On the rotation of a linear combination of harmonic univalent functions

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Abstract

In this paper, a rotation by μ of certain linear combination of two univalent harmonic mappings convex along real axis is studied. Under certain conditions it is shown that this rotation is univalent and sense preserving in the open unit disk $\mathbb{U}$ and convex in the direction of $\overline{\mu}$.

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1 Introduction and preliminaries

Let $\mathcal{H}$ denotes a class of complex-valued functions $f = u + iv$ which are harmonic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real-valued harmonic functions in $\mathbb{U}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{U}$, called the analytic and co-analytic parts of f, respectively. The Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$.

According to the Lewy [6], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{U}$ if and only if $J_f(z) > 0$ in $\mathbb{U}$ which is equivalent to the existence of an analytic function $\omega_f(z) = g'(z)/h'(z)$ in $\mathbb{U}$ such that

$$|\omega_f(z)| < 1 \quad \text{for all} \quad z \in \mathbb{U}. \tag{1.1}$$

The function ω_f is called the dilation of f. For detail study refer [3].

A class of all univalent, sense preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $S_{\mathcal{H}}$. If a function $f = h + \overline{g} \in S_{\mathcal{H}}$, then h and g are of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (|b_1| < 1; z \in \mathbb{U}). \tag{1.2}$$

A subclass of functions $f = h + \overline{g} \in S_{\mathcal{H}}$ with the condition $g'(0) = 0$ (or $\omega_f(0) = 0$) is denoted by $S^0_{\mathcal{H}}$. Further, the subclasses of functions f in $S_{\mathcal{H}}$ $\left(S^0_{\mathcal{H}}\right)$ are denoted by $K_{\mathcal{H}}$ $\left(K^0_{\mathcal{H}}\right)$ if f map the unit disk $\mathbb{U}$ onto a convex region.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction $\gamma \in [0, \pi)$, if for all $t \in \mathbb{C}$, the set $\Omega \cap \{t + re^{i\gamma} : r \in \mathbb{R}\}$ is either connected or empty. A function is said to be convex in the direction γ if it maps $\mathbb{U}$ univalently onto a domain convex in the direction γ.

In 1984, Clunie and Sheil-Small [1] introduced a method, known as shear construction or shearing, for constructing a univalent harmonic mapping from a related conformal map. According to this method of constructing a harmonic univalent map convex in a given direction, a generalized result may be given as follows:

Lemma 1.1. *A locally univalent harmonic function $f = h + \overline{g}$ in $\mathbb{U}$ is a univalent harmonic mapping of $\mathbb{U}$ onto a domain convex in a direction φ if and only if $h - e^{2i\varphi}g$ is a univalent analytic mapping of $\mathbb{U}$ onto a domain convex in the direction φ.*

Dorff and Rolf [2] applied another way of constructing a univalent harmonic map by taking two suitable harmonic maps f_1 and f_2 with same dilations, whose linear combination $f = tf_1 + (1 - t)f_2,\ t \in [0, 1]$ is univalent and convex in the direction of the imaginary axis. Wang et al. [10] derived several sufficient conditions on harmonic univalent functions f_1 and f_2 so that their linear combination $f = tf_1 + (1 - t)f_2,\ t \in [0, 1]$, is univalent and convex in the direction of the real axis. More results on the linear combination f of f_1 and f_2 may also be found in [4, 5, 8, 9, 10] etc. (also see the references cited in these).

We need following Lemmas in proving our results.

Lemma 1.2. *[7]Let f be analytic function in $\mathbb{U}$ with $f(0) = 0$ and $f'(0) \neq 0$. Suppose also that*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})} \quad (\theta \in \mathbb{R}; z \in \mathbb{U}). \tag{1.3}$$

If

$$\Re\left(\frac{zf'(z)}{\varphi(z)}\right) > 0 \quad (z \in \mathbb{U}), \tag{1.4}$$

then f is convex in the direction of real axis.

