Finite Groups, Monomial Ideals and a Subdivision of a Simplex

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Abstract

For $n \geq 1$, let $\mathfrak{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$, $R = k[x_1, \ldots, x_n]$ be the standard polynomial ring over a field k and $\mathbf{x}^\sigma = \prod_{i=1}^n x_i^{\sigma(i)}$ be a monomial in R for $\sigma \in \mathfrak{S}_n$. For any non-empty subset $T \subseteq \mathfrak{S}_n$, $I_T = \langle \mathbf{x}^\sigma : \sigma \in T \rangle$ is a monomial ideal of R. We consider the monomial ideal I_G of R for a subgroup G of $\mathfrak{S}_n$. Many properties of the monomial ideal I_G and its Alexander dual $I_G^{[\mathbf{n}]}$ (with respect to $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^n$) are obtained. Let $\mathcal{A}_n$ be the Alternating subgroup of $\mathfrak{S}_n$. A cellular resolution of the Alexander dual $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ of $I_{\mathcal{A}_n}$ supported on a nice subdivision of an $n-1$-simplex Δ_{n-1} is obtained by modifying the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of the $n-1$-simplex Δ_{n-1}.

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1 Introduction

Let $\mathfrak{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$ and $R = k[x_1, \ldots, x_n]$ be the standard polynomial ring over a field k. The monomial ideal $I_{\mathfrak{S}_n} = \left\langle \mathbf{x}^\sigma = \prod_{i=1}^n x_i^{\sigma(i)} : \sigma \in \mathfrak{S}_n \right\rangle$ of R, called a *permutohedron ideal*, has many combinatorial properties. The convex hull $\mathbf{P}_n = P(\mathfrak{S}_n)$ of $n!$ points $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$; $\sigma \in \mathfrak{S}_n$ is a $(n-1)$-dimensional polytope in $\mathbb{R}^n$, called a *permutohedron*. The minimal resolution of the permutohedron ideal $I_{\mathfrak{S}_n}$ is the cellular resolution supported on the permutohedron $\mathbf{P}_n$ (see [1, 2]).

The Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ of $I_{\mathfrak{S}_n}$ with respect to $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^n$ is the monomial ideal of R given by

$$I_{\mathfrak{S}_n}^{[\mathbf{n}]} = \left\langle \left(\prod_{i \in A} x_i \right)^{n-|A|+1} : \emptyset \neq A \subseteq [n] \right\rangle.$$

The minimal resolution of $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is the cellular resolution supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1}. Therefore, the i^{th} Betti number $\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]})$ of $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is precisely, the number $f_i(\mathbf{Bd}(\Delta_{n-1}))$ of i-dimensional faces (or i-faces) of the simplicial complex $\mathbf{Bd}(\Delta_{n-1})$ (see [5]). We have

$$\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]}) = \beta_{i+1}\left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}\right) = f_i(\mathbf{Bd}(\Delta_{n-1})) = (i+1)!S(n+1, i+2),$$

where $S(n,k)$ is number of k-partitions of $[n]$, called a *Stirling number of the second kind.* The standard monomials of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$ correspond bijectively to the parking functions of length n and $\dim_k\left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}\right) = (n+1)^{n-1}$ [5]. By Cayley's formula, the number of spanning trees of the complete graph K_{n+1} on $[n+1]$ is precisely, $(n+1)^{n-1}$. Thus the monomial ideal $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is called a *tree ideal*. For more on cellular resolutions and Alexander duals of monomial ideals, we refer to [4].

Let G be a subgroup of $\mathfrak{S}_n$. In this paper, we have investigated homological properties of the monomial ideals I_G and its Alexander dual $I_G^{[\mathbf{n}]}$. Let $\mathbf{P}(G)$ be the convex hull of points $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$ for $\sigma \in G$. Then $\mathbf{P}(G)$ is a polytope contained in the $n-1$-dimensional permutohedron $\mathbf{P}_n = \mathbf{P}(\mathfrak{S}_n)$. Let $f_i(\mathbf{P}(G))$ be the number of i-faces of $\mathbf{P}(G)$. We observed that the minimal free resolution of I_G is the cellular resolution supported on the polytope $\mathbf{P}(G)$ and the i^{th} Betti number $\beta_i(I_G) = f_i(\mathbf{P}(G))$ for $0 \leq i \leq \dim(\mathbf{P}(G))$ (Theorem 2.1).

Let $\omega \in \mathfrak{S}_n - G$ and $G\omega$ (or ωG) be the right (or left) coset of G in $\mathfrak{S}_n$ determined by ω. Since the polytope $\mathbf{P}(G\omega)$ (or $\mathbf{P}(\omega G)$) is combinatorially equivalent to $\mathbf{P}(G)$, we have $\beta_i(I_{G\omega}) = \beta(I_{\omega G}) = \beta_i(I_G)$ for $0 \leq i \leq \dim(\mathbf{P}(G))$. We also consider the Alexander dual $I_G^{[\mathbf{n}]}$ of I_G with respect to $\mathbf{n}$. The quotient $\frac{R}{I_G^{[\mathbf{n}]}}$ is an Artinian k-algebra. Further, $\dim_k\left(\frac{R}{I_G^{[\mathbf{n}]}}\right) = \dim_k\left(\frac{R}{I_{G\omega}^{[\mathbf{n}]}}\right)$ (Theorem 2.3). Thus, for a normal subgroup G of $\mathfrak{S}_n$ and $\omega \in \mathfrak{S}_n$, we have (Corollary 2.2)

$$\dim_k\left(\frac{R}{I_G^{[\mathbf{n}]}}\right) = \dim_k\left(\frac{R}{I_{G\omega}^{[\mathbf{n}]}}\right) = \dim_k\left(\frac{R}{I_{\omega G}^{[\mathbf{n}]}}\right).$$

