Chaki-pseudo parallel invariant submanifold of sasakian manifolds with respect to certain connections

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Abstract

This paper deals with the study of Chaki-pseudo parallel submanifolds of Sasakian manifold. In view of Chaki-pseudo parallel condition, totally geodesic property have been investigated with respect to semisymmetric metric connection, Schouten-van Kampen connection and Tanaka Webstar connections.

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1 Introduction

A $(2n+1)$-dimensional smooth manifold $\bar{M}$ is said to have an almost contact metric structure (ϕ, ξ, η, g) if the following relations hold [6]:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{1.1}$$

$$\eta(\xi) = 1, \; g(X, \xi) = \eta(X), \; \eta(\phi X) = 0, \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \; g(\phi X, Y) = -g(X, \phi Y) \tag{1.3}$$

for any vector fields X and Y on $\bar{M}$, where ϕ is a tensor of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on $\bar{M}$. A smooth manifold $\bar{M}$ equipped with an almost contact metric structure (ϕ, ξ, η, g) is called an almost contact metric manifold.

The fundamental 2-form Φ on $\bar{M}$ is defined by $\Phi(X, Y) = g(X, \phi Y)$ and the Nijenhuis tesor of $\bar{M}$ is given by

$$N(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi,$$

where d denotes the exterior derivative. Also the almost contact metric structure (ϕ, ξ, η) is called normal if and only if the Nijenhuis tensor vanishes. An almost contact metric manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$ with normal structure is called Sasakian [6] if it satisfies $\Phi = d\eta$.

The notion of invariant submanifolds that geometry inherits almost all properties of ambient manifold.The geometry of invariant submanifold of Sasakian manifolds is carried out from 1970's by M. Kon [12], K. Yano and M. Kon [23]. Thereafter several authors studied invariant submanifolds of different ambient manifolds such as ([2], [4]).

Semisymmetric linear connection was introduced by Friedmann and Schouten in 1924 [9]. After that in 1932 [10] Hayden introduced the idea of metric connection on a Riemannian manifold. Later, Yano [22] and many others (see, [15], [17] and references therein) studied semisymmetric metric connection in different context.

The Schouten-van Kampen connection introduced for the study of non-holomorphic manifolds ([16], [20]). In 2006, Bejancu [3] studied Schouten-van Kampen connection on foliated manifolds. Recently Olszak [14] studied Schouten-van Kampen connection on almost(para) contact metric structure.

The Tanaka-Webster connection ([18], [21]) is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. Tanno [19] defined the Tanaka-Webster connection for contact metric manifolds. Recently Hui and mandal [11] introduced the notion of Chaki-pseudo parallel manifolds. In this paper we have studied Chaki-pseudo parallel invariant submanifolds of Sasakin manifolds with respect to certain connections namely, Riemannian connection, semisymmetric connection, Schouten-van Kampen connection, Tanaka Webstar connection. An example of such submanifolds with respect to Schouten-van Kampen connection is also constructed. Finally at the end we gave a conclusion section.

2 Preliminaries

In an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$, we have [5]

$$(\bar{\nabla}_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.1}$$

$$\bar{\nabla}_X \xi = -\phi X, \tag{2.2}$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}^{2n+1}$.
The semisymmetric metric connection $\tilde{\bar{\nabla}}$ and $\bar{\nabla}$ on $\bar{M}^{2n+1}$ are related by [22]

$$\tilde{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi. \tag{2.3}$$

The Schouten-van Kampen connection $\hat{\bar{\nabla}}$ and $\bar{\nabla}$ of $\bar{M}^{2n+1}$ are related by [14]

$$\hat{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.4}$$

The Tanaka-Webster connection $\overset{*}{\bar{\nabla}}$ and $\bar{\nabla}$ of $\bar{M}^{2n+1}$ are related by [8]

