Mathematics and Fluid Mechanics

Sunil Datta

Department of Mathematics & Astronomy
University of Lucknow, Lucknow

Abstract

In this presentation we explore the link between mathematics and fluid mechanics providing a glimpse of the historical development of the latter. Some basic equations and the involved mathematics are discussed; but more important we attempt to unearth the influence of fluid mechanics in advancement of mathematics and its techniques.

Subject class [2010]:05E40, 13D02
Keywords: Cellular resolutions; standard monomials;

1 Early History of Fluid Mechanics

Motion of water and winds has been attracting the attention of mankind ever since it is roaming on this earth. One of the most striking features of many fluid flows that have fascinated all observers ancient and modern is the strange mix of order and chaos that is their characteristic. The lines below in **Yoga Vasistha** exhibit the similarity between the fluid motion and wondering of mind.

As water displays itself richly
In current, wave, foam and spray,
So does the mind exhibit
A strange, splendid diversity.

Knowledge and exploitation of fluid motion is clearly indicated in following observations-

- Three aerodynamically correct wooden spears were recently excavated near Hanover, Germany. Archeologists date the carving of those complete spears to about 400,000 years ago.
- Soon after the dawn of civilization and the establishment of an agriculture way of life 8000 years ago, complex systems of irrigation were built to control the water flow.
- Some resourceful citizens of the Roman Empire discovered that adding the right kind of diffuser to the calibrated convergent nozzle installed at outlets of the water main significantly increased the quantity of potable water over that granted by the emperor.
- Farmers knew the value of wind breaks to keep top soil in place and to protect fragile crops.

- Aristotle (384-322 B.C.), in section eight of his book Physica, while investigating the motion of a projectile, concludes that the air imparts a push on it. This erroneous idea ruled the philosophical and the scientific world for centuries.
- Archimedes(287 BC) principle on the action of a heavy fluid at rest on an immersed body (Up thrust).

2 History:15th Century Onwards

Abandoning the Aristotelian view of the action of air assisting motion, Italian genius **Leonardo da Vinci (1452-1519)** seems to be the first to recognize in the air only a resisting action. Leonardo da Vinci had the hands and eyes of an engineer as well as of an artist and drew wonderfully faithful pictures of vortices in turbulent water flow; the words eddies and eddying motions percolate throughout Leonardo's treatise on liquid flows. Leonardo da Vinci correctly deduced the mass conservation equation for incompressible, one dimensional flows. He held the view: Mechanics is the paradise of the mathematical sciences because by means of it one comes to the fruits of mathematics. It was left to **Galilen Galilei (1564-1642)** to demonstrate the resisting action of the air upon the bodies moving through it.

However, the credit for initiating scientific study of the subject of fluid friction goes to the priest **Edme Mariotte (1686)** who was one of the founders of French Academy. His Trait du movement des eaux et des autres corps fluids, published in 1886, dealt with the theory and principles of hydrodynamics offering not only mathematical formulae but also the techniques by which a model is held stationary in a stream of fluid and drag is measured by a balance or by a weighing machine.

Newton in his Principia published in 1687 discussed internal attrition of liquids which was later known as internal friction of fluids. A lesser known fact is that Newton also deduced that the speed in a divergent channel increases that was contradictory to the observed subsonic flows then. But with Supersonic Flow coming into picture it was found that Newton result holds good for a supersonic flow.

Daniel Bernoulli(1700-1782) introduced the term hydrodynamics. Fluid mechanics develops into a science only after **Euler** derived, in 1736, the equations of motion of a material point. **Jean le Rondd'Alembert(1717-1783)** introduced the principle of the conservation of mass(continuity equation).

Lagrange (1788) wrote One owes to Euler the first general formulas for fluid motion presented in the simple and luminous notation of p.d.e. . By this discovery, all fluid mechanics was reduced to a single point of analysis; equations involved were integrable, one would determine completely, in all cases, the motion of a fluid moved by forces.

3 Navier Stokes equations

History of Navier-Stokes equations begins with the 1822 memoir of wherein **Navier** derived equations for homogeneous incompressible fluids from a molecular argument. Using similar arguments, **Poisson (1829)** derived the equations for a compressible fluid. The continuum derivation of the Navier Stokes equation is due to **Saint-Venant (1843)** and **Stokes (1845)** .

