Bi-Univalent function Associated with Fractional derivative operator

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Abstract

The purpose of the present paper is to introduce a new subclass of the function class $\sum$ of bi-univalent functions defined in the open unit disk, which is associated with the fractional derivative operator. Furthermore, we obtain estimates on the Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$ for the functions belonging to this new subclasses.

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1 Introduction

Let A denote the class of function f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U\{z \in C : |z| < 1\}$ and satisfy the normalized condition $f(0) = f'(0) - 1 = 0$. Further, by S we shall denote the class of all function in A which are univalent in U.

Following definition of fractional derivative and fractional integrals are due to Owa [7] and Srivastava and Owa [12]

Definition 1.1. *The fractional integral of order λ is defined for a function $f(z)$ of the form* (1.1)

$$D_z^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\varsigma)d\varsigma}{(z-\varsigma)^{1-\lambda}}, \tag{1.2}$$

where $\lambda > 0$, $f(z)$ in an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\varsigma)^{1-\lambda}$ is removed by requiring $\log(z-\varsigma)$ *to be real when $(z-\varsigma) > 0$.*

Definition 1.2. *The fractional derivative of order λ is defined for a function $f(z)$ of the form* (1.1) *by*

$$D^{=}\lambda_0 f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{d^z} \int_0^z \frac{f(\varsigma)d\varsigma}{(z-\varsigma)^\lambda} d\varsigma \tag{1.3}$$

where $0 \leq \lambda < 1,\ f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $z - \varsigma)^{-\lambda}$ is removed as in Definition 1.1 given above.

Definition 1.3. *Under the hypothesis of Definition 1.2, the fractional derivative of order $n + \lambda$ in defined for function $f(z)$ by*

$$D_z^{n+\lambda} = \frac{d^n}{dz^n} f(z) \tag{1.4}$$

where $0 \leq \lambda < 1$ and $n \in N_0 - \{0, 1, 2, 3, ...\}$.

For f of the form (1.1)*, using the Definition 1.2 and Definition 1.3, Dixit and Porwal [5] introduced a fractional derivative operator as*

$$\begin{aligned} &\Omega^0\lambda(f) = f(z) \\ &\Omega_\lambda^1 f(z) = \Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}D_z^{1+\lambda}f(z) = \Omega_\lambda f(z) \\ &\quad - -- \\ &\Omega_\lambda^n f(z) = \Omega_\lambda(\Omega_\lambda^{n-1} f(z), \end{aligned}$$

one can easily obtain

$$\Omega_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k$$

where

$$\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(()k-\lambda}.$$

It is worthy to note that for $\lambda = 0, \pi^n f(z)$ reduces to familiar Salagean opearator defined by Salagean [10].

For $n \in N_o,\ 0 \leq \beta < 1,\ \mu \geq 0$ and $0 \leq \lambda < 1$, we introduce the subclass $Q(n, \mu, \lambda, \beta)$of S of function of the form (1.1)*satisfying the condition.*

$$Re\left(\frac{(1-\mu)\Omega_\lambda^n f(z) + \mu\Omega_\lambda^{n+1} f(z)}{z}\right) > \beta, z \in U \tag{1.5}$$

where Ω_λ^n stands for fractional derivative operator introduced by Dixit and Porwal [5]. For $n = 0$ and $\lambda = 0$ it reduces to the class $Q_\mu(\beta)$ studied by Ding et al. [4], (see also [1, 2, 6, 13]).

It is known that every $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, (z \in U)$$

and

$$f^{-1}(f(w)) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}() = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4.$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let $\sum$ denote the class of bi-univalent function (1.1) *in U. One may refers Srivastava et al [11] and reference there in for some basic results.*

Recently, several researchers such as [9, 11] found the coefficients $|a_2|, |a_3|$ of bi-univalent class $\sum$. Motivated with their work, we introduce a new subclass of the function class $\sum$ and obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for function in there new subclass of class $\sum$. We have employed the techniques used earlier by Srivastava et al. [11] and Porwal and Darus [9].

In this paper, we required the following Lemma due to [8].

