Coefficient Bounds for New Subclasses of Bi-Univalent Functions Associated With Pseudo-Starlike Functions

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Abstract

In the present investigation, we have introduced and studied two new subclasses of bi-univalent functions associated with pseudo-starlike functions in the disc $|z| < 1$. Furthermore, for these subclasses we find bounds on the initial coefficients $|a_2|$ and $|a_3|$.

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1 Introduction

Let $\mathcal{A}$ denote the class of all functions analytic in the disc $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ and have the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,. \tag{1.1}$$

We let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which consist of univalent functions in U. Also let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the well known classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$ respectively (for more details see [4]).

The Koebe one quarter theorem [4] asserts that the image of U under every univalent function $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. Thus every univalent function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}[f(z)] = z$, $(z \in U)$ and $f\left[f^{-1}(w)\right] = w$, $\left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$. It can also be easily verified that

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \,. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in U. We denote the class of bi-univalent functions in U given by (1.1) by Σ, (for more details see [16]).

This class Σ was first investigated by Lewin [11] and showed that $|a_2| < 1.51$. Brannan and Clunie [2] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$. Also in [10], Kedzierawski proved this conjecture for the starlike functions. On the other hand

Netanyahu [14], showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Further, Tan [17] proved the pre-eminent estimate $|a_2| \leq 1.485$ for functions in the class Σ. The coefficient estimate for each of the coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1,2\})$ for a subclass of Σ was given in 2013 by Jahangiri and Hamidi [7] using Faber polynomial expansions.

Brannan and Taha [3] introduced certain subclasses $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of strongly bi-starlike and bi-convex functions of order α respectively and found the bound on the initial coefficients $|a_2|$ and $|a_3|$ (for details see [3]).

Recently, several researchers introduced and investigated the various subclasses of bi-univalent functions and found the bounds on the initial coefficients $|a_2|$ and $|a_3|$ (see [5], [6], [7], [9], [12], [8], [13], [16]). Also, Babalola [1] defined the class $\mathfrak{L}_\lambda(\beta)$ of λ-pseudo starlike functions of order β and proved that all pseudo-starlike functions are Bazilevič of type $\left(1 - \frac{1}{\lambda}\right)$, order $\beta^{\frac{1}{\lambda}}$ and univalent in U.

Motivated by aforementioned work, we introduce the subclasses $B^\lambda_\Sigma(\alpha,\beta)$ and $B_\Sigma(\lambda,\beta,\gamma)$ of Σ and found the bounds on the initial coefficients $|a_2|$ and $|a_3|$. The techniques used are same as of Joshi *et al.*[8].

To establish our main results, we need the following Lemma.

Lemma 1.1. *[15]. Let $h \in \mathcal{P}$, where $\mathcal{P}$ is the family of all functions h analytic in U with $\Re\{h(z)\} > 0$ and have the form $h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \ \ldots$ for $z \in U$. Then $|h_n| \leq 2$ for each $n \in \mathbb{N}$.*

MAIN RESULTS

2 Coefficient estimates for the functions in the class $B^\lambda_\Sigma(\alpha,\beta)$

Definition 2.1. *A function f given by (1.1) is in the class $B^\lambda_\Sigma(\alpha,\beta)$ if it satisfies the following conditions :*

$$f \in \Sigma\ ,$$

$$\left|arg\left(\frac{z[f'(z)]^\lambda}{(1-\beta)f(z)+\beta z f'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in U) \tag{2.1}$$

$$\text{and} \quad \left|arg\left(\frac{w[g'(w)]^\lambda}{(1-\beta)g(w)+\beta w g'(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in U), \tag{2.2}$$

where $0 < \alpha \leq 1$, $0 \leq \beta < 1$, $\lambda \geq 1$, g is extension of f^{-1} to U defined in (1.2).

Now we state and prove the following theorem.