Inviscid	Viscous
Incompressible stratified, varying concentration supersonic, hypersonic	Compressible subsonic, transonic,
Laminar Low Re, Boundary Layer Transition flows-Stability	Turbulent
Newtonian	Non-Newtonian Visco-elastic, Micropolar
Single phase	Multiphase Fluids

Tab. 1: Fluids and Fluid Flows

Equations of Hydrodynamics:
Navier - Stokes equations

$$\rho\mathbf{u}.grad\ \mathbf{u} = -grad\ p\ + \mu\ div\ grad\ \mathbf{u}$$

Euler's Equations:

$$\rho\ \mathbf{u}.grad\ \mathbf{u} = -grad\ p$$

Stokes equations:

$$0 = -\rho\ grad\ p + \mu\ div\ grad\ \mathbf{u}$$

Ossen's equation:

$$\rho\ \mathbf{U}.grad\ \mathbf{u} = -\ grad\ p + \mu\ div\ grad\ \mathbf{u}$$

Equation of continuity:

$$div\ \mathbf{u} = 0$$

Eulers Equations were integrated in many cases but results disagreed grossly with observations and thus flagrantly contradicted Lagranges opinion. In his Hydrodynamica **Bernoulli** observed-I attribute enormous differences for the greatest part to the adhesion of the water to the sides of the tube, which adhesion can certainly exert an incredible effect on the cases of this kind.

Such apparent inconsistencies between experimental facts and conclusions based on plausible arguments are called paradoxes. Hydrodnamics has an abundance of these; we present a few.

DAlemberts Paradox (1752): In an inviscid fluid of infinite extent the Drag of any body of any shape in uniform stream is zero. This result was in direct Contradiction to known experimental evidence and caused mathematical fluid mechanics to be discredited by engineers. This resulted in an unfortunate split between Hydraulics (observing phenomena which could not be explained) and Theoretical fluid mechanics explaining phenomena which could not be observed

Prandtls Resolution: Although Stokes Equations did yield a finite value of drag but that was too small. **DAlemberts Paradox** is considered to have been resolved by **Ludwig Prandtl** in 1904. In his short report Motion of fluids with very little viscosity he introduced the idea that the effectsof a thin viscous boundary layer were responsible for the drag force.

A New Resolution of DAlembert's Paradox was advanced by **John Hoffman and Claes Johnson (2006))**. Their computation solution of Eulers equations shows that the zero-drag potential solution is unstable and develops into a turbulent solution with substantial drag. This resolves the Paradox.The resolution is fundamentally different from Prandtl's resolution based on boundary layer theory.

Stokes Paradox (1851):
For a uniform flow past an infinite circular cylinderStokes failed to obtain a solution because of Logarithmic singularity in velocity. Lamb (1911) using linearized Oseens equations estimated Drag

$$D = 4\pi\mu U/[1/2 - Y - In(Re/4)]$$

Whitehead Paradox (1889): For uniform flow past a sphere Whitehead applied iterative procedure to improve upon Stokes solution but failed because of Singularity. The non-existence of uniformly valid solution was resolved partially by Oseen (1910). Using his linearized Navier-Stokes equations, he found

$$D = 6\pi\mu aU/[1 + (3/8)Re + O(Re^2)]$$

Reynolds Paradox (1883):

While Analytic and Numeric methods suggest stability of Poiseuille flow for all Reynolds numbers, Reynolds Experiments exhibit Instability for R¿Rc. Reynolds Experiments still defy explanation.