Lemma 1.1. *If $h \in P$ then $|c_k| \leq 2$ for each k, where P is the family of all function h analytic in U for which $Re\{h(z)\} > 0$,*

$$h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ...,\ \text{for } z \in U.$$

2 Coefficient bounds for the function class $B_\Sigma(n, \mu, \lambda, \alpha)$

Definition 2.1. *A function $f(z)$ given by* (1.1) *is said to be in the class $B_\Sigma(n, \mu, \lambda, \alpha)$ if the following condition are satisfied*

$$(2.1) \quad f \in \sum \ \textit{and} \left| \arg \left(\frac{(1-\mu)\Omega_\lambda^n f(z) + \mu\Omega_\lambda^{n+1} f(z)}{z} \right) < \frac{2\pi}{z} \right.$$
$$(n \in N_0,\ 0 < \alpha \leq 1,\ \mu \geq 1,\ 0 \leq \lambda < 1,\ z \in U)$$

$$(2.2) \quad f \in \sum \ \textit{and} \left| \arg \left(\frac{(1-\mu)\Omega_\lambda^n g(w) + \mu\Omega_\lambda^{n+1} g(w)}{z} \right) \right| < \frac{\alpha\pi}{z}$$
$$(n \in N_0,\ 0 < \alpha \leq 1,\ \mu \geq 1,\ 0 \leq \lambda < 1,\ z \in U)$$

where the function g in given by

$$(2.3) \qquad g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4$$

we begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for function in the class $B_\Sigma(n, \mu, \lambda, \alpha)$.

Theorem 2.1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $B_\Sigma(n,\mu,\lambda,\alpha)$ $(n \in N_0,\ 0<\alpha\le 1,\ \mu\ge 1)$ *and* $0\le\lambda<1)$. *Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\begin{array}{c}(1-\mu)\{\pi(2,\lambda)\}^n+\mu\{\phi(2,\lambda)\}^{n+1}+\alpha[2(1-\mu)\{\phi(3,\lambda)\}^n \\ +2\mu\{\phi(3,\lambda)\}^{n+1}]-[(1-\mu)\{\phi(2\lambda)\}^n+\mu\{\phi(3\lambda)\}^{n+1}]^2\end{array}}} \tag{2.4}$$

and

$$|a_3| \le \frac{2\alpha}{[(1-\mu)\{\phi(3,\lambda)\}^n+\mu\phi\{(3,\lambda)\}^{n+1}]} + \frac{4\alpha^2}{[(1-\mu)\{\phi(2,\lambda)\}^n+\mu\{\phi(2,\lambda)\}^{n+1}]^2} \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$\frac{(-\mu)^n_\lambda f(z)+\mu\Omega^{n-1}f(z)}{z}=[p(z)]^\alpha \tag{2.6}$$

and

$$\frac{(-\mu)^n_\lambda g(w)+\mu\Omega^{n+1}_\lambda g(w)}{z}=[q(w)]^\alpha \tag{2.7}$$

where $p(z)$ and $q(w)$ in p and have the form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+... \tag{2.8}$$

$$q(w)=1+q_1w+q_2w^2+q_3w^3+...\ . \tag{2.9}$$

Now equating the coefficients in (2.6) and (2.7), we have

$$[(1-\mu)\{\phi(2,\lambda)\}^n+\mu\{\phi(2,\lambda)\}^{n+1}]a_2=\alpha p_1 \tag{2.10}$$

$$[(1-\mu)\{\phi(3,\lambda)\}^n+\mu\{\phi(3,\lambda)\}^{n+1}]a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2 \tag{2.11}$$

$$-[(1-\mu)\{\phi(2,\lambda)\}^n+\mu\{\phi(2,\lambda)\}^{n+1}]a_2=\alpha q_1 \tag{2.12}$$

$$[(1-\mu)\{\phi(3,\lambda)\}^n+\mu\{\phi(3,\lambda)\}^{n+1}](2a_2^2-a_3)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we obtain