$$\overset{*}{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.5}$$

3 Main Results

Let M^{2m+1}, $m < n$, be an invariant submanifold of a Sasakian manifold $\bar{M}^{2n+1}(\phi, \xi, \eta, g)$. Let ∇, $\widetilde{\nabla}$, $\hat{\nabla}$ and $\overset{*}{\nabla}$ be the induced connection of M from $\bar{\nabla}$, $\bar{\widetilde{\nabla}}$, $\bar{\hat{\nabla}}$ and $\bar{\overset{*}{\nabla}}$ respectively. Then the Gauss formula with respect to above four connections are [24]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3.1}$$

$$\bar{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X, Y), \tag{3.2}$$

$$\bar{\hat{\nabla}}_X Y = \hat{\nabla}_X Y + \hat{h}(X, Y), \tag{3.3}$$

$$\bar{\overset{*}{\nabla}}_X Y = \overset{*}{\nabla}_X Y + \overset{*}{h}(X, Y) \tag{3.4}$$

for all $X, Y \in \Gamma(TM)$ and $h, \widetilde{h}, \hat{h}, \overset{*}{h}$ are the second fundamental forms of M with respect to ∇, $\widetilde{\nabla}$, $\hat{\nabla}$ and $\overset{*}{\nabla}$ respectively. The covariant derivative of the second fundamental form is given by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{3.5}$$

for any vector fields X, Y, Z tangent to M.

On the analogy of almost contact Hermitian manifolds, the invariant and anti-invariant submanifolds are depend on the behaviour of almost contact metric structure ϕ. A submanifold M of an almost contact metric manifold is said to be invariant [4] if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for any vector field X tangent to M at every point of M, that is $\phi(TM) \subset TM$ at every point of M. Now, for invariant submanifolds of Sasakian manifold the following relation hold:

$$h(X, \xi) = 0. \tag{3.6}$$

A submanifold M of a Sasakian manifold $\bar{M}$ with respect to $\bar{\nabla}$ (respectively $\bar{\widetilde{\nabla}}$, $\bar{\hat{\nabla}}$ and $\bar{\overset{*}{\nabla}}$) is called Chaki-pseudo parallel if its second fundamental form h (respectively $\widetilde{h}$, $\hat{h}$, $\overset{*}{h}$) satisfies

$$(\nabla_X h)(Y, Z) = 2\alpha(X)h(Y, Z) + \alpha(Y)h(X, Z) + \alpha(Z)h(X, Y) \tag{3.7}$$

respectively

$$(\widetilde{\nabla}_X \widetilde{h})(Y, Z) = 2\alpha(X)\widetilde{h}(Y, Z) + \alpha(Y)\widetilde{h}(X, Z) + \alpha(Z)\widetilde{h}(X, Y) \tag{3.8}$$

$$(\hat{\nabla}_X \hat{h})(Y, Z) = 2\alpha(X)\hat{h}(Y, Z) + \alpha(Y)\hat{h}(X, Z) + \alpha(Z)\hat{h}(X, Y) \tag{3.9}$$

$$(\overset{*}{\nabla}_X \overset{*}{h})(Y, Z) = 2\alpha(X)\overset{*}{h}(Y, Z) + \alpha(Y)\overset{*}{h}(X, Z) + \alpha(Z)\overset{*}{h}(X, Y) \tag{3.10}$$

for all X, Y, Z on M, where α is a nowhere vanishing 1-form. In particular, if $\alpha(X) = 0$ then h is said to be parallel and M is said to be parallel submanifold of $\bar{M}$. We now prove the following:

Theorem 3.1. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$. Then M is totally geodesic if and only if M is Chaki-pseudo parallel with respect to $\bar{\nabla}$.*

Proof. Suppose that M is a Chaki-pseudo parallel invariant submanifold of $\bar{M}$. Then by virtue of (3.5) we have from (3.7)

$$\begin{aligned} &\nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \\ = \ & 2\alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y). \end{aligned} \tag{3.11}$$

Putting $Z = \xi$ and using (3.6) we compute

$$h(\phi X, Y) = \alpha(\xi)h(X,Y). \tag{3.12}$$

Replacing X by ϕX in (3.12) and using (1.1) we get

$$\alpha(\xi)h(\phi X, Y) = -h(X,Y). \tag{3.13}$$

From (3.12) and (3.13) we have, $[\{\alpha(\xi)\}^2 + 1]h(X,Y) = 0$, which implies that $h(X,Y) = 0$ for all X, Y on M as $\{\alpha(\xi)\}^2 + 1 \neq 0$. Hence M is totally geodesic submanifold.
The converse is trivial. □

