Some Linear Generating Relations Involving Two Polynomials of Bedient

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Abstract

In this paper, we obtain some linear generating functions associated with even and odd degree polynomials of Bedient by using series decomposition technique, in terms of sum of two Kampé de Fériet's double hypergeometric functions with suitable convergence conditions.

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1 INTRODUCTION AND PRELIMINARIES

Throughout the present work, we use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$.
Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$, $\mathbb{R}_-$ denote the sets of positive and negative real numbers respectively and $\mathbb{C}$ denotes the set of complex numbers.
The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(1.1) \qquad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.
The following results will be required in our present investigations:
In our analysis following identity of Srivastava [19, p.4] plays an important role

$$(1.2) \qquad \sum_{m,n=0}^{\infty} f(m+n)\frac{x^m y^n}{m!n!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}$$

The generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters is defined by

$$(1.3) \qquad {}_pF_q\left[\begin{array}{l} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} \; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}.$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that $\beta_j \neq 0, -1, -2, \ldots; \quad j = 1, 2, \ldots, q$.

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction on β_j, the ${}_pF_q$ series in (1.3):
(i) converges for $|z| < \infty$ if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q + 1$,
(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$(1.4) \qquad \omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is
(I) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,
(II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \Re(\omega) \leq 0$,
(III) divergent for $|z| = 1$, if $\Re(\omega) \leq -1$.
The notation $\Delta(\ell; \lambda)$ abbreviates the array of ℓ parameters given by

$$\frac{\lambda}{\ell}, \frac{\lambda + 1}{\ell}, \ldots\ldots, \frac{\lambda + \ell - 1}{\ell}; \qquad \ell = 1, 2, 3, 4, \ldots$$

First Bedient's polynomials [14, p.297 (1); 2, p.15 (2.5)] are given by

$$(1.5) \qquad R_n(\beta, \gamma; x) = \frac{(\beta)_n}{n!} (2x)^n {}_3F_2\left[\begin{array}{r} \triangle(2; -n), \gamma - \beta; \\ \gamma, \quad 1 - \beta - n; \end{array} \; \frac{1}{x^2}\right]$$

Second Bedient's polynomials [14, p.297 (2); 2, p.44 (3.4)] are given by

$$(1.6) \qquad G_n(\alpha, \beta; x) = \frac{(\alpha)_n (\beta)_n}{n!(\alpha + \beta)_n} (2x)^n {}_3F_2\left[\begin{array}{ll} \triangle(2; -n), 1 - \alpha - \beta - n; & \\ 1 - \alpha - n, \qquad 1 - \beta - n; & \end{array} \frac{1}{x^2}\right]$$

First Bedient's polynomials R_n [14, p.297 (4)] are also defined by

$$(1.7) \qquad \sum_{n=0}^{\infty} R_n(\beta, \gamma; x) t^n = (1 - 2xt)^{-\beta} {}_2F_1\left[\begin{array}{l} \beta, \gamma - \beta; \\ \gamma \qquad ; \end{array} \; \frac{-t^2}{1 - 2xt}\right]$$

where $|xt| < \frac{1}{2}$ and $\left|\frac{-t^2}{1-2xt}\right| < 1$ and $\gamma \neq 0, -1, -2, -3, \ldots$.
Or

$$\sum_{n=0}^{\infty} R_n(\beta,\gamma;x)t^n = \sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n}(\gamma-\beta)_n(2xt)^m(-t^2)^n}{(\gamma)_n m!n!} \tag{1.8}$$