Finally, we construct a cellular resolution of the Alexander dual $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ of the monomial ideal $I_{\mathcal{A}_n}$ associated to the Alternating subgroup $\mathcal{A}_n$ of $\mathfrak{S}_n$. Let $\mathcal{A}_n^c = \mathfrak{S}_n - \mathcal{A}_n$ be the set of odd permutations of $\mathfrak{S}_n$. The minimal generators of the monomial ideals $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ and $I_{\mathcal{A}_n^c}^{[\mathbf{n}]}$ are given in Proposition 3.1. We construct a simplicial complex $\mathbf{Y}$ by modifying the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of the $n-1$-simplex Δ_{n-1} as follows. Each facet F of $\mathbf{Bd}(\Delta_{n-1})$ is given by a chain $\mathcal{C}$ of subsets of $[n]$ of the form $\mathcal{C}\ :\ \emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_n = [n]$, where $|A_i| = i$. We also say that the facet F given by the chain $\mathcal{C}$ is spanned by vertices $A_1, \ldots, A_n = [n]$ and write $F = \langle A_1, \ldots, A_n\rangle$. The facets of $\mathbf{Bd}(\Delta_{n-1})$ are in one-to-one correspondence with the permutations of $[n]$. In fact, if $A_i - A_{i-1} = \{b_i\}$, then the permutation σ_F corresponding to the facet $F = \langle A_1, \ldots, A_n\rangle$ is given by $\sigma_F(b_i) = i$ for $1 \leq i \leq n$. If $\sigma_F \in \mathcal{A}_n$, then the facet F of $\mathbf{Bd}(\Delta_{n-1})$ is also a facet of the simplicial complex $\mathbf{Y}$. If $\sigma_F \notin \mathcal{A}_n$, then the centroid v_F of the $n-2$-face $\langle A_1, \ldots, A_{n-1}\rangle$ of the facet F is a vertex added to the simplicial complex $\mathbf{Y}$. Further, the extra vertex v_F is joined to all the vertices of the $n-1$-dimensional facet F. Thus a facet $F = \langle A_1, \ldots, A_n\rangle$ of $\mathbf{Bd}(\Delta_{n-1})$ such that $\sigma_F \notin \mathcal{A}_n$ is subdivided into exactly $n-1$ facets of $\mathbf{Y}$. These $n-1$-facets of $\mathbf{Y}$ obtained by subdividing F are of the form $F_i = \langle v_F, A_1, \ldots, \widehat{A_i}, \ldots, A_n\rangle$ for $1 \leq i \leq n-1$, where $\widehat{A_i}$ indicates that the vertex A_i is deleted.

A vertex of $\mathbf{Y}$ corresponding to a non-empty subset $A \subseteq [n]$ is labelled with the monomial $\left(\prod_{i\in A} x_i\right)^{n-|A|+1}$, while the extra vertex corresponding to the centroid v_F of the $n-2$-face $\langle A_1, \ldots, A_{n-1}\rangle$ of the facet F with $\sigma_F \notin \mathcal{A}_n$ is labelled with the monomial $\mathbf{x}^{\mathbf{n}-\sigma_F} = \prod_{j=1}^{n} x_j^{n-\sigma_F(j)}$. Let $f_i(\mathbf{Y})$ be the number of i-faces of $\mathbf{Y}$. Then we proved that

$$f_i(\mathbf{Y}) = (i+1)!S(n+1, i+2) + \binom{n}{i}\frac{n!}{2}; \quad \text{for } 0 \leq i \leq n-3,$$

while $f_{n-2}(\mathbf{Y}) = \binom{n+1}{2}\frac{n!}{2}$ and $f_{n-1}(\mathbf{Y}) = \binom{n}{1}\frac{n!}{2}$ (Proposition 3.2).

The free complex associated to the labelled simplicial complex $\mathbf{Y}$ is a cellular resolution of the monomial ideal $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ for $n \geq 4$ (Theorem 3.1). But the cellular resolution supported on $\mathbf{Y}$ is non-minimal. For $n = 4$, by deleting appropriate faces of $\mathbf{Y}$, we obtained a labelled polyhedral cell complex $\mathbf{X}$ such that the minimal free resolution of $I_{\mathcal{A}_4}^{[\mathbf{4}]}$ is the cellular resolution supported on $\mathbf{X}$.

2 Finite Groups and Monomial Ideals

Let G be a subgroup of the symmetric group $\mathfrak{S}_n$. We consider the monomial ideal $I_G = \langle \mathbf{x}^\sigma : \sigma \in G\rangle$ of the polynomial ring $R = k[x_1, \ldots, x_n]$ associated to the subgroup G. We consider a polytope $\mathbf{P}(G)$ obtained by the convex hull of points $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$; $\sigma \in G$. As stated in the Introduction, $\mathbf{P}(G)$ is a polytope contained in the permutohedron $\mathbf{P}(\mathfrak{S}_n)$. Let $f_i(\mathbf{P}(G))$ be the number of i-dimensional faces of $\mathbf{P}(G)$ and $\beta_i(I_G)$ be the i^{th} Betti number of the monomial ideal I_G. The polytope $\mathbf{P}(G)$ is naturally a labelled polyhedral cell complex, with monomial label $\mathbf{x}^\sigma$ on the vertex $(\sigma(1), \ldots, \sigma(n))$ corresponding to $\sigma \in G$. The monomial label $\mathbf{x}^{\alpha(F)}$ on a face F of $\mathbf{P}(G)$ is given by the least common multiple (LCM) of labels on vertices of F.

