SOME RESULTS ON W_1- CURVATURE TENSOR ON $(k, \mu)-$ CONTACT SPACE FORMS

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Abstract

The object of the present paper is to study $(k, \mu)-$contact space forms satisfying certain curvature tensor. We also study $\xi - W_1-$projectively flat, W_1-projectively flat and $(k, \mu)-$contact space forms satisfying $\widehat{J}.S = 0$ and $Q.\widehat{J} = 0$. Finally, we studied $\phi-W_1-$ semisymmetric $(k, \mu)-$contact space form.

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1 Introduction

In [4], Blair, Koufogiorgos and Papantoniou introduced $(k, \mu)-$ contact metric manifolds. A class of contact metric manifolds with contact metric structure (ϕ, ξ, η, g) in which the curvature tensor R satisfies the condition

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y),$$

for all $X, Y \in TM$ is called $(k, \mu)-$ contact metric manifolds.

The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a $\phi-$sectional curvature. If the $(k, \mu)-$contact metric manifold M has constant $\phi-$sectional curvature c, then it is called a $(k, \mu)-$ contact space form and is denoted by M(c). $(k, \mu)-$ contact space forms have been studied by K. Arslan, R. Ezentas, I. Mihai, C. Murthan and Özgür, C. [2] and A. Akbar and A. Sarkar [1] and many others.

The W_1- curvature tensor is important tensor from the differential geometric point of view. Let M be a $(2n + 1)-$dimensional Riemannian manifold. M is said to be locally W_1- flat for $n \geq 1$, if and only if the W_1- curvature tensor $\widehat{J}$ vanishes, which is defined by

$$\widehat{J}(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(X, Z)Y - S(Y, Z)X], \tag{1.1}$$

for all $X, Y, Z \in TM$, where R is the curvature tensor and S is the Ricci tensor.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g). Since at each point $p \in M$ the tangent space T_PM can be decomposed into direct sum $T_PM = \phi(T_PM\ ) \oplus \{\xi_P\}$, where $\{\xi_P\}$ is the $1-$ dimensional linear subspace of T_PM generated by $\{\xi_P\}$, the conformal curvature tensor C is a map

$$C : T_PM \times T_PM \times T_PM \longrightarrow \phi(T_PM) \oplus \{\xi_P\}, p \in M.$$

It may be natural to consider the following particular cases:

(1) the projection of the image of C in $\phi(T_PM)$ is zero;

(2) the the projection of the image of C in $\{\xi_P\}$ is zero;

(3) the projection of the image of $C|_{\phi(T_PM)\times\phi(T_PM)\times\phi(T_PM)}$ in $\phi(T_PM)$ is zero.

An almost contact metric manifold satisfying the case (1), (2), and (3) is said to be conformally symmetric [18], $\xi-$conformally flat [19], and $\phi-$conformally flat [7] respectively. In an anlogous way, we define $\xi - W_1-\ flat\ (k,\mu)-$contact space forms.

Definition 1.1. A contact metric manifold is called $W_1-\ flat$ if the manifold satisfies $\widehat{J}(X,Y)\xi = 0$ for all vector fields X,Y.

As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor satisfies

$$R(X,Y).R = 0,$$

where $R(X,Y)Z$ is considered as a field of linear operators acting on R.

A natural extension of such curvature conditions from curvature conditions of pseudosymmetry type. The condition $Q \cdot R = 0$ have been studied by $Verstraelen$ et al. in [15] .

In this paper, we characterize $(k,\mu)-$ contact space forms $Q \cdot P = 0$.

Motivated by the above studied, in this paper we characterize a $(k,\mu)-$contact space form satisfying certain curvature conditions on the W_1- curvature tensor. The paper is organized as follows:

In section 2, we give necessary details about $(k,\mu)-$contact space forms. In section 3, we study W_1- flat $(k,\mu)-$contact space forms. Section 4 deals with the study of $(k,\mu)-$contact space forms satisfying $\widehat{J}.S = 0$. In section 5, $\xi - W_1-$ flat $(k,\mu)-$contact space forms have been studied. Section 6, we study $(k,\mu)-$contact space forms satisfying $Q.\widehat{J} = 0$. Finally, we study $\phi - W_1-$ semisymmetric $(k,\mu)-$contact space form.

2 Preliminaries

A $(2n+1)-$ dimensional differential manifold M is called an almost contact manifold [3] if there is an almost contact structure (ϕ,ξ,η) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$ form η satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta o \phi = 0 \tag{2.1}$$

An almost contact stucture is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$.

The condition for being normal is equivalent to vanishing of the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is,

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.3}$$

or equivalently,

$$g(X, \xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

for all $X, Y \in TM$.