Corollary 3.1. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\bar{\nabla}$. Then M is totally geodesic if and only if M is parallel with respect to $\bar{\nabla}$.*

We now consider M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$. Then we have

Theorem 3.2. *[2] Let M be a submanifold of a Sasakian manifold $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$. Then*
(i) M admits induced semisymmetric metric connection $\widetilde{\nabla}$,
(ii) The second fundamental forms with respect to Levi-Civita connection and semisymmetric connection are equal, i.e. $h = \widetilde{h}$.

We prove the following:

Theorem 3.3. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\bar{\widetilde{\nabla}}$. Then M is totally geodesic if and only if M is Chaki-pseudo parallel with respect to semisymmetric metric connection.*

Proof. Suppose that M is Chaki-pseudo parallel with respect to $\bar{\widetilde{\nabla}}$. Then by virtue of Definition 3.1 and Theorem 3.2, we have

$$(\widetilde{\nabla}_X h)(Y,Z) = 2\alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y). \tag{3.14}$$

In view of (2.3) and (3.6) we have from (3.14) that

$$\begin{aligned} &(\nabla_X h)(Y,Z) + g(h(Y,Z),\xi) - g(X, h(Y,Z))\xi \\ &- \eta(Y)h(X,Z) - \eta(Z)h(X,Y) \\ &= 2\alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y) \end{aligned}$$

i.e.,

$$
\begin{aligned}
(3.15)\qquad & \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) + h(Y, \nabla_X Z) \\
& + g(h(Y,Z),\xi) - g(X, h(Y,Z))\xi \\
& - \eta(Y)h(X,Z) - \eta(Z)h(X,Y) \\
& = 2\alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y).
\end{aligned}
$$

Putting $Z = \xi$ in (3.15) and using (3.6) we get

$$(3.16)\qquad -h(Y, \nabla_X \xi) - h(X,Y) = \alpha(\xi)h(X,Y).$$

By virtue of (2.2), we have from (3.17) that

$$(3.17)\qquad h(Y, \phi X) - \{\alpha(\xi) + 1\}h(X,Y) = 0.$$

Replacing X by ϕX in (3.17) and using (1.1) and (3.6) we get

$$(3.18)\qquad h(Y,X) + \{\alpha(\xi) + 1\}h(\phi X, Y) = 0.$$

From (3.17) and (3.18) we get

$$[\{\alpha(\xi) + 1\}^2 + 1]h(X,Y) = 0,$$

which implies that $h(X,Y) = 0$ as $\{\alpha(\xi) + 1\}^2 + 1 \neq 0$.
The converse part is trivial. □

Thus we can state the following:

Corollary 3.2. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\tilde{\tilde{\bar{\nabla}}}$. Then M is totally geodesic if and only if M is parallel with respect to $\tilde{\tilde{\bar{\nabla}}}$.*

We now consider M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to Schouten-van Kampen connection $\hat{\bar{\nabla}}$. Then we prove the following:

Theorem 3.4. *Let M be a submanifold of a Sasakian manifold $\bar{M}$ with respect to $\hat{\bar{\nabla}}$. Then*
(i) M admits induced Schouten-van Kampen connection $\hat{\nabla}$,
(ii) The second fundamental forms with respect to Levi-Civita connection and Schouten-van Kampen connection are equal, i.e. $h = \hat{h}$.