It is well known that the minimal resolution of the permutohedron ideal $I_{\mathfrak{S}_n}$ is the cellular resolution supported on the permutohedron $\mathbf{P}_n$ (see [2]). Thus, we have $\beta_i(I_{\mathfrak{S}_n}) = f_i(\mathbf{P}_n)$. We recall that an $i-1$-face of the permutohedron $\mathbf{P}_n$ is represented by a chain of subsets of $[n]$ of the form $\emptyset = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_i$ of length i (see [4]).

Theorem 2.1. *The minimal free resolution of the monomial ideal I_G is the cellular resolution supported on the polytope $\mathbf{P}(G)$. In particular, i^{th} Betti number $\beta_i(I_G)$ of I_G equals $f_i(\mathbf{P}(G))$ for $0 \leq i \leq \dim(\mathbf{P}(G))$.*

Proof. In view of Proposition 4.5 of [4], we need to show that the subcomplex $\mathbf{P}(G)_{\leq \mathbf{b}} = \{F \in \mathbf{P}(G) : \mathbf{x}^{\alpha(F)} \text{ divides } \mathbf{x}^{\mathbf{b}}\}$ is either empty or acyclic. The subcomplex $\mathbf{P}(\mathfrak{S}_n)_{\leq \mathbf{b}}$, if non-empty, is contractible (see [2]). On similar lines, it is easy to see that $\mathbf{P}(G)_{\leq \mathbf{b}} \subseteq \mathbf{P}(\mathfrak{S}_n)_{\leq \mathbf{b}}$ is contractible, if nonempty. □ □

Remark 2.1. 1. The subgroup $G = \mathfrak{S}_r \times \mathfrak{S}_s \subseteq \mathfrak{S}_n$ with $r + s = n$ has been considered in [3].

2. It would be an interesting problem to obtain a combinatorial description of the faces of the polytope $\mathbf{P}(G)$ similar to that of permutohedron $\mathbf{P}(\mathfrak{S}_n)$.

Corollary 2.1. *Let $G = \langle \sigma \rangle$ be a cyclic group generated by $\sigma \in \mathfrak{S}_n$.*

1) *Let σ be an r-cycle. Then i^{th} Betti number $\beta_i(I_G) = \binom{r}{i+1}$ for $0 \leq i \leq r-1$.*

2) *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ is a product of disjoint cycles of lengths $r_1, r_2, \ldots, r_t$, respectively. Suppose that $gcd(r_i, r_j) = 1$ for all $1 \leq i < j \leq t$, and $G_j = \langle \sigma_j \rangle$.*

$$\beta_i(I_G) = \sum_{\substack{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t, \\ j_1 + \ldots + j_t = i}} \beta_{j_1}(G_1)\beta_{j_2}(G_2)\cdots\beta_{j_t}(G_t).$$

Proof. 1. We see that the polytope $\mathbf{P}(G)$ is an $r-1$-simplex spanned by the vertices σ^j for $1 \leq j \leq r$. Clearly,

$$\beta_i(I_G) = f_i(\mathbf{P}(G)) = \binom{r}{i+1}.$$

2. In this case, we see that the polytope $\mathbf{P}(G)$ is the product $\mathbf{P}(G_1) \times \cdots \times \mathbf{P}(G_t)$, where $\mathbf{P}(G_j)$ is an $r_j - 1$-simplex for $1 \leq j \leq t$. Now

$$f_i(\mathbf{P}(G)) = \sum_{\substack{(j_1, \ldots, j_t) \in \mathbb{N}^t, \\ j_1 + \ldots + j_t = i}} f_{j_1}(\mathbf{P}(G_1)) \cdots f_{j_t}(\mathbf{P}(G_t)).$$

This completes the proof. □ □

Let $\omega \in \mathfrak{S}_n - G$ and let $G\omega$ (or ωG) be the right (or left) coset of G in $\mathfrak{S}_n$ determined by ω. The monomial ideal $I_{G\omega} = \langle \mathbf{x}^{\sigma\omega} : \sigma \in G \rangle$ (or $I_{\omega G} = \langle \mathbf{x}^{\omega\sigma} : \sigma \in G \rangle$) has the same Betti numbers as I_G. In fact, the polytopes $\mathbf{P}(G\omega)$ and $\mathbf{P}(G)$ are combinatorially equivalent and the minimal resolution of $I_{G\omega}$ is the cellular resolution supported on $P(G\omega)$.

We consider the Alexander dual $I_G^{[\mathbf{n}]}$ of the monomial ideal I_G with respect to $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^n$. Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ such that $\mathbf{b} \preceq \mathbf{n}$(i.e. $b_i \leq n$; $\forall i$). Then $\mathbf{b}$ is a maximal vector such that $\mathbf{x}^{\mathbf{b}} \notin I_G$ if and only if $\mathbf{x}^{\mathbf{n}-\mathbf{b}} \in I_G^{[\mathbf{n}]}$ is a minimal generator (see Proposition 5.23 of [4]).

Theorem 2.2. *The Alexander dual* $I_G^{[\mathbf{n}]}$ *is a monomial ideal of* R *such that the quotient* $\frac{R}{I_G^{[\mathbf{n}]}}$ *is an Artinian* k*-algebra. Further, the group* G *acts on the minimal generators of* $I_G^{[\mathbf{n}]}$*.*