Second Bedient's polynomials G_n [14, p.298 (6)] are also defined by

$$\sum_{n=0}^{\infty} G_n(\alpha,\beta;x)t^n = {}_2F_1\left[\begin{array}{cc} \alpha, & \beta; \\ \alpha+\beta; & \end{array} 2xt-t^2\right] = \sum_{N=0}^{\infty} \frac{(\alpha)_N(\beta)_N}{(\alpha+\beta)_N N!}(2xt-t^2)^N \tag{1.9}$$

where $|2xt-t^2| < 1$ and $\alpha+\beta \neq 0, -1, -2, -3, \ldots$.
Now using Srivastava's identity (1.2) in the right hand side of equation (1.9), we get

$$\sum_{n=0}^{\infty} G_n(\alpha,\beta;x)t^n = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(2xt)^m(-t^2)^n}{(\alpha+\beta)_{m+n} m!n!}. \tag{1.10}$$

The Kampé de Fériet's double hypergeometric function of higher order in the modified notation of Srivastava and Panda [22, p.423 (26), 424 (27)], is given by

$$F^{p:q;k}_{j:m;n}\left[\begin{array}{c} (a_p):(b_q);(d_k); \\ (g_j):(e_m);(h_n); \end{array} x,y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_{r+s}\prod_{i=1}^{q}(b_i)_r\prod_{i=1}^{k}(d_i)_s}{\prod_{i=1}^{j}(g_i)_{r+s}\prod_{i=1}^{m}(e_i)_r\prod_{i=1}^{n}(h_i)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.11}$$

where (a_p) abbreviates the array of p parameters given by $a_1, a_2, \ldots, a_p$ with similar interpretations for $(b_q), (d_k)$ et cetera and for convergence of double hypergeometric series(1.11), we have
(i)$p+q < j+m+1$, $p+k < j+n+1$, $|x| < \infty$ and $|y| < \infty$ or
(ii)$p+q = j+m+1$, $p+k = j+n+1$, and

$$\begin{cases} |x|^{\frac{1}{p-j}} + |y|^{\frac{1}{p-j}} < 1, & \text{if} \quad p > j \\ max\{|x|,|y|\} < 1, & \text{if} \quad p \leq j. \end{cases}$$

The Appell's double hypergeometric functions F_1, F_2, F_3 and F_4 [21, p.53 (4,5,6,7)] are denoted by $F^{1:1;1}_{1:0;0}$, $F^{1:1;1}_{0:1;1}$, $F^{0:2;2}_{1:0;0}$ and $F^{2:0;0}_{0:1;1}$ respectively.
The idea of separation of a power series into its even and odd terms [21, p.200 (1), p.214 Q.N.8; see also 20, p.196], exhibited by the elementary identity

$$\sum_{n=0}^{\infty} \Psi(n) = \sum_{n=0}^{\infty} \Psi(2n) + \sum_{n=0}^{\infty} \Psi(2n+1)\ , \tag{1.12}$$

is at least as old as the series themselves.

$$\sum_{m,n=0}^{\infty} \Phi(m,n) = \sum_{m,n=0}^{\infty} \Phi(2m,2n) + \sum_{m,n=0}^{\infty} \Phi(2m+1,2n+1) + \sum_{m,n=0}^{\infty} \Phi(2m,2n+1) + \sum_{m,n=0}^{\infty} \Phi(2m+1,2n), \tag{1.13}$$

provided that above multiple series are absolutely convergent.
Motivated by the work of Barr [1], Carlson [3], MacRobert [6-7], Manocha [8], Chaudhary et al. [4,5,9], Qureshi and Ahmad [10], Qureshi, Quraishi and Pal [12], Qureshi, Yasmeen and Pathan [13], Qureshi, Kabra and Khan [11], Sharma [15-17] and Srivastava [18,20], we shall obtain some linear generating relations associated with Bedient's polynomials of even and odd degree.