Proof. By virtue of (2.4), (3.1) and (3.3) we have

$$(3.19)\qquad \hat{\nabla}_X Y + \hat{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(Y)\phi X - g(\phi X, Y)\xi$$

for $X, Y \in \Gamma(TM)$. Now by equating tangential and normal components of (3.19) we have

$$(3.20)\qquad \hat{\bar{\nabla}}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi,$$

and

$$(3.21)\qquad \hat{h}(X,Y) = h(X,Y).$$

Hence the theorem is proved. □

Next we prove the following:

Theorem 3.5. *A submanifold M of $\bar{M}$ with respect to $\hat{\bar{\nabla}}$ is totally geodesic if and only if it is Chaki-pseudo parallel with respect to $\hat{\bar{\nabla}}$ connection.*

Proof. Let M is Chaki-pseudo parallel with respect to Schouten-van Kampen connection. Then by virtue of Definition 3.1 and Theorem 3.4 we have

$$(\hat{\nabla}_X \hat{h})(Y,Z) = 2\alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y). \tag{3.22}$$

Putting $Z = \xi$ in (3.22) and using (3.6), (3.20) and (3.21) we compute

$$\alpha(\xi)h(X,Y) = 0,$$

which implies that $h(X,Y) = 0$ as $\alpha(\xi) \neq 0$. Hence M is totally geodesic. The converse part is trivial. □

Corollary 3.3. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\hat{\bar{\nabla}}$. Then M is totally geodesic if and only if M is parallel with respect to $\hat{\bar{\nabla}}$.*

Let $\bar{M} = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7\}$ be a 7-dimensional manifold.
We choose vector fields as

$$e_1 = \frac{\partial}{\partial x_1} + \frac{1}{2}x_2\frac{\partial}{\partial x_7}, \quad e_2 = 4\frac{\partial}{\partial x_2}, \quad e_3 = \frac{\partial}{\partial x_3} + \frac{1}{3}x_4\frac{\partial}{\partial x_7}$$
$$e_4 = 6\frac{\partial}{\partial x_4}, \quad e_5 = 8\frac{\partial}{\partial x_5}, \quad e_6 = \frac{\partial}{\partial x_6} + \frac{1}{4}x_5\frac{\partial}{\partial x_7}, \quad e_7 = \frac{\partial}{\partial x_7} = \xi,$$

and we define a metric on $\bar{M}$ by

$$\begin{aligned} g(e_i, e_j) &= 1, \quad for \;\; i = j \\ &= 0 \quad for \;\; i \neq j. \end{aligned}$$

We take a $(1,1)$ tensor field ϕ in such a way that

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = e_4, \quad \phi(e_4) = -e_j$$
$$\phi(e_5) = -e_6, \quad \phi(e_6) = e_5, \quad \phi(e_7) = 0.$$

Here we define the 1-form η as $\eta(X) = g(X, \frac{\partial}{\partial x_7})$. From the above we show that

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(\bar{M})$, then we say that $(\bar{M}, \phi, \xi, \eta, g)$ is an almost contact metric manifold.
The non-vanishing components of Lie bracket are

$$[e_1, e_2] = -2e_5, \qquad [e_3, e_4] = -2e_5, \qquad [e_5, e_6] = 2e_5$$

Using Koszul formula we have

$$
\begin{aligned}
&\bar{\nabla}_{e_1}e_2 = -e_7, \quad \bar{\nabla}_{e_2}e_1 = e_7, \quad \bar{\nabla}_{e_3}e_4 = -e_7, \quad \bar{\nabla}_{e_4}e_3 = e_7,\\
&\bar{\nabla}_{e_5}e_6 = e_7, \quad \bar{\nabla}_{e_6}e_5 = -e_7, \quad \bar{\nabla}_{e_1}e_7 = e_2, \quad \bar{\nabla}_{e_7}e_1 = e_2,\\
&\bar{\nabla}_{e_2}e_7 = -e_1, \quad \bar{\nabla}_{e_7}e_2 = -e_1, \quad \bar{\nabla}_{e_3}e_7 = e_4, \quad \bar{\nabla}_{e_7}e_3 = e_4,\\
&\bar{\nabla}_{e_4}e_7 = -e_3, \quad \bar{\nabla}_{e_7}e_4 = -e_3, \quad \bar{\nabla}_{e_5}e_7 = -e_6, \quad \bar{\nabla}_{e_7}e_5 = -e_6\\
&\bar{\nabla}_{e_6}e_7 = e_5, \quad \bar{\nabla}_{e_7}e_6 = e_5
\end{aligned} \tag{3.23}
$$

