Convex Sets in Finite and Infinite Dimensional Spaces

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Abstract

The main aim of this expository article is to present the notion of convex sets and to highlight the differences in certain properties of convex sets in finite and infinite dimensional spaces. This article also provides glimpses of some unsolved and recently solved problems regarding convex sets.

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1 Introduction

Convex set theory is a vibrant field of modern mathematics with rich applications in geometry, analysis, economics and optimization. The first monograph on convex sets was published by Bonnesen and Fenchel [2] in 1934. Over the past few decades many useful applications of convex sets were discovered. According to the importance of these applications convexity is a prosperous subject even today. There is a fascination to understand convex sets and several attempts are being made to solve many conjectures and unsolved problems related to convex sets.

There are many good books on convex sets. For finite dimensional treatment one may refer to books by Rockafellar [14], Hiriart-Urruty and Lemarchal [7] and Boyd and Vandenberghe [3]. For a geometrical view of convex sets one may refer to the books by Lay [9], Leonard and Lewis [10] and Soltan [15].

In this presentation the focus is to provide a brief introduction to convex sets from various books listed in the references. The aim is to study various properties of convex sets and highlight the differences in finite and infinite dimensional setting. We only provide the statements of theorems without going into the proof. However, references are provided for the readers interested in their proofs. Many illustrative examples are also provided in the paper. We also provide some unsolved and recently solved problems related to convexity notion.

The rest of the paper is organised as follows. Section 2 deals with the notion of convex sets and various algebraic and topological properties of convex sets. Section 3 deals with some motivating problems related to convex sets.

2 Convex sets

Intuitively, a convex set is connected in the sense one can pass between any two points without leaving the set. We now provide a mathematical definition of a convex set.

Definition 2.1. (Definition 1.10 in [1]) A nonempty set C in a real linear space X is said to be a convex set if for x and y in C, the line segment joining x and y is contained in C, that is,

$$x, y \in C \Rightarrow \{(1-\lambda)x + \lambda y \ : \ \lambda \in [0,1]\} \subseteq C.$$

For example, a circular disk is a convex set but a circle is not a convex set in $\mathbb{R}^2$. Some examples of convex and nonconvex sets in finite dimensional spaces are given in Figure 1.

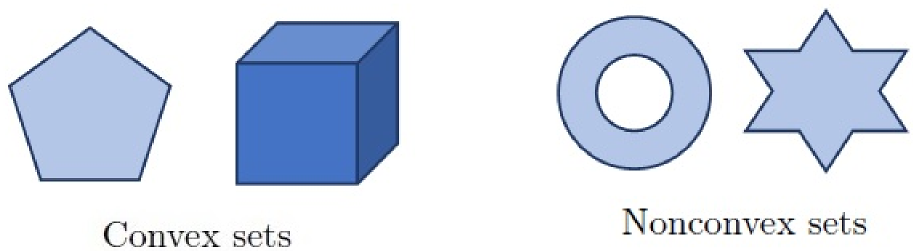

Fig. 1

The following examples are from infinite dimensional spaces.

Example 2.2. *(i) The set* $C = \{u \in L_2[0,1] \ : \ \int_0^1 (u(s))^2 ds = 1\}$ *is a convex set in* $L_2[0,1]$.

(ii) The set $C = \{u \in l_2(\mathbb{N}) : |u_n| \leq 2^{-n}\}$ *is a convex set in* $l_2(\mathbb{N})$.

(iii) The set $C = \{u \in l_2(\mathbb{N}) : u_n = 0$ *except for finitely many* $n\}$ *is a convex set in* $l_2(\mathbb{N})$ *with* $int(C) = \emptyset$ *and* $cl(C) = l_2(\mathbb{N})$.

Affine subspaces (affine sets) are translations of linear subspaces.

Definition 2.3. (Definition 1.11 in [1]) A nonempty set A in a real linear space X is said to be an affine set if for x and y in A, the line passing through x and y is contained in A, that is,

$$x, y \in A \Rightarrow \{(1-\lambda)x + \lambda y \ : \ \lambda \in \mathbb{R}\} \subseteq A.$$

If $x_i, i = 1, 2, \ldots, k$ are points in X, an *affine combination* of $\{x_1, x_2, \ldots, x_k\}$ is any point of the form $\Sigma_{i=1}^k \lambda_i x_i$, $\Sigma_{i=1}^k \lambda_i = 1$. Moreover, if $\lambda_i \geq 0$, the affine combination is called a *convex combination.*

For instance, a singleton, a line, a plane and the space $\mathbb{R}^3$ are all affine subsets of $\mathbb{R}^3$. The set of all affine combinations of three noncollinear points in $\mathbb{R}^3$ is the plane passing through these points whereas the set of all convex combinations is the triangle (including the interior) having vertices at the three points.

