# A problem of thermal stresses in bonded dissimilar micropolar elastic half-spaces containing a penny- shaped crack at the interface

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**Abstract**

A theory of micropolar thermoelasticity which includes heat-flux among the constitutive variables is formulated by [3]. It is found that the linearized version of the theory admits second sound effects. In [4] the distribution of thermal stress in an elastic layer bonded in two half-spaces along its plane surfaces and contains a penny-shaped crack parallel to the interface is analyzed. The crack is situated in the mid-plane of the layer. K.N. Srivastava and R.M. Palaiya [5] investigated the termoelastic equilibrium of a semi infinite solid containing a penny-shaped crack situated parallel to the free boundary. They assumed that the thermal conditions on the upper surface of the crack are identical with those on the lower surface and the free boundary of the solid is kept at zero temperature. Using linear theory of micropolar elasticity [2] studied the problem of a penny-shaped crack opened out by thermal loads. The classical values of the stress intensity factors are recovered as a limiting case. An analysis of the distribution of thermal stresses in a sphere which is bounded to an infinite elastic medium is given by [3]. The penny-shaped crack lies on the diameter as plane of the sphere and the center of the crack. Using suitable solution of the termoelastic displacement differential equation, the problem is then reduced to the solution of the Fredholm integral equation.

**Keywords:** Fredholm Integral Equations, Dual Integral Equation.

## 1 Introduction

The axisymmetric problem of a penny-shaped crack opened out by thermal loads is studied. The deformation in the two half spaces is due to the application of temperature to the faces of the crack. Present study deals with the problem of determining the components of stress and stress intensity factor of apenny-shaped crack situated at the interface of two bonded dissimilar elastic half spaces. The problem is first reduced to a system of dual integral equations which are further transformed to singular integral equations. Calculations have been done in case when constant temperature is prescribed on the crack surface.

## 2 FORMULATION OF THE PROBLEM

Consider homogenous isotropic medium uninterrupted except for the penny-shaped crack, $z = 0$,$0 \leq r \leq 1$. We assume that the crack is opened out by the application of heat to it flat surfaces. Let the thermal conditions on the upper surface of the crack be identical with those on the lower surface. We also assume that the crack is deformed by the application of axially pressure p(r) in addition to the heat flux. We shall consider two cases.
(i) Crack opened out by a prescribed heat flux across its flat surfaces.
(ii) Crack opened out by the application of a prescribed temperature to its flat surfaces.

By symmetry the distribution of stresses and temperature of the crack is identical with that produced in the semi-infinite medium $z \geq 0$ when its boundary surface $z = 0$ is subjected to the conditions :

$$\sigma_{zz}(r, 0+) = \sigma_{zz}(r, 0-) = -p(r) \tag{2.1}$$

$$\sigma_{rz}(r, 0+) = \sigma_{rz}(r, 0-) = 0 \tag{2.2}$$

$$m_{\phi z}(r, 0+) = m_{\phi z}(r, 0-) = 0 \tag{2.3}$$

$$T(r, 0+) = T(r, 0-) = 0 \tag{2.4}$$

For the region of the interface not occupied by the crack. The following continuity conditions are satisfied.

$$u_r(r, 0+) = u_r(r, 0-) \tag{2.5}$$

$$u_z(r, 0+) = u_z(r, 0-) \tag{2.6}$$

$$\phi(r, 0+) = \phi(r, 0-) \tag{2.7}$$

$$T(r, 0+) = T(r, 0-) \tag{2.8}$$

$$\sigma_{rz}(r, 0+) = \sigma_{rz}(r, 0-) \tag{2.9}$$

$$\sigma_{zz}(r, 0+) = \sigma_{zz}(r, 0-) \tag{2.10}$$

$$m_{\phi z}(r,0+) = m_{\phi z}(r,0-) \tag{2.11}$$

$$T_{zy}(r,0+) = T_{zy}(r,0-) \tag{2.12}$$

$$\overline{K}_1 \frac{\partial T}{\partial Z}_{z=0+} = \overline{K}_2 \frac{\partial T}{\partial Z}_{z=0-}$$

where $\overline{K}_1$ and $\overline{K}_2$ are the coefficients of heat conduction. The components of stress displacement, microrotation must vanish as $(r^2+z^2)^{1/2}$ . We take the following conditions

$$u_z(r,0+) = u_z(r,0-)$$

The symmetry due to z-axis provides additional conditions:

$$u_z(r,0+) = u_z(r,0-)$$

$$\phi(r,0+) = \phi(r,0-) \quad for \;\; all \; r \tag{2.13}$$

$$T(r,0+) = T(r,0-)$$

As the solution of the equations of the elastic equilibrium, we take the following appropriate form of the displacement vector stresses.