Lemma 1.3. *[1] Let $\Omega \subset \mathbb{C}$ be a domain convex in the direction of the real axis. Also let p be a real-valued continuous function in Ω. Then the mapping $\omega \mapsto \omega + p(\omega)$ is univalent in Ω if and only if it is locally univalent. If it is univalent, then its range is convex in the direction of the real axis.*

In this paper, we consider a rotation of certain linear combination of two univalent harmonic mappings which are convex along real axis and show that under certain conditions this rotation is univalent, sense preserving in $\mathbb{U}$ and convex in the direction of $\overline{\mu}$.

Let $\mathcal{A}$ denotes a class of functions ϕ analytic in $\mathbb{U}$ such that $\phi(0) = 0$ and $\phi'(0) = 1$.

Definition 1.1 (Rotation by μ)**.** The rotations of f and ϕ by μ $(\mu \in \mathbb{C}, |\mu| = 1)$, denoted by f^μ and ϕ^μ are given, respectively, by

$$f^\mu(z) = \overline{\mu} f(\mu z) \quad \text{and} \quad \phi^\mu(z) = \overline{\mu}\phi(\mu z).$$

If $\phi \in \mathcal{A}$ is convex in the direction of the real axis, then ϕ^μ is convex in the direction of $\overline{\mu}$.

Proposition 1.1. *Let $f = h + \overline{g} \in S_{\mathcal{H}}$ with*

$$h(z) - g(z) = (1+a)\phi(z), \tag{1.5}$$

where $\phi \in \mathcal{A}$ is convex in the direction of the real axis. Then $f^{\mu} = H + \overline{G} \in S_{\mathcal{H}}$, where $H(z) = \overline{\mu}h(\mu z)$ and $G(z) = \mu g(\mu z)$ are such that $H(z) - \overline{\mu}^2 G(z) = (1+a)\phi^{\mu}(z)$ and is convex in the direction of $\overline{\mu}$.

Proof. Let $f = h + \overline{g} \in S_{\mathcal{H}}$. Then

$$f^{\mu}(z) = \overline{\mu} f(\mu z) = \overline{\mu} h(\mu z) + \overline{\mu g(\mu z)} = H + \overline{G}$$

and by (1.5) we have

$$H(z) - \overline{\mu}^2 G(z) = (1+a)\phi^{\mu}(z)$$

which is convex in the direction of $\overline{\mu}$, hence, by shearing (Lemma 1.1), it generates a harmonic function $f^{\mu} = H + \overline{G}$ which is convex in the direction of $\overline{\mu}$. Further, we have

$$\begin{aligned} |\omega_{f^{\mu}}(z)| &= \left| \frac{G'(z)}{H'(z)} \right| \\ &= \left| \frac{\mu^2 g'(\mu z)}{h'(\mu z)} \right| = \left| \frac{g'(\mu z)}{h'(\mu z)} \right| = |\omega_f(\mu z)| < 1 \end{aligned}$$

This proves the proposition. □

Remark 1.1. In Proposition 1.1, harmonic functions f and f^{μ} belong to the class $S^0_{\mathcal{H}}$ if $a = 0$.

In particular, taking $\mu = e^{i\gamma}$ and $\phi(z) = \frac{z}{1-z}$, Proposition 1.1 may be stated as follows:

Corollary 1.1. *Let $f = h + \overline{g} \in S_{\mathcal{H}}$ and let $\gamma \in [0, \pi)$. Suppose for $a \in (-1, 1)$,*

$$h(z) - g(z) = \frac{(1+a)z}{1-z} \quad (z \in \mathbb{U})$$

is convex in the direction of the real axis. Then $e^{-i\gamma} f(e^{i\gamma} z) =: f_{\gamma} = \mathcal{H} + \overline{\mathcal{G}} \in S_{\mathcal{H}}$, where $\mathcal{H}(z) = e^{-i\gamma} h(e^{i\gamma} z)$ and $\mathcal{G}(z) = e^{i\gamma} h(e^{i\gamma} z)$ are such that $\mathcal{H}(z) - e^{-2i\gamma}\mathcal{G}(z) = \frac{(1+a)z}{1-e^{i\gamma}z}$ and is convex in the direction of $e^{-i\gamma}$.