The next proposition characterizes a convex (affine) set in terms of convex (affine) combinations.

Proposition 2.4. (Proposition 1.12 in [1]) A set S in real linear space is convex (affine) if and only if it contains every (convex) affine combination of points of S.

The *convex* (*affine*) *hull* of a nonempty set S in a real linear space, denoted by $\mathrm{co}(S)$ ($\mathrm{aff}(S)$), is the intersection of all convex (affine) sets containing S, that is

$$\mathrm{co}(S) = \bigcap\{A : S \subseteq A, A \text{ is convex}\}$$

and

$$\mathrm{aff}(S) = \bigcap\{A : S \subseteq A, A \text{ is affine}\}.$$

The convex hull of a set is illustrated in Figure 2.

Fig. 2: Set and its convex hull

In fact, $\mathrm{co}(S)$ ($\mathrm{aff}(S)$) is the smallest convex (affine) set containing S. Moreover, it can be seen from the following theorem that the elements of $\mathrm{co}(S)$ ($\mathrm{aff}(S)$) can be represented only with the elements of S.

Theorem 2.5. (Theorem 1.13 in [1]) The convex (affine) hull of a nonempty set S in a real linear space coincides with the set of all convex (affine) combinations of elements belonging to S, that is,

$$\mathrm{co}(S) = \left\{ \sum_{i=1}^{k} \lambda_i s_i : s_i \in S, \lambda_i \in \mathbb{R}, \lambda_i \geq 0, i = 1, 2, ..., k, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N} \right\}$$

and

$$\mathrm{aff}(S) = \left\{ \sum_{i=1}^{k} \lambda_i s_i : s_i \in S, \lambda_i \in \mathbb{R}, i = 1, 2, ..., k, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N} \right\}.$$

Clearly, any affine subset is convex but the converse is not true. It is important to notice that any affine subset A of a real linear space X is the translate of a linear subspace L. For every $a \in A$, the translate $L = A - a$ is a real linear subspace of X and in fact, $L = A - A$. The *dimension of an affine set* A is the dimension of the real linear subspace $A - A$, that is, $\dim(A) = \dim(A - A)$. The *dimension of a convex set* in X is the dimension of its affine hull. The dimension of a disk in $\mathbb{R}^3$ is 2 whereas the dimension of a line segment joining two points in $\mathbb{R}^3$ is 1.

We now recall the notion of an affine map. Let X and Y be two real linear spaces. A map $T : X \to Y$ is called an *affine map* if for $x, y \in X$ we have $T((1-\lambda)x + \lambda y) = (1-\lambda)T(x) + \lambda T(y)$, for $\lambda \in \mathbb{R}$.

The following easily follows from the definition of convexity.

Theorem 2.6. The following hold:

(i) If $(C_i)_{i\in I}$ is a family of convex sets in a real linear space X then $\bigcap_{i\in I} C_i$ is a convex set in X.

(ii) If $C_1, C_2, ..., C_k$ are convex sets in a real linear space X and $\lambda_1, \lambda_2, ..., \lambda_k$ are real numbers then $\lambda_1 C_1 + \lambda_2 C_2 + ... + \lambda_k C_k$ is a convex set in X.

(iii) If $C_1, C_2, ..., C_k$ are convex sets in a real linear space X then $\text{co}(C_1 + C_2 + ... + C_k) \subseteq \text{co}(C_1) + \text{co}(C_2) + ... + \text{co}(C_k)$.

(iv) If C_i is a convex set in a real linear space $X_i, i = 1, 2, ..., k$ then $C_1 \times C_2 \times ... \times C_k$ is a convex set in $X_1 \times X_2 \times ... \times X_k$.

(v) If $T : X \to Y$ is an affine map from a real linear space X to a real linear space Y and C is a convex set in X then $T(C)$ is a convex set in Y. If D is a convex set in Y then $T^{-1}(D)$ is a convex set in X.