$\bar{\nabla}_{e_i}e_j = 0$ for all others i, j. Now we take,

$$
\begin{aligned}
X &= x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7\\
Y &= y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5 + y_6e_6 + y_7e_7
\end{aligned}
$$

and we can show that

$$(\bar{\nabla}_X\phi)Y = g(X, Y)\xi - \eta(Y)X$$

and

$$\bar{\nabla}_X\xi = -\phi X.$$

This proves that $\bar{M}(\phi, \xi, \eta, g)$ is a Sasakian manifold.
Here we calculate

$$
\begin{aligned}
\bar{\nabla}_XY = &-(x_2y_7 + x_7y_2)e_1 + (x_1y_7 + x_7y_1)e_2\\
&-(x_4y_7 + x_7y_4)e_3 + (x_3y_7 + x_7y_3)e_4 + (x_6y_7 + x_7y_6)e_5\\
&-(x_5y_7 + x_7y_5)e_6 - (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 - x_5y_6 + x_6y_5)e_7
\end{aligned} \tag{3.24}
$$

Now using Schouten-van Kampen connection we have

$$\hat{\bar{\bar{\nabla}}}_XY = -x_7y_2e_1 + x_7y_1e_2 - x_7y_4e_3 + x_7y_3e_4 + x_7y_6e_5 - x_7y_5e_6 \tag{3.25}$$

and the non-vanishing components of $\hat{\bar{\bar{\nabla}}}_ij$ are

$$
\begin{aligned}
&\bar{\nabla}_{e_7}e_1 = e_2, \quad \bar{\nabla}_{e_7}e_2 = -e_1, \quad \bar{\nabla}_{e_7}e_3 = e_4,\\
&\bar{\nabla}_{e_7}e_4 = -e_3 \quad \bar{\nabla}_{e_7}e_5 = -e_6, \quad \bar{\nabla}_{e_7}e_6 = e_5.
\end{aligned}
$$

Now, let $f : M \to \bar{M}$ be a mapping defined by

$$f(x_1, x_2, x_5, x_6, x_7) = (x_1, x_2, 0, 0, x_5, x_6, x_7)$$

Then M is a submanifold of $\bar{M}$. Let us take the basis elements as

$$
\begin{aligned}
&e_1 = \frac{\partial}{\partial x_1} + \frac{1}{2}x_2\frac{\partial}{\partial x_7}, \quad e_2 = 4\frac{\partial}{\partial x_2}\\
&e_5 = 8\frac{\partial}{\partial x_5}, \quad e_6 = \frac{\partial}{\partial x_6} + \frac{1}{4}\frac{\partial}{\partial x_7}, \quad e_7 = \xi.
\end{aligned}
$$

Here we have

$$[e_1, e_2] = -2e_5, \quad [e_5, e_6] = 2e_5.$$

Again from Koszul formula we have

$$\begin{aligned}
&\nabla_{e_1}e_2 = -e_7, \quad \nabla_{e_2}e_1 = e_7, \quad \nabla_{e_5}e_6 = e_7,\\
&\nabla_{e_6}e_5 = -e_7, \quad \nabla_{e_1}e_7 = -e_2, \quad \nabla_{e_7}e_1 = -e_2,\\
&\nabla_{e_2}e_7 = e_1, \quad \nabla_{e_7}e_2 = e_1, \quad \nabla_{e_5}e_7 = -e_6,\\
&\nabla_{e_7}e_5 = -e_6, \quad \nabla_{e_7}e_6 = e_5, \quad \nabla_{e_6}e_7 = e_5.
\end{aligned} \tag{3.26}$$

Hence for any $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned}
&\nabla_X Y = -(x_2y_7 + x_7y_2)e_1 + (x_1y_7 + x_7y_1)e_2\\
&+(x_6y_7 + x_7y_6)e_5 - (x_5y_7 + x_7y_5)e_6\\
&-(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 - x_5y_6 + x_6y_5)e_7.
\end{aligned} \tag{3.27}$$

