# A Note on Extension of Nonuniform Wavelet Bessel Sequences to Dual Wavelet Frames in $L^2(\mathbb{R})$

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**Abstract**

Pair of dual frames in a separable Hilbert space are flexible tool to express any element in the space in simple manner. Christensen, Kim, and Kim gave sufficient conditions for extending any pair of Bessel sequence having wavelet structure into pair of dual frames maintaining the same structure. In this paper, we generalize some results by Christensen et al. to nonuniform wavelet Bessel sequences which gives nonuniform dual frames. Sufficient conditions for nonuniform dual frames in terms of support of Fourier transform of window functions are also given.

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## 1 Introduction

Orthonormal basis are building blocks of any separable hilbert space as it allows us to represent any element of the space in simple and convenient way. Wavelets in $L^2(\mathbb{R})$ are also particular type of orthonormal basis which consist of dilation and translation of some finite number of elements in $L^2(\mathbb{R})$. Mallat[12] introduced the Multiresolution Analysis technique for construction of wavelets in $L^2(\mathbb{R})$, it reduced the complexity for finding wavelets. Recently Gabardo and Nashed [5] generalized the concept of MRA which is based on spectral theory.

**Definition 1.1.** *[5, Definition 3.1] Let $N \geq 1$ be a positive integer and $r$ be an odd integer relatively prime to $N$ such that $1 \leq r \leq 2N-1$, an associated nonuniform multiresolution analysis (abbreviated NUMRA) is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:*

*(i)* $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$,

*(ii)* $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$,

*(iii)* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

*(iv)* $f(x) \in V_j$ *if and only if* $f(2Nx) \in V_{j+1}$,

*(v) There exists a function* $\phi \in V_0$, *called the* scaling function, *such that the collection* $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$, *where* $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$, *is a complete orthonormal system for* $V_0$.

In above definition, the translate set $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$ may not be a group. Frames can be considered as generalization of the concept of orthonormal bases that provides series representation (not necessarily unique) of each vector in the Hilbert space. Manchanda and Sharma[4] studied the frame properties of nonuniform wavelets and obtained necessary and sufficient conditions for nonuniform wavelet frames in $L^2(\mathbb{R})$.

As we know all Bessel sequences are not frames, so extension of Bessel sequence into frames is one of the important problem in frame theory, since frames allow us to get more detail information about each element in the space. Practically tight frames are the most convenient frames to deal with, since if $\{f_k\}$ be tight frame for Hilbert space $\mathcal{H}$ with bound $B$ then any $f \in \mathcal{H}$, we can write $f = \sum \langle f, \frac{f_k}{B} \rangle f_k$, where $\{\frac{f_k}{B}\}$ is the dual frame for $\{f_k\}$. Casazza and Leonhard[2] proved that any Bessel sequence can be extended to tight frame in finite dimensional Hilbert space which was further extended in infinite dimensional space by Li and Sun[13], but the major question was that whether any Bessel sequence having some specific structure and bound maintains the same structure and bound after extension to tight frame or not? Unfortunately the answer is not. Han [14] found out there are Bessel sequences having wavelet structure with bound $B$, that can't be extended into tight frame maintaining the same structure and bound. In such cases instead of extending Bessel sequence into tight frame we try to extend pair of Bessel sequences into pair of dual frames, since in application working with pair of dual frames is as convenient as working with tight frames. In this paper, we extend some results of Christensen, Kim, and Kim to nonuniform wavelet structure. Also, sufficient conditions for nonuniform dual frames in terms of support of Fourier transform of window functions have been obtained.