Theorem 1.1. *Let for $j = 1, 2, f_j = h_j + \overline{g_j} \in S_{\mathcal{H}}$ with*

$$h_j(z) - g_j(z) = (1+a_j)\phi(z) \quad (a_j \in (-1,1), \phi \in \mathcal{A}). \tag{1.6}$$

Let $f = tf_1 + (1-t)f_2,\ t \in [0,1]$. If $\omega_{f_1}(z) = \omega_{f_2}(z)$ and ϕ satisfy the condition $\Re\left(\frac{z\phi'(z)}{\varphi(z)}\right) > 0$ for some function $\varphi(z)$ given by (1.3), then for $\mu \in \mathbb{C}(|\mu| = 1), f^{\mu} \in S_{\mathcal{H}}$ and is convex in the direction of $\overline{\mu}$.

Proof. From Lemma 1.2, the condition $\Re\left(\frac{z\phi'(z)}{\varphi(z)}\right) > 0$ for some function $\varphi(z)$ given by (1.3) implies that ϕ is convex in the direction of real axis. Let $f = tf_1 + (1-t)f_2 =: h + \overline{g}$, where

$$h = th_1 + (1-t)h_2 \quad \text{and} \quad g = tg_1 + (1-t)g_2 \tag{1.7}$$

are such that

$$\begin{aligned} h - g &= t\left(h_1 - g_1\right) + (1-t)\left(h_2 - g_2\right) \\ &=: (1+a)\phi \end{aligned} \tag{1.8}$$

is convex in the direction of real axis with

$$a = ta_1 + (1-t)a_2 \in (-1, 1). \tag{1.9}$$

To show $f = h + \overline{g} \in S_{\mathcal{H}}$, we only need to show $|\omega_f(z)| < 1$. Since, for $j = 1, 2, \omega_{f_j}(z) = \frac{g_j'(z)}{h_j'(z)}$, we get when $\omega_{f_1}(z) = \omega_{f_2}(z)$

$$\begin{aligned} |\omega_f(z)| &= \left|\frac{g'(z)}{h'(z)}\right| = \left|\frac{tg_1'(z) + (1-t)g_2'(z)}{th_1'(z) + (1-t)h_2'(z)}\right| \\ &= \left|\frac{t\omega_{f_1}(z)h_1'(z) + (1-t)\omega_{f_2}(z)h_2'(z)}{th_1'(z) + (1-t)h_2'(z)}\right| \\ &= |\omega_{f_1}(z)| < 1. \end{aligned} \tag{1.10}$$

Hence, on applying Proposition 1.1 for $f = h + \overline{g}$, with the condition (1.8), we get the result. □

A generalization of Theorem 1.1 may be given as follows:

Corollary 1.2. *Let for $j = 1, 2, ..., n, f_j = h_j + \overline{g_j} \in S_{\mathcal{H}}$ with the condition (1.6). Let $F = \sum_{j=1}^{n} t_j f_j,\ t_j \in [0,1]$ such that $\sum_{j=1}^{n} t_j = 1$. If $\omega_{f_1}(z) = \omega_{f_2}(z) = ... = \omega_{f_n}(z)$ and ϕ satisfy the condition $\Re\left(\frac{z\phi'(z)}{\varphi(z)}\right) > 0$ for some function $\varphi(z)$ given by (1.3), then for $\mu \in \mathbb{C}(|\mu| = 1), F^{\mu} \in S_{\mathcal{H}}$ and is convex in the direction of $\overline{\mu}$.*

Remark 1.2. Result of Corollary 1.2 may also be obtained by choosing certain special forms of $\phi(z)$, some of these are as follows:

(i) $\phi(z) = \frac{z}{1-z}$

(ii) $\phi(z) = \frac{1}{2}\log\frac{1+z}{1-z}$

(iii) $\phi(z) = A\log\frac{1+z}{1-z} + B\ \frac{z}{1+cz+z^2} \quad (A > 0, B > 0, c \in [-2, 2])$

(iv) $\phi(z) = \int_0^z \frac{d\zeta}{\left(1+ze^{i\theta}\right)\left(1+ze^{-i\theta}\right)} \quad (\theta \in \mathbb{R})$.