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.5}$$

for all $X, Y \in TM$.

Given a contact metric manifold $M(\phi, \xi, \eta, g)$, we define a $(1, 1)$ tensor field h by $h = \frac{1}{2}L_\xi \phi$ where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\xi = 0, h\phi + \phi h = 0, \tag{2.6}$$

$$\bigtriangledown \xi = -\phi - \phi h, trace(h) = trace(\phi h) = 0, \tag{2.7}$$

where $\bigtriangledown$ is the Levi-Civita connection.

A contact metric manifold is said to be an $\eta-$Einstein manifold if

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.8}$$

where a, b are smooth functions and $X, Y \in TM$, S is the Ricci tensor.

A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\bigtriangledown_X \phi) = g(X, Y)\xi - \eta(Y)X. \tag{2.9}$$

On a Sasakian manifold the following relation holds

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

for all $X, Y \in TM$.

Blair, Koufogiorgos and Papantoniou [4] considered the $(k, \mu)-$ nullity condition and gave several reasons for studying it. The $(k, \mu)-$ nullity distribution $N(k, \mu)$ [4] of a contact metric manifold M is defined by

$$N(k, \mu) : p \to N_P(k, \mu) = [U \in T_pM \mid R(X, Y)U = (kI + \mu h)(g(Y, U)X - g(X, U)Y)],$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$.

A contact metric manifold M with $\xi \in N(k,\mu)$ is called a $(k,\mu)-$contact metric manifold. Then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.11}$$

for all $X, Y \in TM$. For $(k,\mu)-$contact metric manifolds, it follows that $h^2 = (k-1)\phi^2$. This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for a $(k,\mu)-$contact metric manifold, the condition of being Sasakian manifold, $K-$contact manifold, $k = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the $(k,\mu)-$nullity distribution $N(k,\mu)$ is reduced to $k-$nullity distribution $N(k)$ [12]. If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)-$contact metric manifold.

The sectional curvature $K(X,\phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a $\phi-$sectional curvature. If the $(k,\mu)-$contact metric manifold M has constant $\phi-$sectional curvature c, then it is called a $(k,\mu)-$contact space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [14]

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] \\ &+\frac{c-1}{4}[2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y \\ &-g(Y,\phi Z)\phi X] + \frac{c+3-4k}{4}[\eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ &+\frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY \\ &-g(\phi hY,Z)\phi hX + g(\phi Y,\phi Z)hX - g(\phi X,\phi Z)hY \\ &+g(hX,Z)\phi^2 Y - g(hY,Z)\phi^2 X] + \mu[\eta(Y)\eta(Z)hX \\ &-\eta(X)\eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi] \end{aligned} \tag{2.12}$$

for all $X, Y, Z \in T(M)$, where $c + 2k = -1 = k - \mu$ if $k < 1$.

From (2.12), we obtain for $(k,\mu)-$contact space forms:

$$\begin{aligned} R(X,Y)\phi Z &= \frac{c+3}{4}[g(Y,\phi Z)X - g(X,\phi Z)Y] \\ &+\frac{c-1}{4}[-2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi - g(X,Z)\phi Y \\ &+\eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X] \\ &+\frac{c+3-4k}{4}[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi] \\ &+\frac{1}{2}[g(hY,\phi Z)hX - g(hX,\phi Z)hY + g(hX,Z)\phi hY \\ &-g(hY,Z)\phi hX - g(\phi Y,Z)hX + g(\phi X,Z)hY \\ &-g(hX,\phi Z)Y + g(hX,\phi Z)\eta(Y)\xi - g(hY,\phi Z)X \\ &-g(hY,\phi Z)\eta(X)\xi] + \mu[g(hY,\phi Z)\eta(X)\xi - g(hX,\phi Z)\eta(Y)\xi], \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
\phi R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)\phi X - g(X,Z)\phi Y] \\
&+\frac{c-1}{4}[-2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi \\
&-g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X \\
&-g(Y,\phi Z)\eta(X)\xi] \\
&+\frac{c+3-4k}{4}[\eta(Z)\eta(X)\phi Y - \eta(Y)\eta(Z)\phi X] \\
&+\frac{1}{2}[g(hY,Z)\phi hX - g(hX,Z)\phi hY - g(\phi hX,Z)hY \\
&+g(\phi hY,Z)hX + g(\phi Y,\phi Z)\phi hX - g(\phi X,\phi Z)\phi hY \\
&-g(hX,Z)\phi Y + g(hY,Z)\phi X] \\
&+\mu[\eta(Y)\eta(Z)\phi hX - \eta(Z)\eta(X)\phi hY], \qquad (2.14)
\end{aligned}
$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \qquad (2.15)$$