Geometrically we illustrate the (Minkowski) sum of two convex sets in Figure 3.

Fig. 3: Sum of two convex sets

We now recall some topological nature of convex sets.

Theorem 2.7. (Theorem 2.2 in [4]) If C is a nonempty convex subset of a real normed linear space X then

(i) $\text{cl}(C)$ and $\text{int}(C)$ are convex sets in X;

(ii) $x \in \text{cl}(C)$ and $y \in \text{int}(C)$ implies that $]x, y] \subseteq \text{int}(C)$;

(iii) $\text{int}(C) \neq \emptyset$ implies that $\text{cl}(C) = \text{cl}(\text{int}C)$ and $\text{int}(C) = \text{int}(\text{cl}(C))$.

In the characterization of convex hull of a set in Theorem 2.5 there is no restriction on the positive integer k and it may have to go upto $+\infty$. However, the following powerful theorem states that any point in convex hull of a set with finite dimension can be represented by restricting the value of k.

Theorem 2.8. (**Carathéodory Theorem**, Theorem 3.1.2 in [12]) Suppose that S is a subset of a real linear space X and $\text{co}(S)$ has dimension n. Then each point in $\text{co}(S)$ can be represented as a convex combination of at most $n + 1$ points of S.

From the above theorem it is clear that, if S is a nonempty subset of a finite dimensional real linear space X having dimension say n, then every element of $\text{co}(S)$ can be represented as a convex combination of at most $n + 1$ elements of S. For the proof of the finite dimensional version refer to Theorem 1.3.6 in [].

We now state Hellys theorem which is based on the intersection of convex sets.

Theorem 2.9. (**Hellys Theorem**, Exercise 7 on Page 108 in []) Let $(C_i)_{i\in I}$ be a finite collection of convex sets in a finite dimensional real linear space X of dimension n where $|I| > n+1$. If every subcollection of at most $n+1$ sets has a nonempty intersection, then the entire collection has a nonempty intersection.

In general, convex hull of a closed set is not always closed. For instance the set $S = \{(x_1, 1) : x_1 \in \mathbb{R}\} \cup \{(0,0)\}$ is a closed set but its convex hull $\text{co}(S) = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 > 0\} \cup \{(0,0)\}$ is not a closed set. This argument leads to the following notion of closed convex hull.

Definition 2.10. (Page 29 in [4], Definition 1.4.1 in [7]) If S is a nonempty subset of a real normed linear space X then the closed convex hull of S, denoted by $\overline{\text{co}}(S)$ is the intersection of all closed convex sets containing S.

In fact, closed convex hull of a set can be obtained by taking the closure of the convex hull of the set.

Proposition 2.11. (Exercise 2.5 in [4], Proposition 1.4.2 in [7]) If S is a nonempty subset of a real normed linear space X then $\overline{\text{co}}(S) = \text{cl}(\text{co}(S))$.

It can be easily observed that the convex hull of closure of a set is contained in the closed convex hull of that set. However, the equality holds in finite dimensional spaces if the given set is bounded.

Lemma 2.12. (Corollary 4.15 in [6], Theorem 1.4.3 in [7]) If X is finite dimensional real normed linear space and S is a compact set in X then its convex hull $\text{co}(S)$ is a compact set.

Using the above lemma we have the following proposition.

Proposition 2.13. (Page 101 in [7]) If X is finite dimensional real normed linear space and S is a bounded set in X then

$$\overline{\text{co}}(S) = \text{cl}(\text{co}(S)) = \text{co}(\text{cl}(S)).$$

The above theorem fails to hold for infinite dimensional spaces as illustrated by the following example.

Example 2.14. The set $S = \{u \in l_2(\mathbb{R}) : u_n = (\frac{e_n}{n})\} \cup \{e_0\}$ is a compact set in $l_2(\mathbb{R})$, where $e_i = (0, 0, \ldots, 1, 0, \ldots)$ for all i, where 1 is at i^{th} place and $e_0 = (0, 0, \ldots)$. It can be seen that $u = (0, \frac{1}{2}, \frac{1}{3}, \ldots) \in l_2(\mathbb{R})$ but $u \notin \text{co}(S)$. Also, $u = \frac{\sum_{i=1}^{n} (\frac{1}{2^i})(\frac{e_i}{i})}{\sum_{i=1}^{n} \frac{1}{2^i}} \in cl(co(S))$. Hence, $\text{co}(S) \neq \text{cl}(\text{co}(S))$, that is, $\text{co}(S)$ is not compact.