2 MAIN GENERATING RELATIONS

Any values of parameters and variables leading to the results given in this section which do not make sense, are tacitly excluded (Suppose α, β, γ and x are real numbers), then

$$\sum_{n=0}^{\infty} R_{2n}(\beta,\gamma;x)t^n = F^{2:0;2}_{0:1;3}\left[\begin{array}{c} \Delta(2;\beta) : -; \Delta(2;\gamma-\beta); \\ - \quad : \frac{1}{2}; \ \frac{1}{2}, \Delta(2;\gamma); \end{array} 4x^2t, t^2\right] + \frac{\beta(\beta-\gamma)t}{\gamma} F^{2:0;2}_{0:1;3}\left[\begin{array}{c} \Delta(2;\beta+1) : -; \Delta(2;\gamma-\beta+1); \\ - \quad : \frac{1}{2}; \ \frac{3}{2}, \Delta(2;\gamma+1); \end{array} 4x^2t, t^2\right], \tag{2.1}$$

where $|4x^2t|^{\frac{1}{2}} + |t^2|^{\frac{1}{2}} < 1$

$$\sum_{n=0}^{\infty} R_{2n+1}(\beta,\gamma;x)t^n = \frac{2\beta(\beta+1)(\beta-\gamma)tx}{\gamma} \times F^{2:0;2}_{0:1;3}\left[\begin{array}{c} \Delta(2;\beta+2) : -; \Delta(2;\gamma-\beta+1); \\ - \quad : \frac{3}{2}; \ \frac{3}{2}, \Delta(2;\gamma+1); \end{array} 4x^2t, t^2\right] + 2\beta x F^{2:0;2}_{0:1;3}\left[\begin{array}{c} \Delta(2;\beta+1) : -; \Delta(2;\gamma-\beta); \\ - \quad : \frac{3}{2}; \ \frac{1}{2}, \Delta(2;\gamma); \end{array} 4x^2t, t^2\right], \tag{2.2}$$

where $|4x^2t|^{\frac{1}{2}} + |t^2|^{\frac{1}{2}} < 1$

$$\sum_{n=0}^{\infty} G_{2n}(\alpha,\beta;x)t^n = F^{4:0;0}_{2:1;1}\left[\begin{array}{c} \Delta(2;\alpha), \Delta(2;\beta) : -; -; \\ \Delta(2;\alpha+\beta) \quad : \frac{1}{2}; \frac{1}{2}; \end{array} 4x^2t, t^2\right] - \frac{\alpha\beta t}{(\alpha+\beta)} F^{4:0;0}_{2:1;1}\left[\begin{array}{c} \Delta(2;\alpha+1), \Delta(2;\beta+1) : -; -; \\ \Delta(2;\alpha+\beta+1) \quad : \frac{1}{2}; \frac{3}{2}; \end{array} 4x^2t, t^2\right], \tag{2.3}$$

where $|4x^2t|^{\frac{1}{2}} + |t^2|^{\frac{1}{2}} < 1$

$$\sum_{n=0}^{\infty} G_{2n+1}(\alpha,\beta;x)t^n = -\frac{2x\alpha(\alpha+1)\beta(\beta+1)t}{(\alpha+\beta)(\alpha+\beta+1)}\times$$

$$\times F^{4:0;0}_{2:1;1}\left[\begin{array}{l} \Delta(2;\alpha+2),\Delta(2;\beta+2):-;-; \\ \Delta(2;\alpha+\beta+2) \quad :\tfrac{3}{2};\tfrac{3}{2}; \end{array} 4x^2t, t^2\right]+$$

$$+\frac{2x\alpha\beta}{(\alpha+\beta)}F^{4:0;0}_{2:1;1}\left[\begin{array}{l} \Delta(2;\alpha+1),\Delta(2;\beta+1):-;-; \\ \Delta(2;\alpha+\beta+1) \quad :\tfrac{3}{2};\tfrac{1}{2}; \end{array} 4x^2t, t^2\right], \tag{2.4}$$

where $|4x^2t|^{\frac{1}{2}} + |t^2|^{\frac{1}{2}} < 1$.