$$R(X,\xi)\xi = k[X - \eta(X)\xi] + \mu hX, \qquad (2.16)$$

$$R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + \mu[g(hY,Z)\xi - \eta(Z)hY], \qquad (2.17)$$

$$
\begin{aligned}
S(Y,Z) &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,Z) \\
&+\frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)]\eta(Y)\eta(Z) \\
&+[2n-2+\mu]g(hY,Z), \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
S(Y,hZ) &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,hZ) \\
&+(k-1)[2n-2+\mu]g(Y,Z) \\
&-(k-1)[2n-2+\mu]\eta(Y)\eta(Z), \qquad (2.19)
\end{aligned}
$$

$$S(Y,\xi) = 2nk\eta(Y), \qquad (2.20)$$

$$S(\xi,\xi) = 2nk, \qquad (2.21)$$

$$QY = \frac{1}{2}[c(n+1)+3(n-1)+2k]Y + \frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]\eta(Y)\xi + [2n-2+\mu]hY, \tag{2.22}$$

$$Q\xi = 2nk\xi \tag{2.23}$$

Definition 2.1. The $M-$projectively curvature tensor $\widehat{J}$ of type (1,3) on $(k,\mu)-$contact metric form M of dimension (2n+1) is defined as

$$\widehat{J}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(X,Z)Y - S(Y,Z)X] \tag{2.24}$$

for any vector field X,Y,Z on M. The manifold is called W_1- flat if $\widehat{J}$ vanishes identically on M.

From (2.24) using (2.15), (2.18), (2.20), (2.21), (2.22) and (2.23) we have

$$\widehat{J}(X,Y)\xi = 2k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{2.25}$$

$$\widehat{J}(\xi,Y)\xi = 2k[\eta(Y)\xi - Y] - \mu hY \tag{2.26}$$

$$\widehat{J}(\xi,Y)Z = k[g(Y,Z)\xi - 2\eta(Z)Y] + \mu[g(hY,Z)\xi - \eta(Z)hY] + \frac{1}{2n}S(Y,Z)\xi \tag{2.27}$$

$$\widehat{J}(\xi,Y)hZ = kg(Y,hZ)\xi + \mu g(hY,hZ)\xi + \frac{1}{2n}S(Y,hZ)\xi \tag{2.28}$$

3 W_1- flat $(k,\mu)-$contact space forms

Theorem 3.1. . *A (2n+1)-dimensional W_1- flat $(k,\mu)-$contact space form is an $\eta-$Einstein manifold.*

Proof. From the definition of W_1- flat $(k,\mu)-$contact space forms we have

$$\widehat{J}(X,Y)Z = 0$$

Applying this in (2.24), we obtain

$$R(X,Y)Z - \frac{1}{2n}[S(X,Z)Y - S(Y,Z)X] = 0 \tag{3.1}$$

Taking the inner product with V of (3.1), we obtain

$$g(R(X,Y)Z,V) = \frac{1}{2n}[S(X,Z)g(Y,V) - S(Y,Z)g(X,V)] \tag{3.2}$$

Putting $X = V = \xi$ in (3.2) and using (2.17), (2.18), (2.19), (2.20) and (2.21), we have

$$g(hY,Z) = 2\frac{k}{\mu}\eta(Y)\eta(Z) - \frac{k}{\mu}g(Y,Z) - \frac{1}{2n\mu}S(Y,Z) \tag{3.3}$$

By using (3.3) in (2.18), we get

$$S(Y,Z) = a_1 g(Y,Z) + b_1\eta(Y)\eta(Z) \tag{3.4}$$

where

$$a_1 = \frac{2n\{\frac{1}{2}[c(n+1) + 3(n-1) + 2k] - k(2n-2+\mu)]\}}{4n-2+\mu}$$

and

$$b_1 = \frac{2n\{\frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] + 2k(2n-2+\mu)]\}}{4n-2+\mu}$$

□

4 $(k,\mu)-$contact space forms satisfying $\widehat{J}.S = 0$

Theorem 4.1. *A (2n+1)-dimensional $(k,\mu)-$contact space forms satisfying $\widehat{J}.S = 0$ is an $\eta-$Einstein manifold.*