## 2 Preliminaries

In the entire paper, we assume $\Lambda = \{0, \frac{r}{N}\} + 2\mathbb{Z}$, where $N \in \mathbb{N}, r$ be an odd integer coprime with $N$ such that $1 \leq r \leq 2N - 1. \mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ denote the set of integers, positive integers and real numbers respectively. First we recall some basic definations. Let $\mathcal{H}$ be a separable hilbert space. A sequence $\{g_k\}_{k\in\mathbb{Z}}$ is called frame for $\mathcal{H}$, if there exits positive constants $A, B$ such that for any $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, g_k \rangle|^2 \leq B\|f\|^2.$$

If we can choose $A = B$,then $\{g_k\}$ is called tight frame and if upper inequality is satisfied, then it is called Bessel sequence with bound $B$. If $\{f_k\}$ and $\{g_k\}$ be frames for Hilbert space $\mathcal{H}$ such that for any $f \in H$ we have

$$f = \sum_{k\in\mathbb{Z}} \langle f, f_k \rangle g_k.$$

Then, $\{f_k\}$ and $\{g_k\}$ called pair of dual frames. Let $\{f_k\}_{\in\mathbb{Z}}$ be frame for Hilbert space $\mathcal{H}$, then the following are important bounded operators associated with $\{f_k\}_{\in\mathbb{Z}}$

$$\begin{aligned}
&T : \ell^2(\mathbb{Z}) \to \mathcal{H}, \quad T\{c_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} c_k f_k \quad \text{(Preframe Operator)},\\
&T^* : \mathcal{H} \to \ell^2(\mathbb{Z}), \quad T^* f = \{\langle f, f_k\rangle\}_{k\in\mathbb{Z}} \quad \text{(Analysis Operator)},\\
&S : TT^* : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k \quad \text{(Frame Operator)}.
\end{aligned}$$

Also we consider the following operators on $L^2(\mathbb{R})$

$$\begin{aligned}
&T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_a f(\gamma) = f(\gamma - a) \quad \text{(Translation by } a\text{)},\\
&L : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad Lf(\gamma) = \sqrt{2N} f(2N\gamma) \quad \text{(N-Dilation operator)}.
\end{aligned}$$

The $j$ fold $N$-dilation, where $j \in \mathbb{Z}$, is given by

$$L^j f(\gamma) = (2N)^{\frac{j}{2}} f((2N)^j \gamma).$$

The Fourier transform of a function $f$ is denoted by $\mathcal{F}f$ or $\hat{f}$, and defined as

$$\mathcal{F}f = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\gamma} dx.$$

For $N \in \mathbb{N}$, $j \in \mathbb{Z}$ and $a \in \mathbb{R}$, we have the following properties.

(*i*) $L^j : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary map.

(*ii*) $L^j T_a = T_{(2N)^{-j}a} L^j$.

(*iii*) $\mathcal{F}L^j = L^{-j}\mathcal{F}$.

(*iv*) $\mathcal{F}T_a = E_{-a}\mathcal{F}$.

**Definition 2.1.** *Let $\{\psi_1, \psi_2, \ldots, \psi_n\} \subset L^2(\mathbb{R})$ be a finite set, where $\psi_\ell \neq 0$, $1 \leq \ell \leq n$. The family*

$$\{L^j T_\lambda \psi_\ell\}_{\substack{j\in\mathbb{Z},\lambda\in\Lambda\\ \ell=1,2,\ldots,n}} = \left\{(2N)^{\frac{j}{2}} \psi_1 (2N)^j \gamma - \lambda\right\}_{j\in\mathbb{Z},\lambda\in\Lambda} \bigcup \cdots \bigcup \left\{(2N)^{\frac{j}{2}} \psi_n (2N)^j \gamma - \lambda\right\}_{j\in\mathbb{Z},\lambda\in\Lambda}$$

*is called nonuniform wavelet system in $L^2(\mathbb{R})$. If this system forms a frame (or Bessel sequence), then it is called nonuniform wavelet frame (or nonuniform wavelet Bessel sequence).*

## 3 Main Results

We begin with a characterization for dual frames in terms of Bessel sequences in separable Hilbert spaces which can be found in [8].