3 DERIVATIONS

To prove the generating relations (2.1) and (2.2), we proceed as follows:

$$\sum_{n=0}^{\infty} R_n(\beta,\gamma;x)t^n = \sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n}(\gamma-\beta)_n(2xt)^m(-t^2)^n}{(\gamma)_n m!n!} \tag{3.1}$$

Using series identities (1.12) and (1.13) in equation (3.1), we get

$$\sum_{n=0}^{\infty} R_{2n}(\beta,\gamma;x)t^{2n} + \sum_{n=0}^{\infty} R_{2n+1}(\beta,\gamma;x)t^{2n+1} =$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\beta}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\gamma-\beta}{2})_n(\frac{\gamma-\beta+1}{2})_n(4x^2t^2)^m(t^2)^{2n}}{(\frac{1}{2})_m(m!)(\frac{1}{2})_n(n!)(\frac{\gamma}{2})_n(\frac{\gamma+1}{2})_n} - \frac{\beta(\beta+1)(\gamma-\beta)2xt^3}{\gamma}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\beta+2}{2})_{m+n}(\frac{\beta+3}{2})_{m+n}(\frac{\gamma-\beta+1}{2})_n(\frac{\gamma-\beta+2}{2})_n(4x^2t^2)^m(t^2)^{2n}}{(\frac{3}{2})_m m!(\frac{3}{2})_n n!(\frac{\gamma+1}{2})_n(\frac{\gamma+2}{2})_n} - \frac{\beta(\gamma-\beta)t^2}{\gamma}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\gamma-\beta+1}{2})_n(\frac{\gamma-\beta+2}{2})_n(4x^2t^2)^m(t^2)^{2n}}{(\frac{1}{2})_m m!(\frac{3}{2})_n n!(\frac{\gamma+1}{2})_n(\frac{\gamma+2}{2})_n} + 2x\beta t\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\gamma-\beta}{2})_n(\frac{\gamma-\beta+1}{2})_n(4x^2t^2)^m(t^2)^{2n}}{(\frac{3}{2})_m m!(\frac{1}{2})_n n!(\frac{\gamma}{2})_n(\frac{\gamma+1}{2})_n} \tag{3.2}$$

Put $t = iT$ or $t^2 = -T^2$ in equation (3.2) and equating real and imaginary parts, we get

$$\sum_{n=0}^{\infty} R_{2n}(\beta,\gamma;x)(-T^2)^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\beta}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\gamma-\beta}{2})_n(\frac{\gamma-\beta+1}{2})_n(-4x^2T^2)^m(T)^{4n}}{(\frac{1}{2})_m(m!)(\frac{1}{2})_n(n!)(\frac{\gamma}{2})_n(\frac{\gamma+1}{2})_n}+$$

$$+\frac{\beta(\gamma-\beta)T^2}{\gamma}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\gamma-\beta+1}{2})_n(\frac{\gamma-\beta+2}{2})_n(-4x^2T^2)^m(T)^{4n}}{(\frac{1}{2})_m m!(\frac{3}{2})_n n!(\frac{\gamma+1}{2})_n(\frac{\gamma+2}{2})_n} \tag{3.3}$$

and

$$\sum_{n=0}^{\infty}R_{2n+1}(\beta,\gamma;x)(-T^2)^n=\frac{2\beta(\beta+1)(\gamma-\beta)xT^2}{\gamma}\times$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta+2}{2})_{m+n}(\frac{\beta+3}{2})_{m+n}(\frac{\gamma-\beta+1}{2})_n(\frac{\gamma-\beta+2}{2})_n(-4x^2T^2)^m(T)^{4n}}{(\frac{3}{2})_m m!(\frac{3}{2})_n n!(\frac{\gamma+1}{2})_n(\frac{\gamma+2}{2})_n}+$$

$$+2x\beta\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\gamma-\beta}{2})_n(\frac{\gamma-\beta+1}{2})_n(-4x^2T^2)^m(T)^{4n}}{(\frac{3}{2})_m m!(\frac{1}{2})_n n!(\frac{\gamma}{2})_n(\frac{\gamma+1}{2})_n} \tag{3.4}$$