Proof. Let M(c) be a (2n+1)-dimensional $(k,\mu)-$contact space forms satisfying $\widehat{J}.S = 0$ which implies that

$$S(\widehat{J}(X,Y)U,V) + S(U,\widehat{J}(X,Y)V) = 0 \tag{4.1}$$

By putting $U = X = \xi$, we get

$$S(\widehat{J}(\xi,Y)\xi,V) + S(\xi,\widehat{J}(\xi,Y)V) = 0 \tag{4.2}$$

By using (2.18), (2.19), (2.20) and (2.24), we obtain

$$g(hY,Z) = A_1 g(Y,V) + B_1\eta(Y)\eta(V) \tag{4.3}$$

where,

$$c_1 = \frac{\frac{1}{2}(k+\mu h)[c(n+1) + 3(n-1) + 2k] - 2nk^2}{2nk\mu - (k+\mu h)(2n-2+\mu)}$$

and

$$d_1 = \frac{\frac{1}{2}(k+\mu h)[-c(n+1) - 3(n-1) + 2k(2n-1)] - 2nk^2}{2nk\mu - (k+\mu h)(2n-2+\mu)}$$

By using (4.3) in (2.18), we get

$$S(Y,V)=A_2g(Y,V)+B_2\eta(Y)\eta(V)$$

where

$$A_2=\frac{1}{2}[c(n+1)+3(n-1)+2k]+(2n-2+\mu)A_1$$

and

$$B_2=\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]+(2n-2+\mu)B_1$$

□

5 $\xi-W_1-$ flat $(k,\mu)-$contact space forms

Theorem 5.1. *Let $M(c)$ be a $\xi-W_1-$ flat $(k,\mu)-$contact space forms. Then M(c) is either a Sasakian space form or a N(k)- contact space form.*

Proof. Assume that M(c) is a $\xi-W_1$ $-$flat $(k,\mu)-$contact space form. Then

$$\widehat{J}(X,Y)\xi=0 \tag{5.1}$$

putting $Z=\xi$ in (1.1), we obtain

$$\widehat{J}(X,Y)\xi=R(X,Y)\xi-\frac{1}{2n}[S(X,\xi)Y-S(Y,\xi)X] \tag{5.2}$$

Using (2.11) and (2.20), we get

$$2k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]=0 \tag{5.3}$$

From (5.3), we may conclude that if $k=0$ then either $\mu=0$ or

$$\eta(Y)hX-\eta(X)hY=0 \tag{5.4}$$

Putting $Y=\xi$ in above equation , we have

$$hX=0$$

If $\mu=0$, then M(c) is a N(k)- contact space form.
If $h=0$, then M(c) is a Sasakian space form. □

6 $(k,\mu)-$Contact Space Forms Satisfying $Q.\widehat{J}=0$

Theorem 6.1. *A $(k,\mu)-$Contact Space Forms Satisfying $Q.\widehat{J}=0$ is either $(0,1)-$contact space form of constant $\phi-$sec tional curvature -1 or N(k)- contact space form or, a Sasakian space form.*

Proof. A $(k,\mu)-$contact space forms satisfying $Q.\widehat{J}=0$, where Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$. Suppoese M(c) be a $(k,\mu)-$ contact space form satisfying $Q.\widehat{J}=0$. Then

$$Q(\widehat{J}(X,Y)Z)-\widehat{J}(QX,Y)Z-\widehat{J}(X,QY)Z-\widehat{J}(X,Y)QZ=0 \tag{6.1}$$

Putting $Z=\xi$ in (6.1) and using (2.25), we have

$$\begin{aligned}&\mu\eta(Y)[Q(hX)-hQX]-\mu\eta(X)[Q(hY)-hQY]+\\&2k[\eta(QX)Y-\eta(QY)X]+\mu[\eta(QX)hY-\eta(QY)hX]-\\&4nk^2[\eta(Y)X-\eta(X)Y]-2nk\mu[\eta(Y)hX-\eta(X)hY]=0\end{aligned} \tag{6.2}$$

Using (2.22), we obtain

$$\begin{aligned}Q(hY)-hQY &= \frac{1}{2}[c(n+1)+3(n-1)+2k]hY\\&+\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]\eta(hY)\xi\\&+[2n-2+\mu]h^2Y-\frac{1}{2}[c(n+1)+3(n-1)+2k]hY\\&-\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]\\&\eta(Y)h\xi-[2n-2+\mu]h^2Y=0\end{aligned} \tag{6.3}$$

and

$$\eta(QX)Y-\eta(QY)X=2nk[\eta(X)Y-\eta(Y)X] \tag{6.4}$$

and

$$\eta(QX)hY-\eta(QY)hX=2nk[\eta(X)hY-\eta(Y)hX] \tag{6.5}$$

Using (6.3), (6.4) and (6.5) in (6.2), we have

$$2k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]=0 \tag{6.6}$$

From (6.6), we may conclude that if $k=0$ then either $\mu=0$ or

$$[\eta(Y)hX-\eta(X)hY]=0 \tag{6.7}$$

Putting $Y=\xi$ in the above equation yields

$$hX=0$$

If $k=0$, then from (2.12) we have $\mu=1$ and constant $\phi-\sec tional$ curvature $c=-1$.
If $\mu=0$, then M(c) is a N(k)- contact space form.
If $h=0$, then M(c) is a Sasakian space form. □