Proof. Let $d_i = \min\{\sigma(i) : \sigma \in G\}$ for $1 \le i \le n$. Consider $\mathbf{b}_i = (n, \dots, d_i - 1, \dots, n) \in \mathbb{N}^n$ (i.e., i^{th} place $d_i - 1$ and elsewhere n). Clearly, $\mathbf{b}_i$ is maximal with $\mathbf{x}^{\mathbf{b}_i} \notin I_G$. Thus, $x_i^{n-d_i+1} = \mathbf{x}^{\mathbf{n}-\mathbf{b}_i} \in I_G^{[\mathbf{n}]}$ is a minimal generator. This shows that $\frac{R}{I_G^{[\mathbf{n}]}}$ is Artinian. Let $\mathbf{x}^{\mathbf{c}} = x_1^{c_1} \cdots x_n^{c_n}$ be a minimal generator of $I_G^{[\mathbf{n}]}$. We shall show that $\sigma\mathbf{x}^{\mathbf{c}} = x_{\sigma(1)}^{c_1} \cdots x_{\sigma(n)}^{c_n}$ for $\sigma \in G$, is also a minimal generator of $I_G^{[\mathbf{n}]}$. As $\mathbf{x}^{\mathbf{n}-\mathbf{c}} \notin I_G^{[\mathbf{n}]}$ if and only if $\sigma\mathbf{x}^{\mathbf{n}-\mathbf{c}} \notin I_G^{[\mathbf{n}]}$, the second assertion follows. □ □

The quotient ring $\frac{R}{I_G^{[\mathbf{n}]}}$ is Artinian. Thus there are only finitely many monomials $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$ of R such that $\mathbf{x}^{\mathbf{b}} \notin I_G^{[\mathbf{n}]}$. Such monomials $\mathbf{x}^{\mathbf{b}}$ are called *standard monomials* of $\frac{R}{I_G^{[\mathbf{n}]}}$. In fact, the set of standard monomials forms a k-basis of the finite dimensional vector space $\frac{R}{I_G^{[\mathbf{n}]}}$.

Theorem 2.3. *For any* $\omega \in \mathfrak{S}_n - G$*, we have* $\dim_k \left(\frac{R}{I_G^{[\mathbf{n}]}}\right) = \dim_k \left(\frac{R}{I_{G\omega}^{[\mathbf{n}]}}\right)$.

Proof. We have $I_G = \langle \mathbf{x}^{\sigma} : \sigma \in G \rangle$ and $I_{G\omega} = \langle \mathbf{x}^{\sigma\omega} : \sigma \in G \rangle$. Now,

$$\mathbf{x}^{\sigma\omega} = \prod_{i=1}^{n} x_i^{\sigma(\omega(i))} = \prod_{j=1}^{n} x_{\omega^{-1}(j)}^{\sigma(j)} = \prod_{j=1}^{n} y_j^{\sigma(j)} = \mathbf{y}^{\sigma},$$

where $y_j = x_{\omega^{-1}(j)}$. Thus the monomial ideal $I_{G\omega}$ coincides with I_G on changing variable $x_{\omega^{-1}(j)}$ with x_j for all $1 \le j \le n$. Also, under the same permutation of variables, the Alexander dual $I_{G\omega}^{[\mathbf{n}]}$ will coincide with Alexander dual $I_G^{[\mathbf{n}]}$. Therefore, the number of standard monomials of $\frac{R}{I_G^{[\mathbf{n}]}}$ and $\frac{R}{I_{G\omega}^{[\mathbf{n}]}}$ are the same. □ □

The number of standard monomials of $\frac{R}{I_{\omega G}^{[\mathbf{n}]}}$ need not equal $\dim_k \left(\frac{R}{I_G^{[\mathbf{n}]}}\right)$.

Let σ be a 3-cycle in $\mathfrak{S}_4$ given by $\sigma = 2314$ in word notation. Let $G = \langle \sigma \rangle$ be the cyclic group of order 3 generated by σ. Choose $\omega = 2431 \in \mathfrak{S}_4$. Then Alexander duals $I_G^{[\mathbf{4}]}$ and $I_{\omega G}^{[\mathbf{4}]}$ are given by

$$I_G^{[\mathbf{4}]} = \langle x_1^4, x_2^4, x_3^4, x_4, x_1^2x_2^3, x_1^3x_3^2, x_2^2x_3^3, x_1^2x_2^2x_3^2 \rangle,$$

and

$$I_{\omega G}^{[\mathbf{4}]} = \langle x_1^3, x_2^3, x_3^3, x_4^4, x_1^2x_2, x_1x_3^2, x_2^2x_3, x_1x_2x_3 \rangle.$$

It can be easily checked that $\dim_k \left(\frac{R}{I_G^{[\mathbf{4}]}}\right) = 44$, while $\dim_k \left(\frac{R}{I_{\omega G}^{[\mathbf{4}]}}\right) = 52$.

Corollary 2.2. *Let G be a normal subgroup of $\mathfrak{S}_n$ and $\omega \in \mathfrak{S}_n$. Then*

$$\dim_k\left(\frac{R}{I_G^{[\mathbf{n}]}}\right) = \dim_k\left(\frac{R}{I_{G\omega}^{[\mathbf{n}]}}\right) = \dim_k\left(\frac{R}{I_{\omega G}^{[\mathbf{n}]}}\right).$$

Proof. For a normal subgroup G of $\mathfrak{S}_n$, $\omega G = G\omega$ for all $\omega \in \mathfrak{S}_n$. In view of Theorem 2.3, we are through. □ □

It is interesting to note that the top Betti number $\beta_{n-1}\left(I_G^{[\mathbf{n}]}\right)$ of the Alexander dual $I_G^{[\mathbf{n}]}$ gives the order $|G|$ of the subgroup G.