Put $T=i\sqrt{t}$ or $T^2=-t$ in equations (3.3) and (3.4), we have

$$\sum_{n=0}^{\infty}R_{2n}(\beta,\gamma;x)(t)^n=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\gamma-\beta}{2})_n(\frac{\gamma-\beta+1}{2})_n(4x^2t)^m(t^2)^n}{(\frac{1}{2})_m(m!)(\frac{1}{2})_n(n!)(\frac{\gamma}{2})_n(\frac{\gamma+1}{2})_n}+$$

$$+\frac{\beta(\beta-\gamma)t}{\gamma}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\gamma-\beta+1}{2})_n(\frac{\gamma-\beta+2}{2})_n(4x^2t)^m(t^2)^n}{(\frac{1}{2})_m m!(\frac{3}{2})_n n!(\frac{\gamma+1}{2})_n(\frac{\gamma+2}{2})_n} \tag{3.5}$$

Now applying the definition of Kampé de Fériet's double hypergeometric function (1.11), we get the desired result (2.1).
and

$$\sum_{n=0}^{\infty}R_{2n+1}(\beta,\gamma;x)(t)^n=\frac{2\beta(\beta+1)(\beta-\gamma)tx}{\gamma}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta+2}{2})_{m+n}(\frac{\beta+3}{2})_{m+n}(\frac{\gamma-\beta+1}{2})_n(\frac{\gamma-\beta+2}{2})_n(4x^2t)^m(t^2)^n}{(\frac{3}{2})_m(m!)(\frac{3}{2})_n(n!)(\frac{\gamma+1}{2})_n(\frac{\gamma+2}{2})_n}+$$

$$+2\beta x\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\gamma-\beta}{2})_n(\frac{\gamma-\beta+1}{2})_n(4x^2t)^m(t^2)^n}{(\frac{3}{2})_m m!(\frac{1}{2})_n n!(\frac{\gamma}{2})_n(\frac{\gamma+1}{2})_n} \tag{3.6}$$

Now applying the definition of Kampé de Fériet's double hypergeometric function (1.11), we get the desired result (2.2).
To prove the results (2.3) and (2.4), we proceed as follows:

$$\sum_{n=0}^{\infty}G_n(\alpha,\beta;x)t^n=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}(2xt)^m(-t^2)^n}{(\alpha+\beta)_{m+n}m!n!} \tag{3.7}$$

Using series identities (1.12) and (1.13) in equation (3.7), we get

$$\sum_{n=0}^{\infty} G_{2n}(\alpha,\beta;x)t^{2n}+\sum_{n=0}^{\infty} G_{2n+1}(\alpha,\beta;x)t^{2n+1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha}{2})_{m+n}(\frac{\alpha+1}{2})_{m+n}(\frac{\beta}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(2x)^{2m}(t^2)^m(t^2)^{2n}}{(\frac{\alpha+\beta}{2})_{m+n}(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{1}{2})_m(m!)(\frac{1}{2})_n(n!)} -$$

$$-\frac{2xt\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha+2}{2})_{m+n}(\frac{\alpha+3}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\beta+3}{2})_{m+n}(2x)^{2m}(t^2)^m(t^2)^{2n}(t^2)}{(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{\alpha+\beta+3}{2})_{m+n}(\frac{3}{2})_m m!(\frac{3}{2})_n n!} -$$

$$-\frac{(\alpha)(\beta)t^2}{(\alpha+\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha+1}{2})_{m+n}(\frac{\alpha+2}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(2x)^{2m}(t^2)^m(t^2)^{2n}}{(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{1}{2})_m m!(\frac{3}{2})_n n!} +$$

$$+\frac{2xt(\alpha)(\beta)}{(\alpha+\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha+1}{2})_{m+n}(\frac{\alpha+2}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(2x)^{2m}(t^2)^m(t^2)^{2n}}{(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{3}{2})_m m!(\frac{1}{2})_n n!} \tag{3.8}$$