Theorem 1.2. *Let for* $j = 1, 2, f_j = h_j + \overline{g_j} \in S_{\mathcal{H}}$ *be convex in the direction of the real axis. Let* $f = tf_1 + (1-t)f_2,\ t \in [0,1]$. *If* $\Re\left\{(1 - \omega_{f_1}\overline{\omega_{f_2}})\, h_1'\overline{h_2'}\right\} \geq 0$, *then for* $\mu \in \mathbb{C}(|\mu| = 1), f^\mu \in S_{\mathcal{H}}$ *and is convex in the direction of* $\overline{\mu}$.

Proof. Similar to Theorem 1.1, we have $f = tf_1 + (1-t)f_2 =: h + \overline{g}$, where h and g are given by (1.7). To show $f = h + \overline{g} \in S_{\mathcal{H}}$, we need to show that $|\omega_f(z)| < 1$. From (1.10), we have

$$|\omega_f| = \left|\frac{t\omega_{f_1}h_1' + (1-t)\omega_{f_2}h_2'}{th_1' + (1-t)h_2'}\right|.$$

Under the conditions: $\left|\omega_{f_j}\right| < 1$ for each $j = 1, 2$ and $\Re\left\{(1 - \omega_{f_1}\overline{\omega_{f_2}})\, h_1'\overline{h_2'}\right\} \geq 0$, we get

$$\begin{aligned}
& \left|th_1' + (1-t)h_2'\right|^2 - \left|t\omega_{f_1}h_1' + (1-t)\omega_{f_2}h_2'\right|^2 \\
= \ & \left(th_1' + (1-t)h_2'\right)\left(t\overline{h_1'} + (1-t)\overline{h_2'}\right) \\
& - \left(t\omega_{f_1}h_1' + (1-t)\omega_{f_2}h_2'\right)\left(t\overline{\omega_{f_1}h_1'} + (1-t)\overline{\omega_{f_2}h_2'}\right) \\
= \ & t^2\left(1 - |\omega_{f_1}|^2\right)\left|h_1'\right|^2 + (1-t)^2\left(1 - |\omega_{f_2}|^2\right)\left|h_2'\right|^2 \\
& + 2t(1-t)\Re\left\{(1 - \omega_{f_1}\overline{\omega_{f_2}})\, h_1'\overline{h_2'}\right\} \\
> \ & 0
\end{aligned}$$

which proves $|\omega_f(z)| < 1$. Now we show that $f = h + \overline{g}$ is convex in the direction of the real axis. For this we use Lemma 1.3. By Lemma 1.1, $F_j := h_j - g_j$ is univalent in $\mathbb{U}$ and $\Omega_j = F_j(\mathbb{U})$ is convex in the direction of the real axis for each $j = 1, 2$. We may write $f_j = F_j + 2\Re(g_j)$ and

$$f_j\left(F_j^{-1}(w)\right) = w + 2\Re\left(g_j\left(F_j^{-1}(w)\right)\right) = w + q_j(w)$$

is univalent in Ω_j for each $j = 1, 2$, where $q_j(w)$ is real valued continuous function. Let $F := h - g$ and $\Omega = F(\mathbb{U})$. Then

$$\begin{aligned}
f\left(F^{-1}(w)\right) &= tf_1\left(F_1^{-1}(w)\right) + (1-t)\, f_2\left(F_2^{-1}(w)\right) \\
&= t\left(w + q_1(w)\right) + (1-t)\left(w + q_2(w)\right) \\
&= w + \left(tq_1(w) + (1-t)\, q_2(w)\right) \\
&= w + q(w)
\end{aligned}$$

is univalent in Ω which by Lemma 1.3 is convex in the direction of the real axis. Hence, on applying Proposition 1.1 to the function $f = h + \overline{g} \in S_{\mathcal{H}}$ which is convex in the direction of the real axis, we get the result. □

Theorem 1.3. *Let for* $j = 1, 2, f_j = h_j + \overline{g_j} \in S_{\mathcal{H}}$ *with the condition that* $h_j - g_j$ *is convex in the direction of the real axis and is given by (1.6). Let* $f = tf_1 + (1-t)f_2,\ t \in [0,1]$. *Then for* $\mu \in \mathbb{C}(|\mu| = 1), f^\mu \in S_{\mathcal{H}}$ *and is convex in the direction of* $\overline{\mu}$.