**Lemma 3.1.** *[8, Lemma 5.6.2] Let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be Bessel sequences in a Hilbert space $\mathcal{H}$. Then, the following conditions are equivalent.*

*(i)* $f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle g_k$ *, for any* $f \in \mathcal{H}$

*(ii)* $\{f_k\}_{k\in\mathbb{Z}}$ *and* $\{g_k\}_{k\in\mathbb{Z}}$ *are dual frames for* $\mathcal{H}$

By using techniques developed in [9, Lemma 4.1], the following theorem gives sufficient conditions for dual nonuniform wavelet frames.

**Theorem 3.1.** *Let* $\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *,*$\{L^jT_\lambda\tilde{\psi_1}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *are nonuniform Bessel sequences in* $L^2(\mathbb{R})$ *with preframe operator* $T$ *and* $U$ *respectively. Assume there exits* $\phi \in L^2(\mathbb{R})$ *satisfying*

*(i)* $\{L^jT_\lambda\phi\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *is nonuniform wavelet frame with dual* $\{L^jT_\lambda\tilde{\phi}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$.

*(ii)* $TU^*T_\lambda\phi = T_\lambda TU^*\phi$ *for any* $\lambda \in \Lambda$.

*Then there exists nonuniform wavelet systems* $\{L^jT_\lambda\psi_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *and* $\{L^jT_\lambda\tilde{\psi_2}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *in* $L^2(\mathbb{R})$ *such that*

$$\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \cup \{L^jT_\lambda\psi_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \text{ and } \{L^jT_\lambda\tilde{\psi_1}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \cup \{L^jT_\lambda\tilde{\psi_2}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$$

*form dual nonuniform wavelet frames for* $L^2(\mathbb{R})$.

*Proof.* Let $f \in L^2(\mathbb{R})$ be arbitrary and $I$ be identity operator in $L^2(\mathbb{R})$. Consider the bounded operator $\Omega = I - UT^*$. Using hypothesis $(i)$, we can write

$$\Omega f = \sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \langle \Omega f, L^jT_\lambda\phi\rangle L^jT_\lambda\tilde{\phi}.$$

For any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} f &= UT^*f + \Omega f \\ &= \sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \langle f, L^jT_\lambda\psi_1\rangle L^jT_\lambda\tilde{\psi_1} + \sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \langle \Omega f, L^jT_\lambda\phi\rangle L^jT_\lambda\tilde{\phi} \\ &= \sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \langle f, L^jT_\lambda\psi_1\rangle L^jT_\lambda\tilde{\psi_1} + \sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \langle f, \Omega^*L^jT_\lambda\phi\rangle L^jT_\lambda\tilde{\phi}. \qquad (3.1) \end{aligned}$$

Since $\{L^jT_\lambda\phi\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ is frame for $L^2(\mathbb{R})$ and $\Omega$ is bounded operator acting on $L^2(\mathbb{R})$, there exists constant $C$ such that for any $f\in L^2(\mathbb{R})$,

$$\sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} |\langle f,\Omega^*L^jT_\lambda\phi\rangle|^2 \le \sum_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} |\langle \Omega f, L^jT_\lambda\phi\rangle|^2 \le C\|f\|^2.$$

Thus, $\{\Omega^*L^jT_\lambda\phi\}_{j\in\mathbb{Z}),\lambda\in\Lambda}$ is a Bessel sequence. By using $L^jTU^*=TU^*L^j$ $(j\in\mathbb{Z})$ and $(ii)$, we compute

$$\begin{aligned}
\Omega^*L^jT_\lambda\phi &= (I-TU^*)L^jT_\lambda\phi\\
&= L^jT_\lambda\phi - TU^*L^jT_\lambda\phi\\
&= L^jT_\lambda\phi - L^jTU^*T_\lambda\phi\\
&= L^jT_\lambda\phi - L^jT_\lambda TU^*\phi\\
&= L^jT_\lambda(I-TU^*)\phi\\
&= L^jT_\lambda\Omega^*\phi. \qquad (3.2)
\end{aligned}$$