Proposition 2.1. *Let G be a subgroup of $\mathfrak{S}_n$. Then $\beta_{n-1}\left(I_G^{[\mathbf{n}]}\right) = |G|$.*

Proof. Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$. Then $(n-1)^{th}$ Betti number $\beta_{n-1,\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right)$ of $I_G^{[\mathbf{n}]}$ in multidegree $\mathbf{b}$ is given by $\beta_{n-1,\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right) = \dim_k \widetilde{H}_{n-2}\left(K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right); k\right) \neq 0$, where $K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right) = \left\{\text{square} - \text{free } \tau \subseteq [n] : \mathbf{x}^{\mathbf{b}-\tau} \in I_G^{[\mathbf{n}]}\right\}$ is the *upper Koszul simplicial complex* of $I_G^{[\mathbf{n}]}$ in degree $\mathbf{b}$ (see [4]). Since $K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right)$ is a subcomplex of the $n-1$-simplex Δ_{n-1}, $\widetilde{H}_{n-2}\left(K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right); k\right) \neq 0$ if and only if $K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right) = \partial\Delta_{n-1}$ is the boundary complex of $n-1$-simplex Δ_{n-1}. This shows that $[n] \notin K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right)$ and for every $A \subsetneq [n], A \in K^{\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right)$. Thus, $\mathbf{b}-(1,1,\ldots,1) \preceq \mathbf{n}$ is maximal with $\mathbf{x}^{\mathbf{b}-(1,1,\ldots,1)} \notin I_G^{[\mathbf{n}]}$. Equivalently, $\mathbf{x}^{\mathbf{n}+\mathbf{1}-\mathbf{b}} \in \left(I_G^{[\mathbf{n}]}\right)^{[\mathbf{n}]} = I_G$ is a minimal generator. Thus, $\beta_{n-1,\mathbf{b}}\left(I_G^{[\mathbf{n}]}\right) \neq 0$ if and only if $\mathbf{b} = \mathbf{n}+\mathbf{1}-\sigma$ for some $\sigma \in G$. As, $\beta_{n-1,\mathbf{n}+\mathbf{1}-\sigma}(I_G^{[\mathbf{n}]}) = 1$ for every $\sigma \in G$, we have $\beta_{n-1}\left(I_G^{[\mathbf{n}]}\right) = |G|$. □ □

Remark 2.2.

1. Proposition 2.1 can also be deduced from a general duality theorem for Betti numbers (see Theorem 5.48 of [4]).

2. If the minimal resolution of $I_G^{[\mathbf{n}]}$ is the cellular resolution supported on an $n-1$-dimensional Polyhedral cell complex $\mathbf{P}$, then G acts on $\mathbf{P}$. In particular, G acts on the facets of $\mathbf{P}$ freely and transitively. Hence, $f_{n-1}(\mathbf{P}) = |G|$.

3 Cellular resolution of $I_{\mathcal{A}_n}$

In this section, we consider the alternating subgroup $\mathcal{A}_n$ of $\mathfrak{S}_n$. We would like to construct a cellular resolution of the Alexander dual $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ of the monomial ideal $I_{\mathcal{A}_n}$. First, we consider the cases for $n \leq 3$.

We see that $I_{\mathcal{A}_1} = \langle x_1 \rangle$ and $I_{\mathcal{A}_2} = \langle x_1 x_2^2 \rangle$ are the monomial ideals in the polynomial rings $k[x_1]$ and $k[x_1, x_2]$, respectively. Clearly, Alexander duals $I_{\mathcal{A}_1}^{[(1)]} = \langle x_1 \rangle, I_{\mathcal{A}_2}^{[(2,2)]} = \langle x_1^2, x_2 \rangle$ and, $I_{\mathcal{A}_3}^{[(3,3,3)]} = \langle x_1^3, x_2^3, x_3^3, x_1x_2^2, x_2x_3^2, x_1^2x_3, x_1x_2x_3 \rangle$. The minimal resolutions of $I_{\mathcal{A}_1}^{[(1)]}$ and $I_{\mathcal{A}_2}^{[(2,2)]}$ are supported on a 0-simplex and 1-simplex, respectively. Further, the minimal resolution of $I_{\mathcal{A}_3}^{[(3,3,3)]}$ is supported on the labelled polyhedral complex shown in figure-1.

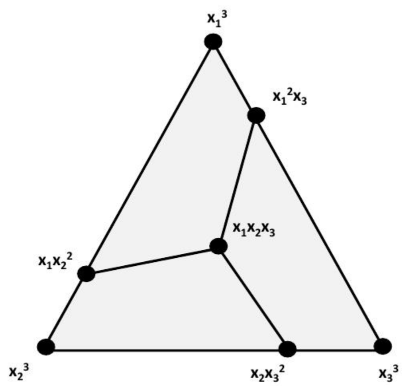

Fig. 1: Labelled polyhedral complex

Proposition 3.1. *For $n \geq 4$, the minimal generators of the monomial ideals $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ and $I_{\mathcal{A}_n^c}^{[\mathbf{n}]}$ are given by*

$$I_{\mathcal{A}_n}^{[\mathbf{n}]} = \left\langle \left(\prod_{i \in A} x_i \right)^{n-|A|+1}, \quad \mathbf{x}^{\mathbf{n}-\tau} : \emptyset \neq A \subseteq [n]; \tau \notin \mathcal{A}_n \right\rangle,$$

and

$$I_{\mathcal{A}_n^c}^{[\mathbf{n}]} = \left\langle \left(\prod_{i \in A} x_i \right)^{n-|A|+1}, \quad \mathbf{x}^{\mathbf{n}-\tau} : \emptyset \neq A \subseteq [n]; \tau \in \mathcal{A}_n \right\rangle.$$

Proof. As $I_{\mathcal{A}_n} \subseteq I_{\mathfrak{S}_n}$, we see that $I_{\mathfrak{S}_n}^{[\mathbf{n}]} \subseteq I_{\mathcal{A}_n}^{[\mathbf{n}]}$. Clearly, $\left(\prod_{i \in A} x_i\right)^{n-|A|+1}$ is also a minimal generator of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$. If $\tau \notin \mathcal{A}_n$, then $\mathbf{x}^{\mathbf{n}-\tau} = \prod_{j=1}^{n} x_j^{n-\sigma(j)}$ is a minimal generator of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ in view of Proposition 5.23 of [4], the second part is proved on the similar lines. □ □