Proof. Similar to Theorem 1.1, we have $f = tf_1 + (1-t)f_2 =: h + \overline{g}$, where h and g given by (1.7). To show $f = h + \overline{g} \in S_{\mathcal{H}}$, we only need to show $|\omega_f(z)| < 1$. From (1.10), we have

$$\omega_f = \frac{t\omega_{f_1} h_1' + (1-t)\omega_{f_2} h_2'}{th_1' + (1-t)h_2'},$$

where from (1.6), for each $j = 1, 2$,

$$h_j' = \frac{(1+a_j)\phi'}{1-\omega_{f_j}}. \tag{1.11}$$

We have

$$\left|\omega_{f_j}\right| < 1 \Leftrightarrow \Re\left(\frac{1+\omega_{f_j}}{1-\omega_{f_j}}\right) > 0 \quad \text{in} \quad \mathbb{U}. \tag{1.12}$$

Thus

$$\begin{aligned}
\omega_f &= \frac{\frac{t(1+a_1)\phi'\omega_{f_1}}{1-\omega_{f_1}} + \frac{(1-t)(1+a_2)\phi'\omega_{f_2}}{1-\omega_{f_2}}}{\frac{t(1+a_1)\phi'}{1-\omega_{f_1}} + \frac{(1-t)(1+a_2)\phi'}{1-\omega_{f_2}}} \\
&= \frac{t(1+a_1)\omega_{f_1}\left(1-\omega_{f_2}\right) + (1-t)(1+a_2)\omega_{f_2}\left(1-\omega_{f_1}\right)}{t(1+a_1)\left(1-\omega_{f_2}\right) + (1-t)(1+a_2)\left(1-\omega_{f_1}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1+\omega_f}{1-\omega_f} &= \frac{1 + \frac{t(1+a_1)\omega_{f_1}\left(1-\omega_{f_2}\right)+(1-t)(1+a_2)\omega_{f_2}\left(1-\omega_{f_1}\right)}{t(1+a_1)\left(1-\omega_{f_2}\right)+(1-t)(1+a_2)\left(1-\omega_{f_1}\right)}}{1 - \frac{t(1+a_1)\omega_{f_1}\left(1-\omega_{f_2}\right)+(1-t)(1+a_2)\omega_{f_2}\left(1-\omega_{f_1}\right)}{t(1+a_1)\left(1-\omega_{f_2}\right)+(1-t)(1+a_2)\left(1-\omega_{f_1}\right)}} \\
&= \frac{t(1+a_1)\left(1+\omega_{f_1}\right)\left(1-\omega_{f_2}\right) + (1-t)(1+a_2)\left(1-\omega_{f_1}\right)\left(1+\omega_{f_2}\right)}{t(1+a_1)\left(1-\omega_{f_1}\right)\left(1-\omega_{f_2}\right) + (1-t)(1+a_2)\left(1-\omega_{f_1}\right)\left(1-\omega_{f_2}\right)} \\
&= \frac{t(1+a_1)\left(1+\omega_{f_1}\right)\left(1-\omega_{f_2}\right) + (1-t)(1+a_2)\left(1-\omega_{f_1}\right)\left(1+\omega_{f_2}\right)}{(1+a)\left(1-\omega_{f_1}\right)\left(1-\omega_{f_2}\right)} \\
&= \frac{1}{1+a}\left[\frac{t(1+a_1)\left(1+\omega_{f_1}\right)}{1-\omega_{f_1}} + \frac{(1-t)(1+a_2)\left(1+\omega_{f_2}\right)}{1-\omega_{f_2}}\right].
\end{aligned}$$