Choose $\psi_2=\Omega^*\phi$ and $\tilde{\psi_2}=\tilde{\phi}$. Then, (3.1), (3.2) and Theorem 3.1 completes the proof. □

The next result extends [9, Theorem 4.2] for the construction of nonuniform wavelet frames from Bessel sequences. First we give some notations: For $\Delta\subseteq L^2(\mathbb{R})$, we write

$$S(\Delta)=\overline{Span}\{T_{\pm\lambda}\phi:\phi\in\Delta,\lambda\in\Lambda\}.$$

Note that

$$S(\{L^jT_\lambda\psi: j\le 0,\lambda\in\Lambda\})=\overline{Span}\{T_{\pm\lambda'}L^jT_\lambda\psi: j\le 0 \ \&\ \lambda,\lambda'\in\Lambda\}.$$

**Theorem 3.2.** *Let* $\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$, $\{L^jT_\lambda\tilde{\psi_1}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *are nonuniform Bessel sequences in* $L^2(\mathbb{R})$, *assume that there exists* $\phi\in L^2(\mathbb{R})$ *such that*

- $(i)$ $\{L^jT_\lambda\phi\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *is nonuniform wavelet frame with dual* $\{L^jT_\lambda\tilde{\phi}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$
- $(ii)$ $\phi\in L^2(\mathbb{R})\ominus S(\{L^jT_\lambda\tilde{\psi_1}: j\le 0,\lambda\in\Lambda\})$

*Then there exists nonuniform wavelet system* $\{L^jT_\lambda\psi_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *and* $\{L^jT_\lambda\tilde{\psi_2}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *in* $L^2(\mathbb{R})$ *such that*

$$\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}\cup\{L^jT_\lambda\psi_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \text{ and } \{L^jT_\lambda\tilde{\psi_1}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}\cup\{L^jT_\lambda\tilde{\psi_2}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$$

*form nonuniform dual wavelet frames for* $L^2(\mathbb{R})$.

*Proof.* Let $T$ and $U$ be preframe operator for given Bessel sequences $\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$, $\{L^jT_\lambda\tilde{\psi}_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ respectively. Using Theorem (3.1) only we need to show $TU^*T_\lambda\phi = T_\lambda TU^*\phi$.

By using $(ii)$, we have

$$\langle \phi, T_{\pm\lambda'}L^jT_\lambda\tilde{\psi}_1\rangle = 0 \quad \text{for } j \leq 0 \text{ and } \lambda, \lambda' \in \Lambda.$$

Therefore

$$\begin{aligned} TU^*T_\lambda\phi &= \sum_{\substack{j'\in\mathbb{Z}\\ \lambda'\in\Lambda}} \langle T_\lambda\phi, L^{j'}T_{\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{\lambda'}\psi_1 \\ &= \sum_{\substack{j'\in\mathbb{Z}\\ \lambda'\in\Lambda}} \langle \phi, T_{-\lambda}L^{j'}T_{\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{\lambda'}\psi_1 \\ &= \sum_{\substack{j'>0\\ \lambda'\in\Lambda}} \langle \phi, T_{-\lambda}L^{j'}T_{\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{\lambda'}\psi_1. \end{aligned}$$

Using $T_\lambda L^j = L^jT_{(2N)^j\lambda}$, we compute

$$\begin{aligned} TU^*T_\lambda\phi = T_\lambda T_{-\lambda} &\sum_{\substack{j'>0\\ \lambda'\in\Lambda}} \langle \phi, T_{-\lambda}L^{j'}T_{\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{\lambda'}\psi_1 \\ = T_\lambda &\sum_{\substack{j'>0\\ \lambda'\in\Lambda}} \langle \phi, L^{j'}T_{-(2N)^{j'}\lambda+\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{-(2N)^{j'}\lambda+\lambda'}\psi_1 \\ = T_\lambda &\sum_{\substack{j'>0\\ \lambda'\in\Lambda}} \langle \phi, L^{j'}T_{\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{\lambda'}\psi_1 \\ = T_\lambda &\sum_{\substack{j'\in\mathbb{Z}\\ \lambda'\in\Lambda}} \langle \phi, L^{j'}T_{\lambda'}\tilde{\psi}_1\rangle L^{j'}T_{\lambda'}\psi_1 \\ = T_\lambda &TU^*\phi. \end{aligned}$$

This concludes the proof. □

The following theorem gives sufficient conditions for nonuniform dual frames in terms of support of Fourier transform of window functions.