An important class of convex sets are the convex cones.

Definition 2.15. (Definition 4.1 in [8]) A set K in a real linear space X is said to be a cone if $\lambda x \in K$ for every $x \in K$ and $\lambda \geq 0$. If K is a convex set then K is called a convex cone.

Convex cones can be characterized as follows.

Theorem 2.16. (Theorem 4.3 in [8]) A cone K in a real linear space is convex if and only if $K+K \subseteq K$.

We now give some examples of cones.

Example 2.17. The following are some examples of cones which are not convex.

(i) $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}$.

(ii) $K = \{x \in l_2(\mathbb{R}) : x_n \neq 0 \; \forall \, n \in \mathbb{N}\}$.

(iii) $K = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = (x_1^2 + x_2^2)^{1/2}\}$.

Example 2.18. The following are some examples of cones which are convex.

(i) $\mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, ..., n\}$.

(ii) $\mathbb{R}^n_{\geq} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$.

(iii) $\mathbb{S}^n_+ = \{A \in \mathbb{S}^n : \langle Ax, x\rangle \geq 0, \forall x \in \mathbb{R}^n\}$ where $\mathbb{S}^n$ is the set of all $n \times n$ real symmetric matrices.

(iv) Ice-cream cone (Lorentz cone) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq (x_1^2 + x_2^2)^{\frac{1}{2}}\}$.

(v) $C_+[0,1] = \{x \in C[0,1] : x(t) \geq 0, \; \forall \, t \in [0,1]\}$.

(vi) $K = \{x \in l_2(\mathbb{R}) : x_n \geq 0, \; \forall \, n \in \mathbb{N}\}$.

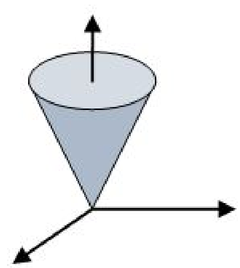

Fig. 4: Ice-cream cone

We now define the notion of relative interior of a nonempty set.

Definition 2.19. (Definition1.16 in [1]) Given a nonempty S in a real normed linear space X, the relative interior of S, denoted by $\mathrm{ri}(S)$, is the set of the interior points of S with respect to the topology relative to $\mathrm{aff}(S)$, that is,

$$\mathrm{ri}(S) = \{x \in S : \exists \varepsilon > 0, B(x, \varepsilon) \cap \mathrm{aff}(S) \subseteq S\}.$$

Thus, $\mathrm{ri}(S) = \mathrm{int}(S)$ if $\mathrm{aff}(S) = X$.

The following theorem states that the relative interior of a nonempty convex set is nonempty in a finite dimensional setting.

Theorem 2.20. (Theorem 5.23 in [6]) If C is a nonempty finite dimensional convex set in a real normed linear space, then $\mathrm{ri}(C) \neq \emptyset$. Moreover,

(i) $\mathrm{ri}(C) = \mathrm{ri}(\mathrm{cl}(C))$;

(ii) $\mathrm{cl}(C) = \mathrm{cl}(\mathrm{ri}(C))$.

The following example illustrates that the relative interior of a convex set may be empty in infinite dimensional space.

Example 2.21. Let $C = \mathrm{co}\{e_0, e_1, e_2, ...\} \subseteq l_2(\mathbb{R})$. For the convex set $C = \{(x_1, x_2, ..., x_m, 0...) : x_i \geq 0, i = 1, 2, ..., m; \sum_{i=1}^{m} x_i = 1; m \in \mathbb{N}\}$ it can be seen that $\mathrm{aff}(C) = \{(x_1, x_2, ..., x_m, 0, ...) : x_i \in \mathbb{R}, i = 1, 2, ..., m; m \in \mathbb{N}\}$. For any $x \in C$, we can see that $y = (x_1, x_2, ..., x_m, -\frac{\varepsilon}{2}, ...) \in B(x, \varepsilon) \cap \mathrm{aff}(C)$ for every $\varepsilon > 0$ but $y \notin C$. Hence $\mathrm{ri}(C)$ is an empty set.