Hence, on using (1.12), we get

$$\begin{aligned}
\Re\left(\frac{1+\omega_f}{1-\omega_f}\right) &= \frac{1}{1+a}\left[t(1+a_1)\Re\left(\frac{1+\omega_{f_1}}{1-\omega_{f_1}}\right) + (1-t)(1+a_2)\Re\left(\frac{1+\omega_{f_2}}{1-\omega_{f_2}}\right)\right] \\
&> 0
\end{aligned}$$

which proves that $|\omega_f(z)| < 1$ and hence, on applying Proposition 1.1 for $f = h + \overline{g}$, with the condition (1.8), we get the result. □

Theorem 1.4. *Let for* $j=1,2, f_j = h_j + \overline{g_j} \in S_{\mathcal{H}}$ *and let* $f = tf_1 + (1-t)f_2,\ t \in [0,1]$. *If*

$$h_j(z) + g_j(z) = \frac{(1+a)z}{1-z} \quad (a \in (-1,1); z \in \mathbb{U}). \tag{1.13}$$

then for $\mu \in \mathbb{C}(|\mu| = 1), f^{\mu} \in S_{\mathcal{H}}$ *and is convex in the direction of* $\overline{\mu}$.

Proof. Similar to Theorem 1.1, we have $f = tf_1 + (1-t)f_2 =: h + \overline{g}$, where h and g are given by (1.7). From (1.13), for each $j = 1,2$, we have

$$\begin{aligned} h_j' - g_j' &= \left(h_j' + g_j'\right)\left(\frac{h_j' - g_j'}{h_j' + g_j'}\right) \\ &= \frac{1+a}{(1-z)^2}\left(\frac{1-\omega_{f_j}}{1+\omega_{f_j}}\right) \end{aligned}$$

which follows by choosing $\varphi(z) = \frac{z}{(1-z)^2}$ in Lemma 1.2, and by using (1.7) and (1.12) that

$$\begin{aligned} \Re\left(\frac{z(h'-g')}{\varphi(z)}\right) &= \Re\left(\frac{z}{\varphi(z)}\left[t\left(h_1' - g_1'\right) + (1-t)\left(h_2' - g_2'\right)\right]\right) \\ &= t\Re\left((1-z)^2\left(h_1' - g_1'\right)\right) + (1-t)\,\Re\left((1-z)^2\left(h_2' - g_2'\right)\right) \\ &= (1+a)\left[t\Re\left(\frac{1-\omega_{f_1}}{1+\omega_{f_1}}\right) + (1-t)\,\Re\left(\frac{1-\omega_{f_2}}{1+\omega_{f_2}}\right)\right] \\ &> 0. \end{aligned}$$

This proves that $h - g$ is convex in the direction of real axis and hence, by Lemma 1.1 $f = h + \overline{g}$ is convex in the direction of real axis. Now to show $f = h + \overline{g} \in S_{\mathcal{H}}$, we only need to show $|\omega_f(z)| < 1$. From (1.10), we have

$$\omega_f = \frac{t\omega_{f_1}h_1' + (1-t)\omega_{f_2}h_2'}{th_1' + (1-t)h_2'}.$$

From (1.13), for each $j = 1,2$,

$$h_j' = \frac{1+a}{\left(1+\omega_{f_j}\right)(1-z^2)} \tag{1.14}$$

which follows that

$$\begin{aligned} |\omega_f| &= \left|\frac{\frac{t\omega_{f_1}}{1+\omega_{f_1}} + \frac{(1-t)\omega_{f_2}}{1+\omega_{f_2}}}{\frac{t}{1+\omega_{f_1}} + \frac{1-t}{1+\omega_{f_2}}}\right| \\ &= \left|\frac{t\omega_{f_1} + (1-t)\omega_{f_2} + \omega_{f_1}\omega_{f_2}}{1 + t\omega_{f_2} + (1-t)\omega_{f_1}}\right|. \end{aligned}$$