Put $t = iT$ or $t^2 = -T^2$ in equation (3.8) and equating real and imaginary parts, we get

$$\sum_{n=0}^{\infty} G_{2n}(\alpha,\beta;x)(-T^2)^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha}{2})_{m+n}(\frac{\alpha+1}{2})_{m+n}(\frac{\beta}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(-4x^2T^2)^m(-T^2)^{2n}}{(\frac{\alpha+\beta}{2})_{m+n}(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{1}{2})_m(m!)(\frac{1}{2})_n(n!)} +$$

$$+\frac{(\alpha)(\beta)T^2}{(\alpha+\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha+1}{2})_{m+n}(\frac{\alpha+2}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(-4x^2T^2)^m(-T^2)^{2n}}{(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{1}{2})_m m!(\frac{3}{2})_n n!} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} G_{2n+1}(\alpha,\beta;x)(-T^2)^n =$$

$$= \frac{2\alpha(\alpha+1)\beta(\beta+1)xT^2}{(\alpha+\beta)(\alpha+\beta+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha+2}{2})_{m+n}(\frac{\alpha+3}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\beta+3}{2})_{m+n}(-4x^2T^2)^m(-T^2)^{2n}}{(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{\alpha+\beta+3}{2})_{m+n}(\frac{3}{2})_m(m!)(\frac{3}{2})_n(n!)} +$$

$$+\frac{2x\alpha\beta}{(\alpha+\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha+1}{2})_{m+n}(\frac{\alpha+2}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(-4x^2T^2)^m(-T^2)^{2n}}{(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{3}{2})_m(m!)(\frac{1}{2})_n(n!)} \tag{3.10}$$

Put $T = i\sqrt{t}$ or $T^2 = -t$ in equations (3.9) and (3.10), we have

$$\sum_{n=0}^{\infty} G_{2n}(\alpha,\beta;x)t^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{\alpha}{2})_{m+n}(\frac{\alpha+1}{2})_{m+n}(\frac{\beta}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(4x^2t)^m(t^2)^n}{(\frac{\alpha+\beta}{2})_{m+n}(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{1}{2})_m(m!)(\frac{1}{2})_n(n!)} +$$

$$+\frac{\alpha\beta(-t)}{(\alpha+\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\alpha+1}{2})_{m+n}(\frac{\alpha+2}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(4x^2t)^m(t^2)^n}{(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{1}{2})_m m!(\frac{3}{2})_n n!} \tag{3.11}$$

Now applying the definition of Kampé de Fériet's double hypergeometric function (1.11), we get the desired result (2.3).
and

$$\sum_{n=0}^{\infty} G_{2n+1}(\alpha,\beta;x)t^n =$$

$$= -\frac{2\alpha(\alpha+1)\beta(\beta+1)xt}{(\alpha+\beta)(\alpha+\beta+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\alpha+2}{2})_{m+n}(\frac{\alpha+3}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(\frac{\beta+3}{2})_{m+n}(4x^2t)^m(t^2)^n}{(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{\alpha+\beta+3}{2})_{m+n}(\frac{3}{2})_m(m!)(\frac{3}{2})_n(n!)}+$$

$$+\frac{2x\alpha\beta}{(\alpha+\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{\alpha+1}{2})_{m+n}(\frac{\alpha+2}{2})_{m+n}(\frac{\beta+1}{2})_{m+n}(\frac{\beta+2}{2})_{m+n}(4x^2t)^m(t^2)^n}{(\frac{\alpha+\beta+1}{2})_{m+n}(\frac{\alpha+\beta+2}{2})_{m+n}(\frac{3}{2})_m(m!)(\frac{1}{2})_n(n!)} \tag{3.12}$$

Now applying the definition of Kampé de Fériet's double hypergeometric function (1.11), we get the desired result (2.4).
We conclude our present investigation, by observing that several other linear generating relations can be obtained from known generating relations, in analogous manner.

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