$$p_1 = q_1 \tag{2.14}$$

and

$$2[(1-\mu)\{\phi(2,\lambda)\}^n + \mu\{\phi(2,\lambda)\}^{n+1}]a_2^2 = \alpha^2(p_1^2+q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we immediately have

$$\begin{aligned}
&2[(1-\mu)\{\phi(3,\lambda)\}^n + \mu\{\phi(3,\lambda)\}^{n+1}]a_2^2 \\
&= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2) \\
&= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\frac{2[(1-\mu)\{\phi(2,\lambda)\}^n + \mu\{\phi(2,\lambda)\}^{n+1}]}{\alpha^2}a_2^2 \\
&= \alpha(p_2+q_2) + \frac{(\alpha-1)}{\alpha}[(1-\mu)\{\phi(2,\lambda)\}^n + \mu\{\phi(2,\lambda)\}^{n+1}]^2 a_2^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&[2\alpha\{(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}\} \\
&\quad - (\alpha-1)\{(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}\}^2]a_2^2 \\
&\qquad = \alpha^2(p_2+q_2)
\end{aligned}$$

or

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{[2\alpha\{(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}\} - (\alpha-1)\{(1-\mu)(\phi(3\lambda))^n + \mu(\phi(3\lambda))^{n+1}\}^2]}$$

or

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{\begin{array}{c}[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]^2 + \alpha[2(1-\mu)(\phi(3\lambda))^n + 2\mu(\phi(3,\lambda))^{n+1}] \\ -[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]^2\end{array}}.$$

Applying Lemma 1.1 for the coefficients p_2 an q_2, we immediately obtain

$$|a_2| \le \frac{2\alpha}{\sqrt{\begin{array}{c}[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]^2 + \alpha[2(1-\mu)(\phi(3,\lambda))^n + 2\mu(\phi(3,\lambda))^{n+1} \\ -\{(1-\mu)\phi(2,\lambda)^n + \mu(\phi(2,\lambda))^{n+1}\}^2]\end{array}}}.$$

Next, in order to find the bound on $|a_3|$ by subtractin (**??**) from (2.11), we have

$$2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}](a_3 - a_2^2)$$

$$= \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)$$

$$2[(1-\mu)(\phi(3,\lambda))^n+\mu(\phi(3,\lambda))^{n+1}]a_3$$
$$=\alpha(p_2-q_2)+\frac{2[(1-\mu)(\phi(3,\lambda))^n+\mu(\phi(3,\lambda))^{n+1}]2^2(p_1^2+q_1^2)}{2[(1-\mu)(\phi(2,\lambda))^n+\mu(\phi(2,\lambda))^{n+1}]^2}$$

or

$$a_3=\frac{\alpha(p_2-q_2)}{2[(1-\mu)(\phi(3,\lambda))^n+\mu\phi(3,\lambda)^{n+1}]}$$
$$+\frac{\alpha^2(p_1^2+q_1^2)}{2[(1-\mu)(\phi(2,\lambda))^n+\mu\phi(2,\lambda)^{n+1}]^2}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we have

$$|a_3|\le\frac{2\alpha}{[(1-\mu)(\phi(3,\lambda))^n+\mu(\phi(3,\lambda))^{n+1}]}$$
$$+\frac{4\alpha^2}{[(1-\mu)(\phi(2,\lambda))^n+\mu(\phi(2,\lambda))^{n+1}]^2}.$$

This complete the proof of Theorem 2.1. □

3 Coefficients bounds for the function class $H_\Sigma(n,\mu,\lambda,\beta)$

Definition 3.1. *A function $f(z)$ given by (1.1) is said to be in the class $H_\Sigma(n,\mu,\lambda,\beta)$ if the following conditions are satisfied*

$$f\in\sum \text{ and } Re\left(\frac{(1-\mu)\Omega_\lambda^n f(z)+\mu\Omega_\lambda^n f(z)}{z}\right)>\beta \tag{3.1}$$
$$(0<\beta\le 1,\ \mu\ge 1,\ n\in N_0,\ 0\le\lambda<1,\ z\in U)$$