Theorem 2.1. *If f given by (1.1) is in the class $B^\lambda_\Sigma(\alpha,\beta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{[(\beta+1)^2 + \alpha(\beta^2 - 2\beta + 2\lambda - 1) + 4\lambda(\lambda - \beta - 1)]}} \tag{2.3}$$

and

$$|a_3| \leq \frac{2\alpha}{(3\lambda - 2\beta - 1)} + \frac{4\alpha^2}{(2\lambda - \beta - 1)^2}. \tag{2.4}$$

Proof. We write the inequalities (2.1) and (2.2) as

$$\frac{z[f'(z)]^\lambda}{(1-\beta)f(z) + \beta z f'(z)} = [p(z)]^\alpha \tag{2.5}$$

and

$$\frac{w[g'(w)]^\lambda}{(1-\beta)g(w) + \beta w g'(w)} = [q(w)]^\alpha \tag{2.6}$$

respectively, where the functions $p(z), q(w) \in \mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ \ldots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ \ldots \ .$$

Clearly,

$$[p(z)]^\alpha = 1 + \alpha p_1 z + \left(\alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2\right) z^2 + \ \ldots$$

and

$$[q(w)]^\alpha = 1 + \alpha q_1 w + \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2\right) w^2 + \ \ldots \ .$$

Also

$$\frac{z[f'(z)]^\lambda}{(1-\beta)f(z) + \beta z f'(z)} = 1 + (2\lambda - \beta - 1)a_2 z + [(2\lambda^2 + \beta^2 - 2\lambda\beta - 4\lambda + 2\beta + 1)a_2^2 + (3\lambda - 2\beta - 1)a_3]z^2 + \ \ldots$$

and

$$\frac{w[g'(w)]^\lambda}{(1-\beta)g(w) + \beta w g'(w)} = 1 - (2\lambda - \beta - 1)a_2 w + [(2\lambda^2 + \beta^2 - 2\lambda\beta + 2\lambda - 2\beta - 1)a_2^2 + (1 + 2\beta - 3\lambda)a_3]w^2 + \ldots \ .$$

Now equating the coefficients in (2.5) and (2.6), we have

$$(2\lambda - \beta - 1)a_2 = \alpha p_1 \ , \tag{2.7}$$

$$(2\lambda^2+\beta^2-2\lambda\beta-4\lambda+2\beta+1)a_2^2+(3\lambda-1-2\beta)a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2\ , \tag{2.8}$$

$$-(2\lambda-\beta-1)a_2=\alpha q_1\ , \tag{2.9}$$

$$(2\lambda^2+\beta^2-2\lambda\beta+2\lambda-2\beta-1)a_2^2+(1+2\beta-3\lambda)a_3=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\ . \tag{2.10}$$

From equations (2.7) and (2.9) we have

$$-p_1=q_1 \tag{2.11}$$

and

$$2(2\lambda-\beta-1)^2a_2^2=\alpha^2(p_1^2+q_1^2)\ . \tag{2.12}$$

By adding equations (2.8) and (2.10), we get

$$(4\lambda^2+2\beta^2-4\lambda\beta-2\lambda)a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right).$$

By using (2.12), we get

$$(4\lambda^2+2\beta^2-4\lambda\beta-2\lambda)a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(\frac{2(2\lambda-\beta-1)^2a_2^2}{\alpha^2}\right)$$

$$\Rightarrow a_2^2=\frac{\alpha^2(p_2+q_2)}{[(\beta+1)^2+\alpha(\beta^2-2\beta+2\lambda-1)+4\lambda(\lambda-\beta-1)]}.$$

By applying Lemma 1.1 for p_2 and q_2 we have

$$|a_2|\leq\frac{2\alpha}{\sqrt{[(\beta+1)^2+\alpha(\beta^2-2\beta+2\lambda-1)+4\lambda(\lambda-\beta-1)]}}.$$

This is the bound on $|a_2|$ as given in (2.3).
Now to find the bound on $|a_3|$, we subtract (2.10) from (2.8) then we have

$$(4\beta-6\lambda+2)a_2^2+2(3\lambda-2\beta-1)a_3=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right).$$

From (2.11) we get $p_1^2=q_1^2$ and also using (2.12) we have

$$a_3=\frac{\alpha(p_2-q_2)}{2(3\lambda-2\beta-1)}+\frac{\alpha^2p_1^2}{(2\lambda-\beta-1)^2}.$$

Again by applying Lemma 1.1 for p_1, p_2 and q_2 we get

$$|a_3|\leq\frac{2\alpha}{(3\lambda-2\beta-1)}+\frac{4\alpha^2}{(2\lambda-\beta-1)^2}.$$

This is bound on $|a_3|$ as given in (2.4). Which established the Theorem 2.1.