$$\overline{u}_r(\xi,z) = \begin{cases} A_1 - \frac{\lambda_1+3\mu_1 B_1}{\lambda_1+2\mu_1\xi} + \frac{\lambda_1+\mu_1 B_1 z e^{-\xi z}}{\lambda_1+2\mu_1\xi} - \frac{v_1\eta_1 C_1 e^{-\xi z}}{2\mu_1}, z \geq 0 \\ -A_2 - \frac{\lambda_2+3\mu_2 B_2}{\lambda_2+2\mu_2} + \frac{\lambda_2+\mu_2 B_2 z e^{\xi z}}{\lambda_2+2\mu_2} - \frac{v_2\eta_2 C_2 e^{\eta} z^z}{2\mu_2}, z \leq 0 \end{cases} \tag{2.14}$$

$$\overline{u}_z(\xi,z) = \begin{cases} A_1 + \frac{\lambda_1+\mu_1 z B_1 e^{-\xi z}}{\lambda_1+2\mu_1} - \frac{v_1 C_1 e^{-\eta_1 z}}{2\mu_1}, z \geq 0 \\ A_2 - \frac{\lambda_2+\mu_2 z B_2 e^{\xi z}}{\lambda_2+2\mu_2} + \frac{v_2 C_2 e^{\eta_2 z}}{2\mu_2}, z \leq 0 \end{cases} \tag{2.15}$$

$$\phi(\xi,z) = \begin{cases} (B_1 e^{-\xi z} + C_1 e^{-\eta_2 z}), z \geq 0 \\ (-B_2 e^{\xi z} - C_1 e^{\eta z}), z \leq 0 \end{cases} \tag{2.16}$$

$$\begin{cases} D_1 e^{-\xi z}, z \geq 0 \\ D_2 e^{\xi z}, z \leq 0 \end{cases} \tag{2.17}$$

$$T(\xi,z)=\begin{cases} H_0[\frac{\phi_1(\xi)e^{-\xi z}}{\alpha_1(1+\eta_1)};r], z\geq 0 \\ H_0[\frac{\phi_2(\xi)e^{\xi z}}{\alpha_2(1+\eta_2)};r], z\leq 0 \end{cases} \tag{2.18}$$

$$\overline{\sigma}_{rz}(\xi,z)\begin{cases} 2\mu_1\big[(-\xi A_1+B_1-\frac{\lambda_1+\mu_1\xi zB_1}{\lambda_1+2\mu_1})e^{-\xi z}+\frac{v_1\xi^2C_1e^{-\eta_1 z}}{2\mu_1}\big] \\ 2\mu_2\big[(-\xi A_2+B_2-\frac{\lambda_2+\mu_2\xi^2 zB_2}{\lambda_2+2\mu_2})e^{\xi z}+\frac{v_2C_2e^{\eta_2 z}}{2\mu_2}\big] \end{cases} \tag{2.19}$$

$$\overline{\sigma}_{rz}(\xi,z)\begin{cases} 2\mu_1\big[(-\xi A_1+B_1-\frac{\lambda_1+\mu_1\xi zB_1}{\lambda_1+2\mu_1})e^{-\xi z}+\frac{v_1\xi^2C_1e^{-\eta_1 z}}{2\mu_1}\big] \\ 2\mu_2\big[(-\xi A_2+B_2-\frac{\lambda_2+\mu_2\xi^2 zB_2}{\lambda_2+2\mu_2})e^{\xi z}+\frac{v_2C_2e^{\eta_2 z}}{2\mu_2}\big] \end{cases} \tag{2.20}$$

$$m_{\phi_z}(\xi,z)=\begin{cases} \big[-v_1(B_1\xi e^{\xi z})+C_1\eta_1e^{-\eta_1 z}\big], z\geq 0 \\ \big[-v_2(B_2\xi e^{-\xi z}+C_2\eta_2e^{\eta_2 z})\big], z\leq 0 \end{cases} \tag{2.21}$$

$$T_{zz}(\xi,z)=\begin{cases} D_1e^{-\xi z}, z\geq 0 \\ D_2e^{\xi z}, z\leq 0 \end{cases} \tag{2.22}$$

$$\frac{\partial T}{\partial z}(\xi,z)=\begin{cases} -\xi D_1e^{-\xi z}, z\geq 0 \\ \xi D_2e^{\xi z}, z\leq 0 \end{cases} \tag{2.23}$$

On putting the value of $z=0$ we get

$$\overline{\sigma}_{zz}(\xi,z)=\begin{cases} 2\mu_1\big[(-\xi A_1+\frac{\mu_1B_1}{\lambda_1+2\mu_1})+\frac{v_1\eta_1C_1}{2\mu_1}\big], z\geq 0 \\ 2\mu_2\big[(\xi A_2+\frac{\mu_1B_2}{\lambda_2+2\mu_2})+\frac{v_2\eta_2C_2}{2\mu_2}\big], z\leq 0 \end{cases} \tag{2.24}$$

$$\overline{\sigma}_{rz}(\xi,z)=\begin{cases} 2\mu_1\big[(-\xi A_1+B_1+\frac{v_1\xi^2C_1}{2\mu_1}\big], z\geq 0 \\ 2\mu_2\big[(-\xi A_2+B_2+\frac{v_2\xi^2C_2}{2\mu_2}\big], z\leq 0 \end{cases} \tag{2.25}$$

$$\overline{m}_{z\phi}(\xi, z) = \begin{cases} \big[-v_1(B_1\xi + C_1\eta_1)\big] & , z \geq 0 \\ \big[-v_2(B_2\xi + C_2\eta_2)\big] & , z \leq 0 \end{cases} \tag{2.26}$$

$$\overline{T}_{zz}(\xi, z) = \begin{cases} D_1, & z \geq 0 \\ D_2, & z \leq 0 \end{cases} \tag{2.27}$$