Reynolds Experiments: In the year 1883 the following classic lines appeared in the paper entitled "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the laws of resistance in parallel channels The author was Osborne Reynolds and he was presenting his experiments with colored filaments in circular tubes. He wrote

"When the velocities are sufficiently low, the streak of color extended in a beautiful straight line across the tube. If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity. As the velocity was increased by small stages, at some point of the tube, always a considerable distance from the trumpet of the intake, the color band would at once mix up with the surrounding water and fill the rest of the tube with a mass of the colored water. Any increase in the velocity caused the point of the break-down to approach the trumpet, but with no velocities that were tried did it reach this. On viewing the tube by the light of electric spark, the mass of color resolved itself into a mass of more or less distinct curls showing eddies

The phenomena depends on the non-dimensional parameter $Re = \rho U d/\mu$, where U is the mean velocity, d the diameter of the tube, r the density and m the coefficient of viscosity of the fluid. At a critical value of this parameter, dependent on the smoothness of the entry conditions, the flow pattern changes from laminar to turbulent. Stokes (President Royal Society) (30th November 1888), singled out this exceptional paper on the occasion of the presentation of a Royal Medal to Osborne Reynolds 'for his investigations in mathematical and experimental physics, and on the application of scientific theory to engineering.
Sommerfeld's Mystery:

Mathematics predicts that the simplest of all flows, Couette flow with a stationary linear velocity profile is stable and thus should exist. But nobody has observed this flow in a fluid with small viscosity.

Sommerfeld was a great admirer and supporter of Indian physics Visited India in second half of 1928. He was awarded honorary D. Sc. by Calcutta University He advocated the use of pure mathematics by engineers

4 Mathematics of Slow Flow

Reynolds number plays a vital role in the whole of viscous flow theory:

- Exact Solutions: Valid for all Re
- Slow Viscous Flow: Re small
- Boundary Layer Flow: Re large
- Turbulent Flow: Re Very high
- Inviscid Flow: $Re \to \infty$

Boundary Layer Flow-*Flow at high Reynolds number* Air and water, the most abundant and most widely used fluids by engineers, have low viscosity and hence high Reynolds number. But the classical mathematical theory of Potental Flow, involving zero viscosity, miserably failed as it predicted no drag force. Engineers depended on rules of thumb and experiences, obtained from experimental model-building. With the advancement of aviation industry Boundary layer theory came into prominence as it offered a useful drag estimates for the speedy airplanes, a high Reynolds number situation; whence the low Reynolds number hydrodynamics went into oblivion for almost 100 years after Stokes equations (1851) came into existence.

It was only in mid fifties of the preceding century, led by biological and chemical sciences, that the interest in low Reynolds number hydrodynamics got revived. It is not difficult to visualize that these disciplines abound in situations involving tiny particles moving at low speeds in fluids of high viscosity. Thus we may cite the swimming of microscopic organisms, Brownian motion, water drops in cloud formation, electrically charged particles in colloids, protein molecules and synthetic polymers, red and white cells in blood, movement of mucus in lungs, flow of dusty gas, smoke, dyes and the electrophoresis.

Because of their technological and biomedical applications, the science of slow flow could no longer be ignored. Attention to such problems was first drawn by Taylor through his two papers (1951, 1952). Sir M.J. Lighthills publication (1976) on flagellar motion and biofluid dynamics propelled the Low Reynolds Number Hydrodynamics from the province of mathematical curiosity to the kingdom of practical applications in biosciences and industry. Stokes equations are simple looking linear equations but construction of the solution in diverse situations is not straight forward and simple. A whole battery of mathematical techniques is required. This becomes more acute when we take up the natural extension to the equations for non-zero low Reynolds number problems. The following techniques are usually applied.

Separation of variables: the eigenfunction expansion method.

Semiseperable solutions: It is instructive to know that while individual terms of these expansions are not solutions of the Stokes equations, the complete expansion does satisfy the equations. Such solutions may be termed as Semiseperable Solutions.

Singular perturbation method: The method is also known as the method of matched asymptotic expansions. It was left to the genius of Saul Kaplun (1957) to recognize the analogy between the theory of flow at small Reynolds number and boundary layer theory. In the same year (1957) Proudman and Pearson published their remarkable paper that presented a very vivid and detailed description of the matching technique.

Singularity Method: The basic singularity of the Stokes equations is the fundamental solution of Stokes equations, namely, Stokeslet, a name coined by Hancock (1953) to honor Stokes. Stokeslet determines the velocity field generated by a point force. The field of a point couple is termed Rotlet. Another important singularity is Stresslet.