Corollary 3.1. *For $n \geq 2$,*

$$\dim_k \left(\frac{R}{I_{\mathcal{A}_n}^{[\mathbf{n}]}} \right) = \dim_k \left(\frac{R}{I_{\mathcal{A}_n^c}^{[\mathbf{n}]}} \right) = (n+1)^{n-1} - \frac{n!}{2}.$$

Proof. As $\mathcal{A}_n$ is a normal subgroup of $\mathfrak{S}_n$, we have $\dim_k\left(\frac{R}{I_{\mathcal{A}_n}^{[\mathbf{n}]}}\right) = \dim_k\left(\frac{R}{I_{\mathcal{A}_n^c}^{[\mathbf{n}]}}\right)$. Also, we recall that $I_{\mathfrak{S}_n}^{[\mathbf{n}]} \subseteq I_{\mathcal{A}_n}^{[\mathbf{n}]}$ and the standard monomials of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$ are of the form $\mathbf{x}^{\mathbf{p}}$ for ordinary parking functions $\mathbf{p}$ of length n. In view of Proposition 3.1, we see that $\mathbf{x}^{\mathbf{p}}$ is not a standard monomial of $\frac{R}{I_{\mathcal{A}_n}^{[\mathbf{n}]}}$ if and only if $\mathbf{p} = \mathbf{n} - \tau$, for some $\tau \notin \mathcal{A}_n$. This completes the proof. □ □

The Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ of the permutohedron ideal $I_{\mathfrak{S}_n}$ is a *generic monomial ideal* and hence its minimal resolution is the cellular resolution supported on its *Scarf complex* (see Theorem 6.13 of [4]). Since the Alexander dual $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ is not a generic monomial ideal, construction of its minimal resolution is not straight forward. We now proceed to construct an explicit cellular resolution of the Alexander dual $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ for $n \geq 4$. Let $\mathbf{Y}$ be the simplicial complex obtained by modifying the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ as described in the Introduction. Let $f_i(\mathbf{Y})$ be the number of i-faces of $\mathbf{Y}$ and $S(n,k)$ be the Stirling number of the second kind.

Proposition 3.2. *Let $n \geq 4$. Then*

$$f_i(\mathbf{Y}) = (i+1)!S(n+1, i+2) + \binom{n}{i}\frac{n!}{2}; \quad \text{for } 0 \leq i \leq n-3,$$

while $f_{n-2}(\mathbf{Y}) = \binom{n+1}{2}\frac{n!}{2}$ and $f_{n-1}(\mathbf{Y}) = \binom{n}{1}\frac{n!}{2}$.

Proof. Since, $f_i(\mathbf{Bd}(\Delta_{n-1})) = (i+1)!S(n+1, i+2)$, we need to show that the number of i-faces of $\mathbf{Y}$ not in $\mathbf{Bd}(\Delta_{n-1})$ is precisely $\binom{n}{i}\frac{n!}{2}$ for $0 \leq i \leq n-3$. Consider a facet $F = \langle A_1, \ldots, A_n\rangle$ of $\mathbf{Bd}(\Delta_{n-1})$ such that permutation $\sigma_F \notin \mathcal{A}_n$. Then centroid v_F is an extra vertex of $\mathbf{Y}$ and $F_i = \langle v_F, A_1, \ldots, \widehat{A}_i, \ldots, A_n\rangle$ for $1 \leq i \leq n-1$ are the facets of $\mathbf{Y}$ containing v_F. An i-face of $\mathbf{Y}$ contained in any facets F_i $(1 \leq i \leq n)$, must contain the vertex v_F, otherwise it will be a face of $\mathbf{Bd}(\Delta_{n-1})$. Each such i-face of $\mathbf{Y}$ is obtained by choosing i vertices out of $A_1, \ldots, A_n$. Thus the number of i-faces of $\mathbf{Y}$ containing v_F is precisely $\binom{n}{i}$. Since there are exactly $\frac{n!}{2}$ vertices of the form v_F, we get the first part.

An $n-2$-face of $\mathbf{Y}$ containing v_F is obtained by choosing $n-2$ vertices out of $A_1, \ldots, A_n$, but the $n-2$-face $\langle A_1, \ldots, A_{n-1}\rangle$ of $\mathbf{Bd}(\Delta_{n-1})$ is no longer a face of $\mathbf{Y}$. Thus

$$f_{n-2}(\mathbf{Y}) = (n-1)!S(n+1, n) - \frac{n!}{2} + \binom{n}{n-2}\frac{n!}{2}.$$

As $S(n+1, n) = \binom{n+1}{2}$, we see that $f_{n-2}(\mathbf{Y}) = \binom{n+1}{2}\frac{n!}{2}$. Similarly, an $n-1$-face of $\mathbf{Y}$ containing v_F is obtained by choosing $n-1$ vertices out of $A_1, \ldots, A_n$ except $\{A_1, \ldots, A_{n-1}\}$. Also, the $n-1$-face $F = \langle A_1, \ldots, A_n\rangle$ of $\mathbf{Bd}(\Delta_{n-1})$ with $\sigma_F \notin \mathcal{A}_n$ is not a face of $\mathbf{Y}$. Since, there are $\frac{n!}{2}$ such facets F, we have

$$f_{n-1}(\mathbf{Y}) = n!S(n+1, n+1) - \frac{n!}{2} + \left[\binom{n}{n-1} - 1\right]\frac{n!}{2}.$$

As $S(n+1, n+1) = 1$, we have $f_{n-1}(\mathbf{Y}) = n\left(\frac{n!}{2}\right)$. □ □