Now from the equation (2.1) and (2.5) and with the help of equations (2.24) and (2.25) we get,

$$\sigma_{zz}(r, 0+) = \sigma_{zz}(r, 0-)$$

$$\sigma_{rz}(r, 0+) = \sigma_{rz}(r, 0-)$$

And we get the following equations:

$$-\Gamma_1\xi A_1 + \Gamma_1(1 - Q_1)B_1 + \Gamma_1 L_1^2\xi\eta_1 C_1 = \xi A_2 + (1 - Q_2)B_2 + L_2^2\xi\eta_2 C_2 \tag{2.28}$$

$$-\Gamma_1\xi A_1 + \Gamma_1 B_1 + \Gamma_1 L_1^2\xi^2 C_1 = -\xi A_2 - B_2 - L_2^2\xi^2 C_2 \tag{2.29}$$

$$\Gamma_2\xi B_1 + \Gamma_2\eta_1 C_1 = \xi B_2 + \eta_2 C_2 \tag{2.30}$$

$$-2\Gamma_1 A_1 + \Gamma_1(2 - Q_1)B_1 + \Gamma_1 L_1^2\xi(\eta_1 - \xi)C_1 = -Q_2 A_2 + L_2^2\xi(\eta_1 - \xi)C_2 \tag{2.31}$$

Solving these equations for $A_2$, $B_2$, $C_2$ in terms of $A_1$, $B_1$, $C_1$, we get the following values of $A_2$, $B_2$ and $C_2$.

$$\begin{aligned} A_2 = \{\Gamma_1 + 2\xi k_1^{-1}(L_2^2 - \eta_2)\}A_1 + \{k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1)(\eta_2 - L_2^2\xi^2) - \xi^{-1}(\Gamma_1 + \Gamma_2)\}B_1 \\ -\Big[\Gamma_1 L_1^2 + \Gamma_2\eta_1\xi^{-1} + \{Q_2\Gamma_2\eta_1 - L_1^2\Gamma_1^2\xi(\eta_1 + \xi)\}\{L_1^2\xi - \eta_2 k_1^{-1}\xi^{-1}\}\Big]C_1 \end{aligned}$$

$$\begin{aligned} B_2 = 2\Gamma_1\xi^{-1}\eta_2 k_1^{-1}A_1 + \{(\Gamma_2 - \xi\eta_2 k_1^{-2}(\Gamma_2 Q_2 - \Gamma_1 Q_1))\}B_1 \\ +\Big[\Gamma_2\eta_2 - \xi^1 - \eta_2 k_1^{-1}\{Q_2\Gamma_2\eta_1 - \Gamma_1 L_1^2\xi^2(\eta_1 + \xi)\}\Big]C_1 \end{aligned}$$

$$C_2 = K_1^{-1}\Big[-2\Gamma_1\xi^2 A_1 + (\Gamma_2 Q_2 - \Gamma_1 Q_1)\xi B_1 + \{\Gamma_2 Q_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_1 + \xi)\}C_1\Big]$$

where

$$P_i = \frac{\lambda_i + 3\mu_i}{\lambda_i + 2\mu_i}, \qquad Q_i = \frac{\lambda_i + 3\mu_i}{\lambda_i + 2\mu_i}$$

$$L_1^2 = v_i/2\mu, \qquad \Gamma_1 = \mu_1/\mu_2, \qquad \Gamma_2 = v_1/v_2$$

$$1 - Q_i = \frac{\mu_i}{\lambda_i} + 2\mu_i, k_1 = L_2^2\xi^2(\eta_2 - \xi) + \eta_2 Q_2$$

Now from the equations (2.15), (2.16) and boundary conditions (2.5) and equations (2.20) and (2.21), we get the following equations.

$$A_1 - L_1^2\xi C_1 = A_2 + L_2^2\xi C_2$$

$$B_1 + C_1 = -B_2 - C_2$$

$$-\Gamma_1 A_1 + \Gamma_1 B_1 + \Gamma_1 L_1^2\xi^2 C_1 = -\xi A_2 - B_2 - L_2^2\xi^2 C_2$$

$$\Gamma_2 B_1\xi + \Gamma_2\eta_1 C_1 = B_2\xi + \eta_2 C_2 \tag{2.32}$$

Putting the values of $A_2, B_2, C_2$ in the above equations (2.32), we get the following equations.

$$\begin{aligned} a_1A_1 + b_1B_1 + c_1C_1 &= \psi(\xi) \\ a_2A_1 + b_2B_1 + c_2C_1 &= \Phi(\xi) \\ a_3A_1 + b_3B_1 + c_3C_1 &= \chi(\xi) \end{aligned} \tag{2.33}$$

where

$$a_1 = 1 + 2\Gamma_1 L_2^2 k_1^{-1}\xi^3 - \Gamma_1 - 2\xi k_1^{-1}(L_2^2\xi^2 - \eta_2)$$

$$b_1 = \Big[(\Gamma_1 Q_1 - \Gamma_2 Q_2)L_2^2\xi^2 - k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1)(\eta_2 - L_2^2\xi^2) - \xi^{-1}(\Gamma_1 + \Gamma_2)\Big]$$

$$\begin{aligned} c_1 = \Gamma_1 L_1^2 + 2\eta_1\xi^{-1} + \{Q_2 L_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_2 + \xi)\}\{L_2^2\xi - \eta_2 k_1^{-1}\xi^{-1}\} - L_1^2\xi \\ -\{\Gamma_2 Q_2 L_2^2\xi\eta_1 - L_1^2 L_2^2\Gamma_1\xi^3(\eta_1 + \xi)\} \end{aligned}$$

$$a_2 = 2\Gamma_1\xi^2 k_1^{-1}(\eta_2 - 1)$$

$$b_2 = 1 + \Gamma_2\xi k_1^{-1}(\Gamma_1 Q_1 - \Gamma_2 Q_2)(\eta_2 - 1)$$

$$c_2 = 1 + \Gamma_2\eta_1\xi^{-1} + \{Q_2\Gamma_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_1+\xi)\}\{1-\eta_2 k_1^{-1}\}$$