5 Fluid Mechanics and Mathematics

Many ideas and concepts of modern Mathematics may be traced to Fluid Mechanics. The remnants of these ideas and concepts lie dormant in the mathematical structures linked to flows, currents, circulation, frames etc. and in the names of the distinguished mathematician and scientists associated with both fields - Fluid Mechanics and Mathematics.

Mathematicians have abstracted and vastly generalized basic fluid mechanical concepts and have created a deep and powerful body of knowledge that is unfortunately now mostly not accessible to fluid mechanists, while mathematicians themselves have lost all but a passing knowledge of the physical origins of many of their basic notions. In this section we shall attempt to unearth the influence of fluid mechanics in advancement of mathematics and its techniques.

Differentiability of fluid motion: The laws of mechanics are expressed using derivatives, and indeed, the development of mechanics went hand in hand with the invention of calculus. The assumption of differentiability of fluid motion means that on small enough scales the fluid map looks like a linear map of a vector space to itself.

Diracs Delta Function:

$$\delta(x) = \left\{ \begin{array}{cc} 0 & x \neq 0 \\ \infty & x = 0 \end{array} \right\}$$

with $\int f(x)\delta(x)dx = f(0)$.

The delta function appeared in the early days of 19th century, in works of the Poission (1815), Fourier (1822) and Cauchy (1823). Subsequently O Heaviside (1883) and G Kirchoff (1891) gave the idea of the delta function. P A M Dirac (1926) introduced delta function in his classic and fundamental work on quantum mechanics.

Distributions: Mathematically, the delta function is not a function. Skeptical mathematicians not only shunned it but also ridiculed it. The state continued till L Schwartz (1950) in his Theorie des Distributions gave a sound mathematical footing to it. Since then we have the class of entities known as Distributions or Generalized Functions.

Stokes Equation:

$$0 = -\nabla p + \nabla^2 u + 8\pi\alpha\delta(x).$$

In the above equation the last term represents the effect of a force of Strength α.

The primary fundamental solution of Stokes equation is Stokeslet, a name given by Hancock (1953). It has the same significance for low Reynolds number problems as 1/r has for potential problems. It is the free space Green's function for the Stokes equations. Stokeslet determines the velocity field generated by a point force. Its derivatives generate higher order singularities. The field of a point couple is termed Rotlet Another important singularity is Stresslet. Singularity method consists of ordered exploitation of singularities in constructing the solution and obtaining useful information. The systematic study of this method was undertaken by Chwang et. al in a series of papers (1974, 1975). Known solutions of slow flow problems were presented in an elegant and simple fashion, and new problems defying solution in conventional methods were tackled. The singularity method not only leads to qualitative insight into the physical problem but also provides a way for quantitative estimate particularly taking advantage of modern computational techniques. Thus a review of the solutions of Stokes equation has led to development of the mathematical tecqunique of Singularity Method.

Singular perturbation method:

The method is also known as the method of matched asymptotic expansions. In the boundary layer flow, the idea of inner and outer limit processes led to provide the approximation to Navier Stokes Equation tenable at large Reynolds number. An outer limit that yields the inviscid solution and an inner limit that yields the boundary layer solution. It was left to the genius of Saul Kaplun (1957) to recognize the analogy between the theory of flow at small Reynolds number and boundary layer theory and to apply to it the singular perturbation method. In the same year (1957) Proudman and Pearson published their remarkable paper that presented a very vivid and detailed description of the matching technique. It gave a great impetus to the study and development of the Singular Perturbation Method in connection with theory of low Reynolds number flow. The main result of their calculations is

$$D = 6\pi\rho v a U[1 + (3/8)Re + (9/40)Re^2 InRe + O(Re^2)]$$

Semiseperable solutions:

While Stokes equation in terms of Stokes Stream function

$$E^4\psi = 0$$

is separable in Cartesian, cylindrical and spherical polar coordinates, it is not so in spheroidal coordinates $(\tau\zeta\varphi)$ Stokes flow problems involving spheroidal particles are of considerable analytical and applied interest because particles such as bacteria, crystals, grains and macromolecules bear close resemblance to these shapes. Dassios et.al. (1994) circumvented the difficulty of non seperability by representing the general Stokes stream function as a complete expansion comprising of a sum of two parts: one from the 0-eigen space and the other from the generalized 0-eigenspace of the operator

$$E^2 = \frac{1}{C^2(\tau^2 - \zeta^2)}[(\tau^2 - 1)\frac{\partial^2}{\partial\tau^2} + (\zeta^2 - 1)\frac{\partial^2}{\partial\zeta^2}]$$

