infty(P)\times V(P(d),\rho)$. For (b), H is realized as an infinite phylon reprn. action of E.