Following the method adopted in [10], we let for each $j=1,2, \omega_{f_j}=\rho_j e^{i\theta_j}$ $(0\leq \rho_j<1, \theta_j\in\mathbb{R})$ and

$$\begin{aligned}
\Psi(t) &= \left|1+t\omega_{f_2}+(1-t)\omega_{f_1}\right|^2-\left|t\omega_{f_1}+(1-t)\omega_{f_2}+\omega_{f_1}\omega_{f_2}\right|^2\\
&= \left|1+t\rho_2\cos\theta_2+(1-t)\rho_1\cos\theta_1+i\left\{t\rho_2\sin\theta_2+(1-t)\rho_1\sin\theta_1\right\}\right|^2\\
&\quad -\left|t\rho_1\cos\theta_1+(1-t)\rho_2\cos\theta_2+\rho_1\rho_2\cos\left(\theta_1+\theta_2\right)\right.\\
&\quad \left.+i\left\{t\rho_1\sin\theta_1+(1-t)\rho_2\sin\theta_2+\rho_1\rho_2\sin\left(\theta_1+\theta_2\right)\right\}\right|^2\\
&= \left[1+t\rho_2\cos\theta_2+(1-t)\rho_1\cos\theta_1\right]^2+\left[t\rho_2\sin\theta_2+(1-t)\rho_1\sin\theta_1\right]^2\\
&\quad -\Big[\left\{t\rho_1\cos\theta_1+(1-t)\rho_2\cos\theta_2+\rho_1\rho_2\cos\left(\theta_1+\theta_2\right)\right\}^2\\
&\quad +\left\{t\rho_1\sin\theta_1+(1-t)\rho_2\sin\theta_2+\rho_1\rho_2\sin\left(\theta_1+\theta_2\right)\right\}^2\Big]\\
&= 1+t^2\rho_2^2+(1-t)^2\rho_1^2+2t(1-t)\rho_1\rho_2\cos\left(\theta_1-\theta_2\right)\\
&\quad +2t\rho_2\cos\theta_2+2(1-t)\rho_1\cos\theta_1-\left[t^2\rho_1^2+(1-t)^2\rho_2^2+\rho_1^2\rho_2^2\right.\\
&\quad \left.+2t(1-t)\rho_1\rho_2\cos\left(\theta_1-\theta_2\right)+2(1-t)\rho_1\rho_2^2\cos\theta_1+2t\rho_1^2\rho_2\cos\theta_2\right]\\
&= 2t\left[\rho_2^2-\rho_1^2+\rho_2\left(1-\rho_1^2\right)\cos\theta_2-\rho_1\left(1-\rho_2^2\right)\cos\theta_1\right]\\
&\quad +\left(1-\rho_2^2\right)\left(1+\rho_1^2+2\rho_1\cos\theta_1\right)
\end{aligned}$$

which is a linear polynomial function of $t\in[0,1]$. Since,

$$\begin{aligned}
\Psi(0) &= \left(1-\rho_2^2\right)\left(1+\rho_1^2+2\rho_1\cos\theta_1\right)\\
&= \left(1-\rho_2^2\right)\left\{\left(\rho_1+\cos\theta_1\right)^2+\sin^2\theta_1\right\}>0
\end{aligned}$$

and

$$\Psi(1)=\left(1-\rho_1^2\right)\left\{\left(\rho_2+\cos\theta_2\right)^2+\sin^2\theta_2\right\}>0,$$

we get $\Psi(t)>0$ for each $t\in[0,1]$. This proves that $|\omega_f|<1$ and hence, on applying Proposition 1.1 for $f=h+\overline{g}$, we get the result. □

Similar to Theorem 3, instead of right half-plane mapping $\frac{(1+a)z}{1-z}$, we may take asymmetric vertical strip mapping $\frac{1+a}{2i\sin\theta}\ \log\left(\frac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right)$ and prove following theorem.

Theorem 1.5. *Let for* $j=1,2, f_j=h_j+\overline{g_j}\in S_{\mathcal{H}}$ *and let* $f=tf_1+(1-t)f_2,\ t\in[0,1]$. *If*

$$h_j(z)+g_j(z)=\frac{1+a}{2i\sin\theta}\ \log\left(\frac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right)\ \left(a\in(-1,1),\theta\in(0,\pi)\right). \tag{1.15}$$

then for $\mu\in\mathbb{C}(|\mu|=1), f^{\mu}\in S_{\mathcal{H}}$ *and is convex in the direction of* $\overline{\mu}$.

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