and

$$Re\left(\frac{(1-\mu)\Omega_\lambda^n g(w)+\mu\Omega_\lambda^n g(w)}{w}\right)>\beta \tag{3.2}$$
$$(0<\beta\le 1,\ \mu\ge 1,\ n\in N_0,\ 0\le\lambda<1,\ z\in U)$$

when the function g is defined by (**??**), *we note that for $\lambda=0$, the class $H_\Sigma(n,\mu,\lambda,\beta)$ reduces to the class $H_\Sigma(n,\mu,\beta)$ studied by Porwal and Darus [9] and for $\lambda=0$, $n=0$ and for $n=0$, $\lambda=0$, $\mu=1$, the class $H_\Sigma(n,\mu,\lambda,\beta)$ reduces to the classes $H_\Sigma(\mu,\beta)$ and $H_\Sigma(\beta)$ studied by Frasin and Aouf [**?**] and Srivastava et al. [11] respectively. Theorem let the function $f(z)$ given by (1.1) be in the class $H_\Sigma(n,\mu,\lambda,\beta)$, $n\in N_0$, $0\le\beta<1$, $\mu\ge 1$ and $0\le\lambda<1$. Then*

$$|a_2|\le\frac{\sqrt{2(1-\beta)}}{\sqrt{[(1-\mu)(\phi(3,\lambda))^n+\mu(\phi(3,\lambda))^{n+1}]}} \tag{3.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]^2} + \frac{2(1-\beta)}{[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]^2} \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist $p(z) \in P$ and $g(z) \in P$ such that

$$\frac{(1-\mu)\Omega_\lambda^n f(z) + \mu\Omega^{n+1} f(z)}{z} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{(1-\mu)\Omega_\lambda^n g(w) + \mu\Omega^{n+1} g(w)}{w} = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(z)$ have the form (2.8) and (2.9) respectively.

Now equating coefficient in (3.5) and (3.6), we obtain

$$[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]a_2 = (1-\beta)p_1, \tag{3.7}$$

$$[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]a_3 = (1-\beta)p_2, \tag{3.8}$$

$$[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]a_2 = (1-\beta)q_1, \tag{3.9}$$

and

$$[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}][2a_2 - a_3] = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we have

$$p_1 = q_1 \tag{3.11}$$

and

$$2[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]a_2^2 = (1-\beta)(p_2 + q_2) \tag{3.13}$$

or

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]}$$

or

$$|a_2^2| \leq \frac{2(1-\beta)}{2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]}$$

which is the bound on $|a_2|$ as given by (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}][a_3 - a_2^2]$$
$$= (1-\beta)(p_2 - q_2)$$

or, equivalently

$$a_3 = a_2^2 + \frac{(1-\beta)(p_z - q_z)}{2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]}.$$

upon substituting the value of a_2^2 from (**??**), we find that

$$a_3 = \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]^2}$$
$$+ \frac{(1-\beta)(p_2 - q_2)}{2[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]}.$$

Applying Lemma 1.1 for the coefficients $p_1, p_2,\ q_1$, and q_2, we have

$$a_3 = \frac{4(1-\beta)^2}{[(1-\mu)(\phi(2,\lambda))^n + \mu(\phi(2,\lambda))^{n+1}]^2}$$
$$+ \frac{2(1-\beta)}{[(1-\mu)(\phi(3,\lambda))^n + \mu(\phi(3,\lambda))^{n+1}]},$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Remark 3.1. *If we put $\lambda = 0$ in Theorem 2.1 and 3.1 we obtain the corresponding results due to Porwal and Darus [9].*

Remark 3.2. *If we put $\lambda = 0$ and $n = 0$ in Theorem 2.1 and 3.1 we obtain the corresponding results due to Frasin and Aouf [**?**].*

Remark 3.3. *If we put $\lambda = 0$, $n = 0$ and $\mu = 1$ in Theorem 2.1 and 3.1 we obtain the corresponding results due to Srivatava et al. [11].*

Remark 3.4. *If would be of interest to find the estimates (not necessarily sharp) for $a_n|$, $n \geq 4$.*

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