7 $\phi - W_1-$ Semisymmetric $(k,\mu)-$contact space forms

Definition 7.1. A $(k,\mu)-$contact space form is said to be $\phi - W_1-$ semisymmetric if $\widehat{J}(X,Y)\cdot\phi = 0$ for all $X, Y \in TM$.

Proposition 7.1. *Let M(c) be a $\phi - W_1-$ semisymmetric $(k,\mu)-$contact space form, then $\mu = 2(1-n)$.*

Proof. Suppose M(c) be a $\phi - W_1-$ semisymmetric $(k,\mu)-$contact space form. Then

$$\widehat{J}(X,Y)\phi Z - \phi(\widehat{J}(X,Y)Z) = 0 \tag{7.1}$$

From (1.1), it follows that

$$\widehat{J}(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{2n}[S(X,\phi Z)Y - S(Y,\phi Z)X \tag{7.2}$$

Using (2.18) in (7.2), we get

$$\begin{aligned}\widehat{J}(X,Y)\phi Z &= R(X,Y)\phi Z - \\ &\frac{1}{2n}\{\frac{1}{2}[c(n+1)+3(n-1)+ \\ &2k][g(X,\phi Z)Y - g(Y,\phi Z)X] + [2n-2 \\ &+\mu][g(hX,\phi Z)Y - g(hY,\phi Z)X\}\end{aligned} \tag{7.3}$$

Again,

$$\begin{aligned}\phi(\widehat{J}(X,Y)Z) &= \phi R(X,Y)Z \\ &-\frac{1}{2n}\{[c(n+1)+3(n-1)+2k][g(X,Z)\phi Y \\ &-g(Y,Z)\phi X] + \frac{1}{2}[-c(n+1)-3(n-1)+ \\ &2k(2n-1)][\eta(Z)\eta(X)\phi Y - \eta(Y)\eta(Z)\phi X] + \\ &[2n-2+\mu][g(hX,Z)\phi Y - g(hY,Z)\phi X]\}\end{aligned} \tag{7.4}$$

Using (7.3) and (7.4) in (7.2), we have

$$\begin{aligned}(\widehat{J}(X,Y)\cdot\phi)Z &= R(X,Y)\phi Z - \phi R(X,Y)Z - \\ &\frac{1}{2n}\{\frac{1}{2}[c(n+1)+3(n-1)+2k][g(X,\phi Z)Y \\ &-g(Y,\phi Z)X - g(X,Z)\phi Y + g(Y,Z)\phi X] + \\ &\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)] \\ &[\eta(Y)\eta(Z)\phi X - \eta(Z)\eta(X)\phi Y] + [2n-2+\mu] \\ &[g(hX,\phi Z)Y - g(hY,\phi Z)X - g(hX,Z)\phi Y + g(hY,Z)\phi X]\}\end{aligned} \tag{7.5}$$

Putting the vcalue of $R(X,Y)\phi Z$ and $\phi R(X,Y)Z$ in (7.5) and taking inner product with W of (7.5) and contracting Y and W, we obtain

$$\begin{aligned} &\{\frac{c+3}{4}(1-2n)+\frac{c-1}{4}(2n-1)-\frac{1}{4n}(2n+1)\\ &[c(n+1)+3(n-1)+2k]\}g(\phi Z,X)+\\ &\{\frac{[2n-2+\mu]}{2n}(2n+1)\}g(\phi Z,hX)=0 \end{aligned} \tag{7.6}$$

Putting $X=hX$ in the above equation yields

$$\begin{aligned} &\{\frac{c+3}{4}(1-2n)+\frac{c-1}{4}(2n-1)-\frac{1}{4n}(2n+1)\\ &[c(n+1)+3(n-1)+2k]\}g(\phi Z,hX)+\\ &\{\frac{[2n-2+\mu]}{2n}(2n+1)\}g(\phi Z,h^2X)=0 \end{aligned} \tag{7.7}$$

Taking trace in both sides of (7.7) and using $trace(h)=0$, we get

$$\mu=2(1-n)$$

□

From the above proposition we can state the following:

Theorem 7.1. *A three dimensional $\phi-W_1-$ semisymmetric $(k,\mu)-$contact space form reduces to an $N(k)-$contact space form.*

8 References

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