$$a_3 = \Gamma_1 + \{\xi\Gamma_1 + 2\xi^2 k_1^{-1}(L_2^2\xi^2 - \eta_2)\} + 2\Gamma_1\xi^2\eta_2 k_1^{-1} - 2\Gamma_1 L_2^2\xi^4 k_1^{-1}$$

$$b_3 = \Gamma_1 - \{\xi k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1)(\eta_2 - L_2^2\xi^2) - (\Gamma_1+\Gamma_2)\} + \Gamma_2 - \xi\eta_2 k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1) + L_2^2\xi^3 k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1)$$

$$c_3 = \Gamma_1 L_1^2\xi^2 - \Big[L_1^2\Gamma_1\xi^2 + \Gamma_2\eta_1 + \xi\{Q_2\Gamma_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_1+\xi)\}\{L_2^2\xi - k_1^{-1}\eta_2\xi^{-1}\}\Big] + \Gamma_2\eta_1\xi^{-1}\eta_2 k_1^{-1}\{\Gamma_2 Q_2\eta_1 - \Gamma_1 L_1^2\xi^2(\eta_1+\xi)\} + L_2^2\xi^2 k_1^{-1}\{\Gamma_2 Q_2\eta_1 - L_1^2\Gamma_1^2(\eta_1+\xi)\}$$

Now from the equations (2.19) (2.21) we have

$$\begin{aligned} -\xi A_1 + (1-Q_1)B_1 + L_1^2\xi\eta_1 C_1 &= \frac{p(r)}{\mu_1}, r<1 \\ -\xi A_1 + B_1 + L_1^2\xi^2 C_1 &= 0 \\ -\xi B_1 + \eta_1 C_1 &= 0 \end{aligned} \tag{2.34}$$

Substituting the values of $A_1, B_1$ and $C_1$ we get

$$-\xi a^{-1}(\xi)\phi_3(\xi) + (1-Q_1)b^{-1}(\xi)\phi_2(\xi) + L_1^2\eta_1 C^{-1}(\xi)\phi_1(\xi) = p(r)/\mu_1$$

$$-a^{-1}(\xi)\phi_3(\xi) + b^{-1}(\xi)\phi_2(\xi) + L_1^2\xi^2 C^{-1}(\xi)\phi_1(\xi) = 0$$

$$b^{-1}(\xi)\phi_2(\xi) + \eta_1 C^{-1}(\xi)\phi_1(\xi) = 0$$

$$H_0\Big[A(\xi)\phi_2(\xi) + B(\xi)\psi(\xi) + C(\xi)\eta(\xi)\Big] = f_1(r)$$

$$H_1\Big[A(\xi)\phi_2(\xi) + B(\xi)\psi(\xi) + C(\xi)\eta(\xi)\Big] = f_2(r)$$

For all $0 \le r \le 1$

$$H_2\Big[B(\xi)\psi(\xi) + C(\xi)\eta(\xi)\Big] = 0 \tag{2.35}$$

or

$$H_0\Big[\xi^{-1}\phi(\xi)\Big] = 0$$
$$H_1\Big[\xi^{-1}\phi(\xi)\Big] = 0$$

for all $r > 1$

$$H_2\Big[\xi^{-1}\eta(\xi)\Big] = 0 \tag{2.36}$$

Now these equations are reduced to the following equations.

$$\int_0^\infty \sum_{j=1}^{3} c_{ij}(\xi)\psi_j(\xi)J_{\mu i}(\xi\eta)d\xi = f_i(\eta); \eta\epsilon I_1 \tag{2.37}$$

$$\int_0^\infty \psi_j(\xi)J_{\mu i}(\xi\eta)d\xi = 0; \eta\epsilon I_2 \tag{2.38}$$

In the above equations it is assumed that the functions $f_i(\eta)$ and $C_{ij}(\xi)$ are known $(i, J = 1, 2, 3)$ and is required to find the functions $\psi_j(\xi)$ $[51, p, 129]$.
We take

$$\psi_j(\xi) = \xi^{1-\beta_j} \sum_{m=1}^{\infty} A_{jm}J_{\mu j} + 2m + \beta_j(\xi) \tag{2.39}$$

Which ensure that the equation (2.38) is satisfied if $\beta_j > 0$ and $Re(\mu_j + m + 1) > 0$ For the moment the $\beta_j > (j = 1, 2, 3)$ are arbitary. Substitution form (2.39) into (2.37) we get

$$\int_0^\infty \eta^{1+\mu_j}(1-\eta_2)^{\beta_j-1}F_s(\mu_j+\beta_j, \mu_j+1, \eta_2)J_{\mu_j}(\xi\eta)d\eta = \xi^{-\beta_j}J_{\mu_j+2m+\beta_j}(\xi)2^{\beta_j-1}\frac{\Gamma(1+\mu_i)\Gamma(s+\beta_j)}{\Gamma(\mu_j+s+1)}$$

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{j=1}^{3} A_{jm}\int_0^\infty \xi^{1-\beta_i-\beta_j}C_{ij}(\xi)J_{\mu_j+2s+\beta_i}(\xi)d\xi \\ = \beta_j(\mu_i, \beta_i, s)\end{aligned} \tag{2.40}$$

Where

$$\beta_i(\mu_i, \beta_i, s) = \frac{\Gamma(\mu_i+s+1)}{(2^{\beta_i-1})\Gamma(\mu_i+1)\Gamma(\beta_i+s)}\int_0^\infty f_i(\eta)\eta^{1+\mu_i}(1-\eta)^{\beta_i-1}F_s(\mu_i+\beta_i, \quad \mu_i+1, \eta^2)d\eta$$