It is easy to observe that if S_1 and S_2 are nonempty convex subsets of a real normed linear space such that $\mathrm{ri}(S_1) \cap \mathrm{ri}(S_2) \neq \emptyset$, then the following hold:

(i) $S_1 \subseteq S_2 \Rightarrow \mathrm{ri}(S_1) \subseteq \mathrm{ri}(S_2)$,

(ii) $\mathrm{ri}(S_1 \cap S_2) = \mathrm{ri}(S_1) \cap \mathrm{ri}(S_2)$,

(iii) $\mathrm{cl}(S_1 \cap S_2) = \mathrm{cl}(S_1) \cap \mathrm{cl}(S_2)$.

One may refer to Proposition 2.1.10 in [6] for the proof of these facts in finite dimensional spaces.

We now recall the notion of recession cone of a nonempty convex set.

Definition 2.22. (Definition 1.1.15 in [11]) Given a nonempty convex set C in a real normed linear space X, the recession cone to C, denoted by $0^+(C)$ is defined as

$$0^+(C) := \{d \in X : d + x \in C, \forall x \in C\}.$$

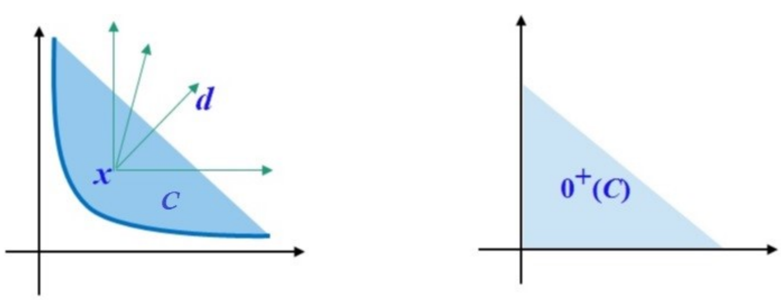

Fig. 5: Set and its recession cone

Proposition 2.23. (Proposition 1.1.16 in [11]) If C is a nonempty closed convex set in a real normed linear space X then $0^+(C)$ is a closed convex cone. If C is bounded then $0^+(C) = \{0\}$. Converse is true if X has finite dimension.

The following example illustrates that the converse of the above theorem fails in infinite dimensional spaces.

Example 2.24. For $C = \text{co}\{e_0, e_1, e_2, ...\} \subseteq l_2(\mathbb{R})$, we have $0^+(C) = \{0\}$.

Remark 2.25. In every infinite dimensional real normed linear space there exists an unbounded closed convex set with trivial recession cone (see Phung [13]).

Hence a space has finite dimension if every bounded set in the space has a trivial recession cone.

Definition 2.26. (Definition 3.3.4 in [12]) Given a nonempty set S in a real linear space X, a point $x \in S$ is said to be an extreme point of S, if there are no pints $x_1, x_2 \in S, x_1 \neq x_2$ such that $x = \alpha x_1 + (1-\alpha)x_2$, for $\alpha \in]0,1[$.

We denote the set of extreme points of S by $\text{ext}(S)$.

The extreme points of a triangle (with its interior) are its vertices. More generally, every polytope $S = \text{co}(\{a_1, a_2, \ldots, a_k\})$ has finitely many extreme points, and they are among the points $a_1, a_2, \ldots, a_k$. All boundary points of a disc $D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} \subseteq \mathbb{R}^2$ are extreme points of D.

We now state Krein–Milman theorem which provides a characterization of a compact convex subset in terms of its extreme points.

Theorem 2.27. (**KreinMilman**, Theorem 10.1.2 in [5]) A compact convex set C in a real normed linear space coincides with the closed convex hull of its extreme points, that is, $C = \overline{\text{co}}(\text{C})$.

From the above theorem it follows that every nonempty compact convex set in a real normed linear space has an extreme point. Finite dimensional version of Krein–Milman theorem is Minkowski theorem (Theorem 3.3.5. in [12]). The following example illustrates that Krein–Milman theorem fails to hold if the set C is closed and bounded but not compact.