Thus, with in terms of Gegenbauer functions, we have the expansion

$$\psi(\tau,\zeta) = g_0(\tau)G_0(\zeta) + g_1(\tau)G_1(\zeta) + \sum_{n=2}^{\infty}\{g_n(\tau)G_n(\zeta) + h_n(\tau)H_n(\zeta)\}$$

where $g's$ and $h's$ are suitable collections of Gegenbauers functions. It is instructive to know that while individual terms of the above expansion are not solutions of Stokes equation, the complete expansion does satisfy the equation. Such solutions may be termed as *Semiseperable Solutions.*

Developing the Fluid Model:

Mathematics is the study of sets with structures and of the transformations between them. In Fluid Mechanics the basic model is the set consisting of the fluid body and the various additional structures that allows one to discuss such properties as continuity, velocity and deformation. The fluid flow is a transformation of the set consisting of fluid particles to itself and is the transport of structures under the flow is investigated and analysed. We develop the fluid model step by step, introducing a model to simulate certain process. When possible coordinate free terminology and notation is used, as anything fundamental should not depend on the choice of coordinates. Being forced to treat Computational objects are treated as global entities providing a new and sometimes valuable and in depth view of familiar operations. The goal of an engineering science is to make the system as efficient as possible and mathematics plays a major role in achieving it. Mathematics enables to know what the efficiency is and how to increase that. Without the use of calculus, geometry and statistics one cannot advance in engineering science.

Methods used in Fluid Mechanics:

- Analytical methods of applied mathematics to solve the basic flow equations along with the boundary conditions
- Numerical methods applied mathematics using Computers
- Experimental methods applying similarity laws for deriving results from model flow investigations

Computational Fluid Mechanics:

Considerable developments in applied mathematics have taken place to solve partial differential equations numerically. Also, the computational power of modern high speed computers has tremendously increased..Computer programs are available that solve practical flow problems numerically. Computational fluid mechanics (CFD) has become an important sub-field of fluid mechanics and its use will increase exponentially.

Super-supercomputer:

Inspired by the original Feynman's vision of smaller submicroscopic computers, in 1994 Adleman introduced and actually demonstrated the novel concept of Direct Numerical Simulations (DNS) and DNA (deoxyribonucleic acid) computing systems. The idea is attracting considerable attention from computer scientists as well as microbiologists.

Thus, the same genetic machinery that generates living organisms may be exploited to solve previously unapproachable mathematical problems. A human body contains about

300 gm of DNA. Crude estimates indicate that a mere 500 gm of DNA molecules suspended in 1000 liters of fluid would have the equivalent memory to all the electronic computers ever made.

Future of Fluid Mechanics Teaching:

Before the turn of the century the discipline of fluid mechanics is bound to take a dramatic transformation on account of the rapidly advancement of computer technology. Super-super computer may have the power to numerically integrate any problem in fluid mechanics. This will necessitate an enormous change in the teaching pattern of fluid mechanics. This will obviate the need for teaching of numerous theorems, methods and approximations employed in the present fluid mechanics course. Students will be using more of prepackaged software and less of analysis to solve the numerous challenging problems they may face. This may lead to unfortunate disappearance of the present day fluid mechanics discipline. The use of computational soft-wares is welcome but the loss of an insight of the problem is regrettable.

It is wise to remember what Galileo Galilei has said, **"Thinking is one of the greatest joys of humankind'**. That is certainly a quality which would wane with the ever more powerful computers unless simultaneous attention is not paid to observation and analysis that constitute the core of Fluid Mechanics. Keep in mind the sayings-

- *Blow a soap bubble and observe it. You may study it all your life and draw one lesson after another in physics from it. (Lord Kelvin)*
- *Now I think hydrodynamics is to be the root of all physical science, and is at present second to none in the beauty of its mathematics. (Lord Kelvin)*