This is an infinite set of equations for $A_{jm}(j = 1, 2, 3, m = 0, 1, 2)$ to find the solution of this set of equations it is convenient to first transform them by adding and subtracting the following sum of integrals to and from the left side:

$$\sum_{m=0}^{\infty} A_{im} \int_0^{\infty} J_{\mu_i+2s+\beta_i}(\xi) J_{\mu_i+2m+\beta_i}(\xi) d\xi = \frac{A_{is}}{2(\mu_i + 2s + \beta_i)}$$

$$\sum_{j=1}^{3} A_{im} \int_0^{\infty} J_{\mu_i+2s+\beta_i}(\xi) J_{\mu_i+2m+\beta_i}(\xi) \xi^{-1} d\xi$$

$$= \sum_{m=0}^{\infty} \sum_{j=1}^{3} \frac{2A_{jm} \sin(1/2)\Big(2s - 2m + \mu_i - \mu_j\Big)\pi}{\pi(\mu_i + 2s + \beta_i)^2 - (\mu_j + 2m + \beta_j)}$$

Now making all the integrals on the left side of this equations convergent, we get

$$\int_0^{\infty} \xi^{-\delta} J_{\mu}(\xi) J_v(\xi) d\xi = \frac{\Gamma(\delta).\Gamma[(1/2\mu) + (1/2v) - (1/2)\delta + 1/2]}{2^{\delta}\Gamma[(-1/2\mu) + (1/2v) + (1/2\delta) + (1/2)]\Gamma[(1/2\mu) + (1/2v) + (1/2\delta)]}$$

## 3 TEMPERATURE FIELD

$$T(r,z) \begin{cases} H_0\Big[\frac{\phi_1(\xi)e^{-\xi z}}{\alpha_1(1+\eta_1)}\Big], & z \geq 0 \\ H_0\Big[\frac{\phi_2(\xi)e^{\xi z}}{\alpha_2(1+\eta_2)}, r\Big], & z \leq 0 \end{cases} \tag{3.1}$$

On putting z = 0 in the above equations (3.1) we obtain

$$T(r, 0+) = H_0\Big[\frac{\phi_1(\xi)}{\alpha_1(1 + \eta_1)}, r\Big]$$

$$T(r, 0-) = H_0\Big[\frac{\phi_2(\xi)}{\alpha_2(1 + \eta_2)}, r\Big]$$

$$\frac{\partial T}{\partial z}(r, 0+) = H_0\Big[\frac{-\xi\phi_1(\xi)}{\alpha_1(1 + \eta_1)}, r\Big]$$

$$\frac{\partial T}{\partial z}(r, 0-) = H_0\Big[\frac{\xi\phi_2(\xi)}{\alpha_2(1 + \eta_2)}, r\Big]$$

Now applying the condition (2.12) we get.

$$\phi_2(\xi) = \frac{\alpha_2(1+\eta_2)}{\alpha_1(1+\eta_1)}\phi_1(\xi) \tag{3.2}$$

And

$$\int_0^\infty \xi^2 J_0(\xi, r)\Big[\frac{\overline{K_2}\phi_1(\xi)}{\alpha_1(1+\eta_1)} + \frac{\overline{K_2}\phi_2(\xi)}{\alpha_2(1+\eta_2)}\Big]d\xi = 0, r > 1$$

From (3.2) substituting the value of $\phi_2(\xi)$ in above equation we get,

$$\int_0^\infty \xi^2\phi_1(\xi)J_0(\xi, r)d\xi = 0, r > 1$$

Hence from the equations (2.4) and (2.12) we get the following pair of dual integral equations.

$$\int_0^\infty \xi\phi_1(\xi)J_0(\xi, r)d\xi = -\alpha_1(1+\eta_1)T(r), r < 1 \tag{3.3}$$

$$\int_0^\infty \xi\phi_1(\xi)J_0(\xi, r)d\xi = 0, r > 1 \tag{3.4}$$

Now taking

$$\xi\phi_1(\xi) = \int_0^1 \beta(t)Cos\ \xi\ tdt \tag{3.5}$$

Equation (3.4) is satisfied and (3.3) is reduced to

$$\beta(t) = \frac{-2\alpha_1(1+H_1)}{\pi}\frac{d}{dt}\int_0^t \frac{rT(r)}{(t^2-r^2)^{1/2}}dr \tag{3.6}$$

## 4 CONSTANT TEMPERATURE

If we consider the physical important case in which the crack is opened out by the application of prescribed constant temperature at the surface, then
$T(r) = T_0$=constant
And from the equation (3.5) we get.