Example 2.28. *The closed unit ball $L^1([0,1])$ is both closed and bounded (but not compact) and has no extreme points. Let B be the closed unit ball of $L^1([0,1])$. It is easy to see that, $f \in B$ with $\|f\|_1 < 1$ is not an extreme point of B. Let $f \in B$ with $\|f\|_1 = 1$, that is, $\int_0^1 |f(t)|\, dt = 1$. Let $c \in]0,1[$ be such that*

$$\int_0^c |f(t)|\, dt = \int_c^1 |f(t)|\, dt = \frac{1}{2}.$$

Define

$$g(x) = \begin{cases} 2f(x), & x \in [0,c[, \\ 0, & x \in [c,1], \end{cases}$$

and

$$h(x) = \begin{cases} 0, & x \in [0,c[, \\ 2f(x), & x \in [c,1], \end{cases}$$

Then, it can be seen that $\|g\|_1 = \|h\|_1 = 1$ and $f = \frac{1}{2}(g+h)$.

We now consider the notions of proximinal and Chebyshev sets.

Definition 2.29. (Definition 2.7 in [8]) A set S in a real normed linear space X is referred to as proximinal if for each $x \in X$ there exists $u \in S$ such that

$$\|x-u\| \leq \|x-y\|\,, \text{ for all } y \in S.$$

The point u is referred to as best approximation of x from S. If the best approximation point is unique for every point $x \in X$ then the set S is called as a *Chebyshev set.*

If we define $P_S(x) := \{u \in S : \|x-u\| = \inf_{y\in S} \|x-y\|\}$, then S is a proximinal set if $P_S(x) \neq \emptyset$ and Chebyshev set if $P_S(x)$ is a singleton.

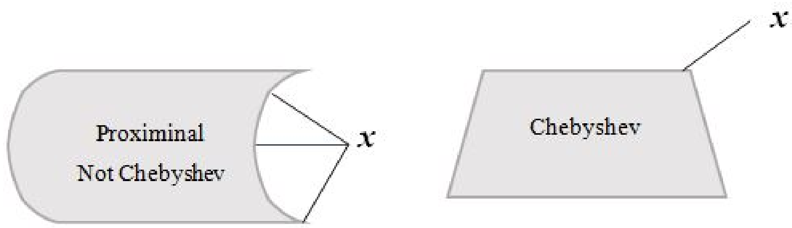

Fig. 6: Proximinal and Chebyshev sets

The following theorem shows that, the reflexivity of the Banach space plays an important role for the solvability of approximation problems.

Theorem 2.30. (Theorem 2.9 in [8]) A real Banach space is reflexive if and only if every nonempty convex closed subset is proximinal.

The following theorem shows that the space needs to be a Hilbert space for a nonempty closed convex set to be Chebyshev.

Theorem 2.31. (Theorem 3.2.1 in [12]) A nonempty closed convex subset is Chebyshev if the underlying space is a Hilbert space.

The above theorem may fail to hold if the space is not a Hilbert space.

Example 2.32. *Let $\mathbb{R}^2$ be endowed with $\|\cdot\|_\infty$ defined by $\|(x,y)\|_\infty = \max(|x|, |y|)$. Clearly, $C = \{(1,t) : t \in \mathbb{R}\}$ is a (closed) convex subset of $\mathbb{R}^2$. For all $x \in C$ we have $\|x\|_\infty \geq 1$. However, for all $t \in [-1,1], (1,t) \in C$ and $\|(1,t)\|_\infty = 1$, so that there are infinite number of elements with minimal norm in C. Hence, C is proximinal but not Chebyshev.*

In finite dimensional normed linear spaces with smooth and strictly convex unit spheres, the closed convex sets coincide with the Chebyshev sets. The following theorem provides a partial converse of Theorem 2.31.

Theorem 2.33. (Theorem 3.2.2 in [12]) Every Chebyshev subset of $\mathbb{R}^n$ is convex.

We now recall a related notion of convexity namely star convexity.

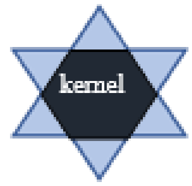

Fig. 7: Star shaped set and its kernel

Definition 2.34. (Page 235 in [10]) A nonempty set C in a real linear space X is said to be a star shaped set at $x \in C$ if for any y in C, the line segment joining x and y is contained in C, that is,

$$(1-\lambda)x + \lambda y \in C,\ \mathit{for}\ \lambda \in [0,1].$$

The collection of all points x with this property is called the *kernel* of C.

The set considered in Figure 1 is a star shaped set and its kernel is the dark shaded portion in Figure 5.

Theorem 2.35. (Theorem 4.3.1. in [11]) A polygon in the plane is star shaped if and only if for every three edges of the polygon, there is a point in the set from which all three edges are visible.