Hence by virtue of (3.1) we have from (3.24) and (3.27) that

$$h(X, Y) = 0 \quad for\ all\ X, Y \in \Gamma(TM) \tag{3.28}$$

Hence the submanifold is totally geodesic with respect to Levi-Civita connection. Agin, from (3.25) we have

$$\hat{\nabla}_X Y = -x_7y_2e_1 + x_7y_1e_2 + x_7y_6e_5 - x_7y_5e_6. \tag{3.29}$$

And it can be represented as

$$\begin{aligned}
\hat{\nabla}_X Y &= \nabla_X Y + (x_2e_1 - x_1e_2 - x_6e_5 + x_5e_6)y_7\\
&\quad -(x_2y_1 - x_1y_2 - x_6y_5 + x_5y_6)e_7\\
&= \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi
\end{aligned}$$

Also from (3.25) and (3.29) we have

$$\hat{h}(X, Y) = h(X, Y) = 0.$$

Hence M admits Schouten-van Kampen connection and it is totally geodesic with respect to schouten-van Kampen connection.

We now consider M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to Tanaka Webstar connection $\overset{*}{\bar{\nabla}}$. Then we have

Theorem 3.6. *Let M be a submanifold of a Sasakian manifold $\bar{M}$ with respect to Tanaka Webstar connection. Then*
(i) M admits induced Tanaka Webstar connection $\overset{*}{\nabla}$,
(ii) The second fundamental forms with respect to Levi-Civita connection and Tanaka Webstar connection are equal, i.e. $h = \overset{*}{h}$.

Proof. By virtue of (2.5), (3.1) and (3.4) we have

$$\overset{*}{\bar{\nabla}}_X Y + \overset{*}{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(X)\phi Y - \eta(Y)\phi X - g(\phi X, Y)\xi \tag{3.30}$$

for $X, Y \in \Gamma(TM)$. Since $\xi \in \Gamma(TM)$ so, by equating tangential and normal components of (3.30) we have

$$\overset{*}{\bar{\nabla}}_X Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{3.31}$$

and

$$\overset{*}{h}(X,Y) = h(X,Y). \tag{3.32}$$

Hence the theorem is proved. □

Next we prove the following:

Theorem 3.7. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\overset{*}{\bar{\nabla}}$. Then M is totally geodesic if and only if M is Chaki-pseudo parallel with respect to $\overset{*}{\bar{\nabla}}$.*

Proof. Suppose that M is Chaki-pseudo parallel with respect to Tanaka Webstar connection. Then by virtue of Definition 3.1 and Theorem 3.6 we have

$$(\overset{*}{\nabla}_X \overset{*}{h})(Y,Z) = 2\alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y). \tag{3.33}$$

In view of (3.6), (3.31) and (3.32), putting $Z = \xi$ we have from (3.33)

$$2h(\phi X, Y) = \alpha(\xi)h(X,Y). \tag{3.34}$$

Replacing X by ϕX and using (1.1) we compute

$$\alpha(\xi)h(\phi X, Y) = -2h(X,Y). \tag{3.35}$$

From (3.34) and (3.35) we get $[\{\alpha(\xi)\}^2 + 4]h(X,Y) = 0$. which imply $h(X,Y) = 0$ if $\{\alpha(\xi)\}^2 + 4 \neq 0$, i.e. M is totally geodesic.
The converse is trivial. □

Corollary 3.4. *Let M be an invariant submanifold of a Sasakian manifold $\bar{M}$ with respect to $\overset{*}{\bar{\nabla}}$. Then M is totally geodesic if and only if M is parallel with respect to $\overset{*}{\bar{\nabla}}$.*

4 Conclusion

In this paper we have studied Chaki-pseudo parallel invariant submanifold of Sasakian manifold with respect to Levi-civita connection, semisymmetric metric connection, Schouten-van Kampen connection and Tanaka Webstar connection. We found that an invariant submanifold of Sasakian manifold is Chaki-pseudo parallel if and only if it is totally geodesic with respect to the said connections. Also we observed that an invariant submanifold of Sasakian manifold is totally geodesic if and only if it is parallel. Thus we can say that the concept of Chaki-pseudo parallel and parallel submanifolds are equivalent with respect to the totally geodesic property irrespective of the connections chosen. One can also study these conditions with respect to any other connections.

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