The simplicial complex $\mathbf{Y}$ is labelled. The monomial label on a vertex corresponding to a non-empty subset $A \subseteq [n]$ is $\left(\prod_{i\in A} x_i\right)^{n-|A|+1}$, while monomial label on a vertex of the form v_F for a facet F of $\mathbf{Bd}(\Delta_{n-1})$ with $\sigma_F \notin \mathcal{A}_n$ is $\mathbf{x}^{\mathbf{n}-\sigma_F}$. The monomial label $\mathbf{x}^{\alpha(F')}$ on a face F' of $\mathbf{Y}$ is the least common multiple of labels on the vertices of F'. Clearly, the monomial ideal generated by the vertex labels of $\mathbf{Y}$ is precisely the Alexander dual $I_{\mathcal{A}_n}^{[\mathbf{n}]}$. If the free complex $\mathbb{F}(\mathbf{Y})$ associated to the labelled simplicial complex (or polyhedral cell complex) $\mathbf{Y}$ is exact, then we say that $\mathbb{F}(\mathbf{Y})$ is a *cellular resolution* of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ supported on $\mathbf{Y}$ [2, 4].

Theorem 3.1. *The free complex $\mathbb{F}(\mathbf{Y})$ is a cellular resolution of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ for $n \geq 4$.*

Proof. We need to verify that for any $\mathbf{b} \in \mathbb{N}^n$, the subcomplexes $\mathbf{Y}_{\leq \mathbf{b}}$ are either empty or acyclic. Since $\mathbf{Bd}(\Delta_{n-1})_{\leq \mathbf{b}}$ is either empty or contractible, it can be easily verified that $\mathbf{Y}_{\leq \mathbf{b}}$ is also contractible, if non-empty. □ □

The cellular resolution of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ supported on $\mathbf{Y}$ is never minimal because there are faces $F' \subsetneq F''$ of $\mathbf{Y}$ such that monomial labels $\mathbf{x}^{\alpha(F')} = \mathbf{x}^{\alpha(F'')}$. Further, the i-th Betti number $\beta_i(I_{\mathcal{A}_n}^{[\mathbf{n}]}) = f_i(\mathbf{Y})$, provided monomial levels on all $i+1$-dimensional faces of $\mathbf{Y}$ are different from monomial levels on their proper subfaces.

Corollary 3.2. *For $n \geq 4$, the i^{th} Betti number $\beta_i(I_{\mathcal{A}_n}^{[\mathbf{n}]})$ satisfies*

$$\beta_i(I_{\mathcal{A}_n}^{[\mathbf{n}]}) \leq f_i(\mathbf{Y}) \quad \text{for } 0 \leq i \leq n-1.$$

Proof. Since $\mathbf{Y}$ supports a cellular resolution $\mathbb{F}(\mathbf{Y})$ of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$, the minimal resolution of $I_{\mathcal{A}_n}^{[\mathbf{n}]}$ is contained in the cellular resolution. Thus, $\beta_i(I_{\mathcal{A}_n}^{[\mathbf{n}]}) \leq f_i(\mathbf{Y})\ \forall i$. □ □

Clearly, $\beta_0(I_{\mathcal{A}_n}^{[\mathbf{n}]}) = f_0(\mathbf{Y})$. It may be an interesting problem to determine the Betti numbers $\beta_i(I_{\mathcal{A}_n}^{[\mathbf{n}]})$ for all $1 \leq i \leq n-1$.

Let $\mathbf{X}$ be a subdivision of an $n-1$-simplex Δ_{n-1} obtained from the simplicial complex $\mathbf{Y}$ by deleting all proper faces F such that $\mathbf{x}^{\alpha(F)} = \mathbf{x}^{\alpha(F')}$ for some face $F' \supsetneq F$ of $\mathbf{Y}$. A face F of $\mathbf{Y}$ with no deleted proper subfaces remains a face of $\mathbf{X}$. Further, if a deleted face F of $\mathbf{Y}$ is a common maximal proper face of the faces F' and F'' of $\mathbf{Y}$, then the faces F' and F'' are merged along the common deleted face F. On merging faces of $\mathbf{Y}$ along deleted common faces, we get the other faces of $\mathbf{X}$. Let $f_i(\mathbf{X})$ be the number of i-faces of $\mathbf{X}$.

Proposition 3.3. *For $n \geq 4$, $f_{n-1}(\mathbf{X}) = \frac{n!}{2}$. Also, for $n \geq 6$,*

$$f_1(\mathbf{X}) = f_1(\mathbf{Y}) = 3^n + 1 - 2^{n+1} + \frac{n(n!)}{2}.$$

Proof. Consider a facet $F = \langle A_1, \ldots, A_n \rangle$ of $\mathbf{Bd}(\Delta_{n-1})$ such that $A_1 = \{2\}$ and $A_i = [i]$ for $2 \leq i \leq n$. We see that the permutation σ_F is the transposition interchanging 1 and 2. Then all the facets of $\mathbf{Y}$ containing v_F are of the form $F_i = \langle v_F, A_1, \ldots, \widehat{A_i}, \ldots, A_n \rangle$ for $1 \leq i \leq n-1$. The monomial label $\mathbf{x}^{\alpha(F_i)}$ is same as the label $\mathbf{x}^{\alpha(F'_i)}$, where $F'_i = F_i - v_F$ is the maximal subface of F_i not containing v_F. This shows that F'_i is not a face of $\mathbf{X}$ for all $i \leq i \leq n-1$. Hence, all the facets F_i of $\mathbf{Y}$ are no longer faces of $\mathbf{X}$. Since, the Alternating group $\mathcal{A}_n$ acts on $\mathbf{Y}$, the same holds for all facets G of $\mathbf{Y}$ such that $\sigma_G \notin \mathcal{A}_n$. This shows that $f_{n-1}(\mathbf{X}) = f_{n-1}(\mathbf{Y}) - (n-1)\frac{n!}{2} = \frac{n!}{2}$. Now, we show that if $n \geq 6$, then both $\mathbf{X}$ and $\mathbf{Y}$ have the same edges. Again as above, it is enough to see that no edges contained in the facet F_i $(1 \leq i \leq n-1)$ get deleted, where $F = \langle A_1, \ldots, A_n \rangle$. If the edge $L_{ij} = \langle A_i, A_j \rangle$ gets deleted, then the monomial label on L_{ij} is same as the label on the 2-face $\langle A_i, A_j, v_F \rangle$. This is not possible for $n \geq 6$. Since $S(n+1, 3) = \frac{3^n+1}{2} - 2^n$, the second part follows. □ □