$$\beta(t) = \frac{-2\alpha_1(1+\eta_1)}{\pi}.T_0 = T_1(say) \tag{4.1}$$

and hence

$$\phi_1(\xi) = \xi^{-2} T_1 \sin(\xi) \tag{4.2}$$

We shall also assume that there is no internal pressure applied to the surface of the crack, so that $p(r) = 0$, this is not all a severe limitation since the solution for non-zero function $p(r)$ is known for the non-thermal problem [6] and the general solution may be obtain by superposition.
We have,

$$\widehat{f_1}(x) = \overline{e_1}\sqrt{\frac{2}{\pi}}\int_0^\infty \phi_1(\xi)\sin(\xi x)d\xi$$

$$= \overline{e_1}\sqrt{\frac{2}{\pi}}\int_0^1 \beta(t)\int_0^\infty \frac{\sin(\xi x)\cos(\xi t)}{\xi}d\xi\ dt$$

$$= \overline{e_1}\sqrt{\frac{2}{\pi}}T_1 x \tag{4.3}$$

and

(4.4)

$$\widehat{f_1}(x) = -\overline{e_2}xA_1\Big[H_1\{\phi_1(\xi), r\}, x\Big] + C_1 = \frac{-\overline{e_2}T_1}{2}\sqrt{\frac{\pi}{2}}\Big[\log|1-x^2| + x\log\Big|\frac{(1+x)}{(1-x)}\Big|\Big] + C_1$$

$$g(x) = -\overline{e_1}T_1\sqrt{\frac{\pi}{2}}x = \frac{iT_1e_2}{\sqrt{2\pi}}\Big[\log(1-x^2) + x\log\Big|\frac{(1+x)}{1-x}\Big|\Big] + C_1$$

Since $x < 1$ to a sufficient degree of approximation, we can neglect higher powers of $x$. Hence we get

$$g(x) = e_1T_1x\sqrt{\frac{\pi}{2}}x = \frac{iT_1e_2\Big(x^2 + \frac{x^4}{6}\Big)}{\sqrt{2\pi}} + C_1$$

Higher accuracy may be achieved by including few more terms in the expansion of $g(x)$. Substituting the values of $g(t)$ and $x(t)$ in $\frac{tg(t)}{x(t)(t-\xi)}$ and expanding in power of $t$.

We get

$$L(\xi) = \xi e_1T_1\sqrt{\frac{\pi}{2}} + \frac{e_2T_1}{6\sqrt{2\pi}}\Big[i\xi^4 + 2\omega\xi^3 + 2i\xi^3(3-\omega^2)\Big] + \frac{2\xi\omega(19-2\omega^2)}{3}$$

Now

$$\lambda(x) = \Lambda^+(x) - \Lambda^-(x), |x| < 1$$

$$= (2\alpha)^{-1}[L(x) + C_2 + iC_3][X^+ - Y^-]$$

$$= \frac{2}{\pi}(\beta^2 - \alpha^2)^{1/2}[i(Ax^4 + Cx^2 + C_3) + (Bx^3 + Dx + C_2)]e^{i\omega\theta}$$

Where

$$A = e_2 T_1 e^{-\pi\omega/12}, B = 2\omega A, C = 2(3 - \omega^2)A$$

$$D = 1/2 e_1 T_1 \pi e^{-\pi\omega} + (2\omega A/3)(19 - 2\omega^2), \theta = \log\left(\frac{1+x}{1-x}\right)$$

And $C_2$, $C_3$ are constants.

Since $\lambda(x) = s(x) + ir(x)$, we have

$$s(x) = \sqrt{\frac{2}{\pi}}(\beta^2 - \alpha^2)^{-1/2}\Big[(Bx^3 + Dx + C_2)\cos\omega\theta - (Ax^4 + Cx^2 + C_3)\sin\omega\theta\Big] \tag{4.5}$$

$$r(x) = \sqrt{\frac{2}{\pi}}(\beta^2 - \alpha^2)^{-1/2}\Big[(Bx^3 + Dx + C_2)\cos\omega\theta - (Ax^4 + Cx^2 + C_3)\sin\omega\theta\Big] \tag{4.6}$$

Since $s(x)$ is an odd and $r(x)$ is an even function we have $C_2 = 0$. The other constant $C_3$ can be calculated by using $\int_0^\infty r(x)dx = 0$.

## 5 FORMULA FOR STRESS AND DISPLACEMENT

From the equation (2.15) we have

$$u_z(r, 0+) - u_z(r, 0-) = A_1 - A_2 - L_1^2\xi C_1 - L_2^2\xi C_2$$

Substituting the values of $A_1, A_2, C_1$ and $C_2$ in the above equation, we have

$$u_z(r, 0+) - u_z(r, 0-) = p_1 a^{-1}(\xi)\phi_3(\xi) + p_2 b^{-1}(\xi)\phi_2(\xi) + p_3 c^{-1}(\xi)\phi_1(\xi)$$

$$= H_0 p_1 a^{-1}(\xi)\phi_3(\xi) + p_2 b^{-1}(\xi)\phi_2(\xi) + p_3 c^{-1}(\xi)\phi_1(\xi)$$

Using the values of $A_1(\xi), B_1(\xi)$ and $\psi_c(\xi) = r_1(x)H(1-x)$

$$Q_s(x) = s_1(x)\ H(1-x), x > 0 \tag{5.1}$$

It can be written as

$$u_z(r, 0+) - u_z(r, 0-) = \sqrt{\frac{2}{\pi}}\int_r^1 \frac{s_1(x)}{(x^2 - r^2)^{1/2}}dx \tag{5.2}$$

which gives the difference of the normal component of displacement of the upper and lower surfaces of the crack. Similarly from the equation (2.14) we get

$$u_r(r,0+) - u_r(r,0-) = p_1 a^{-1}(\xi)\phi_3(\xi) + p_2 b^{-1}(\xi)\phi_2(\xi) + p_3 c^{-1}(\xi)\phi_1(\xi)$$

$$= H_1 p_1 a^{-1}(\xi)\phi_3(\xi) + p_2 b^{-1}(\xi)\phi_2(\xi) + p_3 c^{-1}(\xi)\phi_1(\xi)$$