**Theorem 3.3.** *Let* $\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$, $\{L^jT_\lambda\tilde{\psi}_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *are nonuniform Bessel sequences in* $L^2(\mathbb{R})$*, assume that there exists* $\phi \in L^2(\mathbb{R})$ *such that*

*(i)* $\{L^jT_\lambda\phi\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *is nonuniform wavelet frame with dual* $\{L^jT_\lambda\tilde{\phi}\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$.

*(ii)* *Supp* $\hat{\phi} \subseteq \mathbb{R}\setminus[-1,1]$ *and Supp* $\hat{\tilde{\psi}}_1 \subseteq [-1,1]$.

*Then, there exists nonuniform wavelet systems* $\{L^jT_\lambda\psi_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *and* $\{L^jT_\lambda\tilde{\psi}_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$ *in* $L^2(\mathbb{R})$ *such that*

$$\{L^jT_\lambda\psi_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \cup \{L^jT_\lambda\psi_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \text{ and } \{L^jT_\lambda\tilde{\psi}_1\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} \cup \{L^jT_\lambda\tilde{\psi}_2\}_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}}$$

*form dual nonuniform wavelet frames for* $L^2(\mathbb{R})$.

*Proof.* Using Theorem(3.2), only we need to show that

$$\phi \in L^2(\mathbb{R}) \ominus S(\{L^jT_\lambda\tilde{\psi}_1 : j \leq 0, \lambda \in \Lambda\}).$$

For any $\lambda', \lambda \in \Lambda$ and $j \in \mathbb{Z}$, we have

$$\begin{aligned} \widehat{T_{\pm\lambda'}L^jT_\lambda}\tilde{\psi}_1(\bullet) &= E_{\mp\lambda'}L^{-j}E_{-\lambda}\hat{\tilde{\psi}}_1(\bullet) \\ &= (2N)^{-\frac{j}{2}}e^{\mp 2\pi i\lambda'(\bullet)}e^{-2\pi i\lambda(\frac{\bullet}{(2N)^j})}\hat{\tilde{\psi}}_1(\frac{\bullet}{(2N)^j}). \end{aligned}$$

Since $Supp\,\hat{\tilde{\psi}}_1 \subseteq [-1,1]$, for $\lambda', \lambda \in \Lambda$ and $j < 0$, we have

$$Supp\,\widehat{T_{\pm\lambda'}L^jT_\lambda}\tilde{\psi}_1(\bullet) \subseteq Supp\,\hat{\tilde{\psi}}_1(\frac{\bullet}{(2N)^j}) \subseteq [-\frac{1}{2N}, \frac{1}{2N}].$$

Also for $\lambda', \lambda \in \Lambda$ and $j = 0$, $\text{Supp}\,\widehat{T_{\pm\lambda'}L^jT_\lambda}\tilde{\psi}_1(\bullet) \subseteq Supp\,\hat{\tilde{\psi}}_1(\bullet) \subseteq [-1,1]$. Therefore, for $j \leq 0$ and $\lambda', \lambda \in \Lambda$, we have

$$\langle \phi, T_{\pm\lambda'}L^jT_\lambda\tilde{\psi}_1\rangle = \langle \widehat{\phi}, \widehat{T_{\pm\lambda'}L^jT_\lambda}\tilde{\psi}_1\rangle = 0.$$

This gives

$$\phi \in L^2(\mathbb{R}) \ominus S(\{L^jT_\lambda\tilde{\psi}_1 : j \leq 0, \lambda \in \Lambda\}).$$

This proves the result. □

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