We see that $\beta_1(I^{[\mathbf{n}]}_{\mathcal{A}_n}) = f_1(\mathbf{Y}) = 3^n + 1 - 2^{n+1} + \frac{n(n!)}{2}$. Also, we have already seen that $\beta_{n-1}(I^{[\mathbf{n}]}_{\mathcal{A}_n}) = |\mathcal{A}_n| = \frac{n!}{2}$.

Finally, for $n = 4$ the subdivision $\mathbf{X}$ of a 3-simplex Δ_3 is a labelled polyhedral cell complex and the cellular resolution supported on the polyhedral cell complex $\mathbf{X}$ gives the minimal free resolution of $I^{[\mathbf{4}]}_{\mathcal{A}_4}$.

For $n = 4$, the simplicial complex $\mathbf{Y}$ obtained by modifying the first barycentric subdivision of a 3-simplex has $f_0(\mathbf{Y}) = 27, f_1(\mathbf{Y}) = 98, f_2(\mathbf{Y}) = 120, f_3(\mathbf{Y}) = 48$. The edges of $\mathbf{Y}$ represented by a chain $\emptyset = A_0 \subsetneq A_1 \subsetneq A_2$ of subsets of $[4] = \{1, 2, 3, 4\}$ are deleted if $(|A_1|, |A_2|)$ is either $(1, 3)$, or $(2, 3)$, or $(2, 4)$. There are exactly 12 edges of first type, 12 edges of second type and 6 edges of third type. Thus all together these 30 edges get deleted and so $f_1(\mathbf{X}) = 98 - 30 = 68$. On deleting these edges from $\mathbf{Y}$, the 2-faces containing these edges get merged. For an edge of first or second type, the two faces of the form $\langle A_1, A_2, [4] \rangle$ or $\langle v_F, A_1, A_2 \rangle$ get merged with another 2-faces. There are exactly $2(12+12) = 48$ such 2-faces. Now consider an edge $\langle A_1, A_2 \rangle$ of the third type. A 2-face of $\mathbf{Y}$ containing an edge of the third type is of the form either $\langle A, A_1, A_2 = [4] \rangle$ with $\emptyset \neq A \subsetneq A_1$ or $\langle v_F, A_1, A_2 = [4] \rangle$. Note that 2-faces of the form $\langle A_1, B, A_2 = [4] \rangle$ with $A_1 \subsetneq B \subsetneq A_2 = [4]$ has already been counted. The number of these 2-faces is $(2+1)(6) = 18$. This shows that a total number of 2-faces of $\mathbf{Y}$ that get merged with another 2-faces is 66. Thus $f_2(\mathbf{X}) = 120 - 66 = 54$. We have already seen that $f_3(\mathbf{X}) = 12$. The polyhedral cell complex $\mathbf{X}$ is shown in figure-2(a), while one of its facet is described in the figure-2(b).

Every facet of $\mathbf{X}$ is a 3-dimensional polytope bounded by a pentagonal face, a quadrilateral face and five triangular faces as shown in figure-2(b). The polyhedral cell complex $\mathbf{X}$ gives a nice subdivision of a regular tetrahedron.

References

[1] Bayer D., Peeva I. and Sturmfels B., *Monomial resolutions*, Mathematical Research Letters **5** (1998), 31-46.

[2] Bayer D. and Sturmfels B., *Cellular resolution of monomial modules*, Journal für die Reine und Angewandte Mathematik **502** (1998), 123-140.

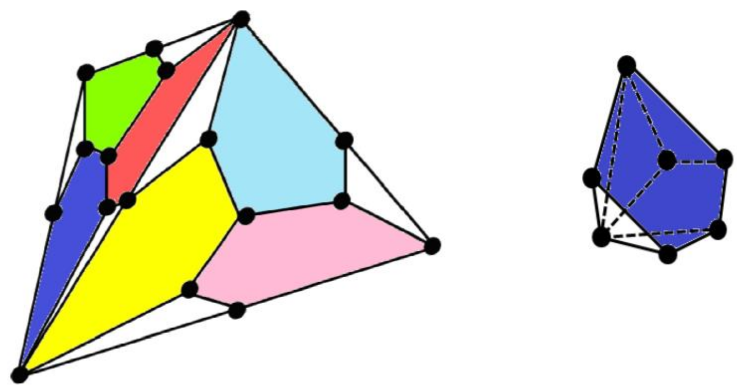

Figure -2(a) **Figure-2(b)**

[3] Kumar A. and Kumar C., *Certain variants of multipermutohedron ideals*, Proc. Indian Acad. Sci.(Math Sci.) Vol.**126** , No.4, November 2016, 479-500.

[4] Miller E. and Sturmfels B., *Combinatorial commutative algebra*, Graduate Texts in Mathematics Vol **227**, Springer 2004.

[5] Postnikov A. and Shapiro B., *Trees, parking functions, syzygies, and deformation of Monomial ideals*, Trans. Amer. Math. Soc. **356**, (2004), 3109-3142.