where

$$p_1 = (1-\Gamma_1)(1-2\xi^3 L_2^2 k_1^{-1}) = 2\xi\eta_2 k_1^{-1}$$

$$p_2 = (\Gamma_1+\Gamma_2)\xi^{-1} - L_2^2\xi^2(\Gamma_2 Q_2 - \Gamma_1 Q_1) - k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1)(\eta_2 - L_2^2\xi^2)$$

$$p_3 = (\Gamma_1-1)L_1^2\xi + 2\eta_1\xi^{-1} + \{\Gamma_2 Q_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_1\xi)\}(L_2^2\xi - \eta_2 k_1^{-1}\xi^{-1}) - L_2^2\xi\{\Gamma_2 Q_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_1+\xi)\}$$

$$p_1\,' = 1 + \Gamma_1 + 2\xi k_1^{-1}(L_2^2\xi - \eta_2) + 2\Gamma_1 P_2\xi\eta_2 k_1^{-1} - 2\Gamma_1 L_2^2\eta_2 k_1^{-1}\xi^2$$

$$\begin{aligned} p_2\,' = k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1)(\eta_2 - L_2^2\xi^2) - \xi^{-1}(\Gamma_1+\Gamma_2) - \xi^{-1}P_1 + \Gamma_2 - \xi^{-1}P_2 \\ -P_2\xi\eta_2 k_1^{-1}(\Gamma_2 Q_2 - \Gamma_1 Q_1) + L_2^2\eta_2(\Gamma_2 Q_2 - \Gamma_1 Q_1)\xi \end{aligned}$$

$$\begin{aligned} p_3\,' = \xi^{-1}P_2\Big[\Gamma_2\eta_1\xi^{-1} - \eta_2 k_1^{-1}\{Q_2\Gamma_2\eta_1 - \Gamma_1 L_1^2\xi^2(\eta_1+\xi)\}\Big] - \Big[\Gamma_1 L_1^2\xi + \Gamma_2\eta_1\xi^{-1} \\ +\{Q_2\Gamma_2\eta_1 - L_1^2\Gamma_1\xi^2(\eta_1+\xi)\}\{L_1^2\xi - \eta_2\xi^{-1}k_1^{-1}\}\Big] - L_1^2\eta_1 - L_1^2 L_2^2\Gamma_1\eta_2\xi(\eta_1+\xi) \end{aligned}$$

Now from the equation (1), we have

(5.3)
$$H_1\Big[p_1\,'a^{-1}(\xi)\phi_3(\xi) + p_2\,'b^{-1}(\xi)\phi_2(\xi) + p_3\,'c^{-1}(\xi)\phi_1(\xi), r\Big] = \int_0 1 r_1(x)dx - H(1-r)\int_0^r \frac{x r_1(x)}{(r^2-x^2)}dx$$

(5.4)
$$u_r(r,0+) - u_z(r,0-) = \sqrt{\frac{2}{\pi}}\Big[\int_0^1 r_1(x)dx - H(1-r)\int_0^r \frac{x r_1(x)}{(r^2-x^2)}dx\Big]$$

Hence we obtain

(5.5)
$$\int_0^1 r_1(x)dx = 0,\ for\ r > 1$$

This condition determines the arbitrary constant.

Now

$$H_1\Big[p_1\,'a^{-1}(\xi)\phi_3(\xi)+p_2\,'b^{-1}(\xi)\phi_2(\xi)+p_3\,'c^{-1}(\xi)\phi_1(\xi),r\Big]=\frac{1}{r}.\sqrt{\frac{2}{\pi}}\int_0^1\frac{xs_1(x)H(r-x)}{(r^2-x^2)^{1/2}}dx$$

And

$$H_1\Big[p_1a^{-1}(\xi)\phi_3(\xi)+p_2b^{-1}(\xi)\phi_2(\xi)+p_3c^{-1}(\xi)\phi_1(\xi),r\Big]=\sqrt{\frac{2}{\pi}}\int_0^1\frac{r_1(x)H(r-x)}{(r^2-x^2)^{1/2}}dx$$

Now substituting the values of $\phi_1(\xi)$ from (4.2) we get

$$H_0[\phi_1(\xi),r]=\int_0^1\frac{\beta(t)H(r-t)}{(r^2-t^2)^{1/2}}dt, r<1$$

And

$$H_0[\phi_1(\xi),r]=\Big(\frac{1}{r}\Big)\int_0^1\beta(t)dt, r>1$$

Now using (3) and (5), we obtain

$$\sigma_{rz}(r,0+)=\frac{-K\beta\sqrt{\frac{2}{\pi}}}{r}\frac{\partial}{\partial r}\int_0^1\frac{xs_1(x)}{(r^2-x^2)^{1/2}}dx+\int_0^1\frac{Ke_2\beta(t)}{r}dt \tag{5.6}$$

$$\sigma_{zz}(r,0+)=K\beta\Big(\sqrt{\frac{2}{\pi}}\Big)\frac{\partial}{\partial r}\int_0^1\frac{r_1(x)}{(r^2-x^2)^{1/2}}dx+\int_0^1\frac{Ke_2\beta(t)}{r}dt \tag{5.7}$$