3 Some Solved and Unsolved Problems

Theorem 2.35 in the previous section provides an answer to the famous art gallery problem which relates to finding a place in the art gallery from where all the paintings can be viewed. This problem was first in 1973 by geometer and topologist Victor Klee. The above theorem implies that if for each three paintings in an art gallery, there is a place from which all three can be viewed, then there must be a place in the gallery from which all of its paintings can be viewed, that is, art gallery is star shaped.

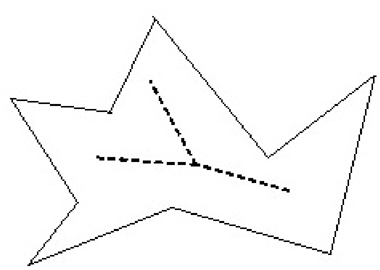

Fig. 8: Art gallery problem

The Carathodory conjecture, dating as early as 1920's, states that any closed convex surface in 3-dimensional Euclidean space must have at least 2 umbilic points (points where the surface curves equally in all directions). A solution to the smooth case was announced by Brendan Guilfoyle and Wilhelm Klingenberg in a paper posted on the arXiv in 2008 (https://arxiv.org/abs/0808.0851). That paper has been revised twice since then, but does not appear to have been accepted for publication yet.

Another conjecture involving Chebyshev is the Chebyshev conjecture which is most commonly posed as follows. Every Chebyshev subset of a Hilbert space is convex. This is still and an unsolved open problem related to convexity notion.

Packing problems are a class of problems that deals with packing objects together in a given space as densely as possible. These problems can easily be related to real life packaging, storage and transportation issues. Packing problems are also analogous to the problem of constructing optimal codes and the problem of understanding the structure of crystals. It is known that dense packing of circles is the hexagonal packing. In three-dimensional space, a Platonic solid is regular, convex polyhedron. It is constructed by congruent regular polygon faces with the same number of faces meeting at each vertex. There are five platonic solids namely tetrahedron, cube, octahedron, dodecahedron and icosahedron. Complete packing in three dimensions is achieved by cubic platonic solids. Also it is achieved using combination of both the platonic solids tetrahedrons and octahedrons.

It is also known that the best way to pack spheres together is the face-centred cubic packing, which consists of layers of spheres such that each layer is positioned so that the spheres rest on the 'holes' of the layer below. This packing using spheres is termed as "Kepler's conjecture" after it was conjectured by Johannes Kepler who is best known for his work on planetary orbits. This arrangement is not unique, even the hexagonal close packing constructed in a similar way is also equally efficient.

In 1998 Thomas Hales, of the University of Michigan and his student Sam Ferguson announced a 250 page proof in 1998. The proof was not accepted as a dozen referees gave it up after spending some years. According to them the proof seemed to be correct, but they just did not have the time or energy to verify everything comprehensively. The proof was published in 2005 but it was still unsatisfactory and the proof was beyond the ability of the mathematics community to check thoroughly. To address this situation and establish certainty, Prof. Hales used computer program to check the proof in 2014 and the proof was accepted by the journal Forum of Mathematics, Pi in 2017.The paper not only settles a centuries-old mathematical problem, but is also a major advance in computer verification of complex mathematical proofs.

The "happy ending problem", states that any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral. The name happy ending problem was coined by Paul Erds as it led to the marriage of two Hungarian mathematicians George Szekeres and Esther Klein in 1937. The couple in fact led a happy life till end and died on the same day at a ripe age of 94 and 95 within a gap of one hour on August 28, 2005.

The question of the existence of a convex and homogenous body with one unstable and one stable equilibria in three dimensions was first raised by the Russian mathematician Vladimir Arnold. Mathematicians were aware that no such shapes exist in two dimensions, and the fact that every three-dimensional object has at least two equilibria. Two Hungarian mathematicians GborDomokos and PterVrkonyinot only proved that its existence and also built one. This convex body is termed as Gmbc and moves forward and then manages to get back on its feet after it has been toppled over. When a Gmbc is placed on a horizontal surface, it starts wobbling around until it reaches the equilibrium position, a bit like a Weeble toy. In theory, it was known that one can balance it on the unstable equilibrium point, but in practice it seemed as impossible since the slightest nudge will make it fall over.

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