## 6 STRESS INTENSITY FACTOR

Using the equation (5.7), (5.5) and (5.6) and component of stress can be written in the following form:

$$\begin{aligned}\sigma_{zz}(r,0+)=2K\cosh\pi\omega\Big[&-(Ar^4+Cr^2+C_3)J_2(C+2Ar^2)J_1-AJ_0+(D+Br^2)r^2I_2\\&-(D+2Br^2)I_2+BI_0\Big]-K'\int_0^1\frac{\beta(t)}{(r^2-t^2)^{1/2}}dt,\ r>1\end{aligned} \tag{6.1}$$

$$\begin{aligned}\sigma_{rz}(r,0+)=2K\cosh\pi\omega\Big[&-(D+Br^2)J_2-BJ_1+AI_0(2r^2+C)I_1+(Ar^4+Cr^2+C_3)I_2\Big]\\&+\frac{Ke_2}{r}\int_0^1\beta(t)dt\end{aligned} \tag{6.2}$$

When $I_K$ and $J_K$ have their usual values as given in (4.3.11a). In case $T(r) = T_0$, then using (4.1) we get

$$\int_0^1 \frac{\beta(t)}{(r^2-t^2)^{1/2}}dt = T_1 \sin^{-1}\left(\frac{1}{r}\right)$$

And

$$\int_0^1 \beta(t)dt = T_1$$

The integral $I_K$ and $J_K$ have been evaluated in [53] on putting $K = 0, 1, 2$ The stress component can be calculated from (6.1) and (6.2), it is a simple matter to show that $I_0, J_0, I_1, J_1$ are bounded

$$I_1 = \left|\frac{\cos\left[\omega \log(\frac{1+x}{1-x})\right]}{(r^2-x^2)}dx\right| \leq \int_0^1 \frac{dx}{(r^2-x^2)^{1/2}} = \sin^{-1}\left(\frac{1}{r}\right)$$

Since the integrals $I_0, J_0, I_1, J_1$ do not have any influence on the singularity of the stress at the crack rim, by virtue of the relation (6.1) and (6.2) we have when $r \to 1+$

$$\sigma_{zz}(r,0+) = \frac{2K\cosh\pi\omega}{\pi}\Big[(Br^4+Dr^2)I_2 - (Ar^4+Cr^2+C_3)J_2\Big] + 0(1) \tag{6.3}$$

$$\sigma_{rz}(r,0+) = \frac{2K\cosh\pi\omega}{\pi}\Big[(D+Br^2)I_2 - (Ar^4+Cr^2+C_3)J_2\Big] + 0(1) \tag{6.4}$$

From [53] we get

$$I_2 = \frac{\omega\pi}{(r^2-1)^{1/2}\sinh\pi\omega}(PX-QY) \tag{6.5}$$

$$I_2 = \frac{\omega\pi}{(r^2-1)^{1/2}\sinh\pi\omega}(PY-QX) \tag{6.6}$$

Where $X+iY, P+iQ$ and $\psi$ have their usual values as given in $X+iY = 2F_1[i\omega, 1/2, 1, 4r/(r+1)^2]P+iQ = \cos\psi\omega + \sin\psi\omega,\ \psi = \log\left(\frac{r-1}{r+1}\right)$

With the help of these equations, we obtain

$$\sigma_{zz}(r,0+) + i\sigma_{rz}(r,0+) = 2K\cosh\pi\omega[Br^3+Dr-i(Ar^4+Cr^2+c_3)] + (rI_2 - iJ_2) = \frac{K\omega\cot\pi\omega}{r(r^2-1)^{1/2}}$$
$$\Big[Br^3+Dr-i(Ar^4+Cr^2+C_3)\Big]$$

$$(X + +iY)(P + iQ)X(1 + r) + 0(r - 1)^{1/2} \tag{6.7}$$

The stress intensity factor is defined as

$$N_1 + iN_2 = \lim_{r \to 1} \left[ (r - 1)^{1/2} (\sigma_{zz} + i\sigma_{rz}) e^{-i\omega\psi} \right] \tag{6.8}$$

Hence we get

$$\begin{aligned} N_1 + iN_2 &= \frac{KT_1 e^{-\pi\omega}}{45\sqrt{2\pi}} \{5w(1 - \omega^2)e_2 + 45\pi e_1\} + i\{45\pi\omega e_1 - 4(\omega^4 - 20\omega^2 + 9)e_2\} \\ & \{\Gamma(1 + i\omega)(1/2 + i\omega)\} \end{aligned} \tag{6.9}$$

The stresses in the plan of the crack in the vicinity of the rim are given by

$$\sigma_{zz}(r, 0+) = \frac{1}{(r - 1)^{1/2}} [N_1 \cos\omega\psi - N_2 \cos\omega\psi] + 0(r - 1)^{1/2} \tag{6.10}$$

$$\sigma_{rz}(r, 0+) = \frac{1}{(r - 1)^{1/2}} [N_1 \cos\omega\psi - N_2 \cos\omega\psi] + 0(r - 1)^{1/2} \tag{6.11}$$

Hence stress components have the singularity of the form

$$\frac{1}{(r - 1)^{1/2}} \begin{matrix} \cos \\ \sin \end{matrix} \left[ \omega \log \left( \frac{r - 1}{r + 1} \right) \right] \textit{ at the rim to the crack.}$$

## 7 CONCLUSION

In this paper, we consider a problem of thermal stress in bonded dissimilar micropolar elastic half spaces containing a penny shaped crack at the interface. In this problem a homogenous isotropic medium uninterrupted except for the penny shaped crack, $z = 0, 0 < r \leq 1$ is considered.

The crack is opened out by the application of heat to its flats surfaces. The thermal conditions on the upper and lower surface of the crack are identical. We also assume that the crack is deformed by the application of axially pressure $p(r)$ in addition the heat flux. Then two cases arise:

(i) Crack is opened out by a prescribed heat flux across its flat surfaces.

(ii) Crack is opened out by the application of a prescribed temperature to its flat surfaces.

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