3 Coefficient estimates for the functions in the class $B_\Sigma(\lambda, \beta, \gamma)$

Definition 3.1. *A function f given by (1.1) is in the class $B_\Sigma(\lambda, \beta, \gamma)$ if it satisfies the following conditions :*

$$f \in \Sigma \ ,$$

$$\Re\left(\frac{z[f'(z)]^\lambda}{(1-\beta)f(z)+\beta z f'(z)}\right) > \gamma \tag{3.1}$$

$$\text{and} \quad \Re\left(\frac{w[g'(w)]^\lambda}{(1-\beta)g(w)+\beta w g'(w)}\right) > \gamma \ , \tag{3.2}$$

where $z \in U$, $w \in U$, $0 \le \gamma < 1$, $0 \le \beta < 1$, $\lambda \ge 1$ and g is defined in (1.2).

The following coefficient estimate holds for bi-univalent functions in the class $B_\Sigma(\lambda, \beta, \gamma)$.

Theorem 3.1. *If function f given by (1.1) is in the class $B_\Sigma(\lambda, \beta, \gamma)$, then*

$$| a_2 | \le \sqrt{\frac{2(1-\gamma)}{\beta^2 + 2\lambda^2 - 2\lambda\beta - \lambda}} \tag{3.3}$$

and

$$| a_3 | \le \frac{2(1-\gamma)}{(3\lambda - 2\beta - 1)} + \frac{4(1-\gamma)^2}{(2\lambda - \beta - 1)^2}. \tag{3.4}$$

Proof. We write inequalities (3.1) and (3.2) as :

$$\frac{z[f'(z)]^\lambda}{(1-\beta)f(z)+\beta z f'(z)} = \gamma + (1-\gamma)p(z) \tag{3.5}$$

and

$$\frac{w[g'(w)]^\lambda}{(1-\beta)g(w)+\beta w g'(w)} = \gamma + (1-\gamma)q(w) \tag{3.6}$$

respectively.
Where $p(z)$, $q(w) \in \mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots \quad .$$

Clearly,

$$\gamma + (1-\gamma)p(z) = 1 + (1-\gamma)p_1 z + (1-\gamma)p_2 z^2 + \ldots$$

and

$$\gamma + (1-\gamma)q(w) = 1 + (1-\gamma)q_1 w + (1-\gamma)q_2 w^2 + \ldots \quad .$$

Also

$$\frac{z[f'(z)]^\lambda}{(1-\beta)f(z) + \beta z f'(z)} = 1 + (2\lambda - \beta - 1)a_2 z + [(2\lambda^2 + \beta^2 - 2\lambda\beta - 4\lambda + 2\beta + 1)a_2^2 + (3\lambda - 2\beta - 1)a_3]z^2 + \ldots$$

and

$$\frac{w[g'(w)]^\lambda}{(1-\beta)g(w) + \beta w g'(w)} = 1 - (2\lambda - \beta - 1)a_2 w + [(2\lambda^2 + \beta^2 - 2\lambda\beta + 2\lambda - 2\beta - 1)a_2^2 + (1 + 2\beta - 3\lambda)a_3]w^2 + \ldots \quad .$$

Equating the coefficients in (3.5) and (3.6), we get

$$(2\lambda - \beta - 1)a_2 = (1-\gamma)p_1 \ , \tag{3.7}$$

$$(2\lambda^2 + \beta^2 - 2\lambda\beta - 4\lambda + 2\beta + 1)a_2^2 + (3\lambda - 2\beta - 1)a_3 = (1-\gamma)p_2 \ , \tag{3.8}$$

$$-(2\lambda - \beta - 1)a_2 = (1-\gamma)q_1 \ , \tag{3.9}$$

$$(2\lambda^2 + \beta^2 - 2\lambda\beta + 2\lambda - 2\beta - 1)a_2^2 + (2\beta - 3\lambda + 1)a_3 = (1-\gamma)q_2 \ . \tag{3.10}$$

From equations (3.7) and (3.9), we have

$$-p_1 = q_1 \tag{3.11}$$

and

$$2(2\lambda - \beta - 1)^2 a_2^2 = (1-\gamma)^2(p_1^2 + q_1^2) \ . \tag{3.12}$$

Now, by adding equations (3.8) and (3.10), we get

$$(4\lambda^2 + 2\beta^2 - 4\lambda\beta - 2\lambda)a_2^2 = (1-\gamma)(p_2 + q_2)$$

$$\Rightarrow |a_2^2| \le \frac{(1-\gamma)(|p_2| + |q_2|)}{2(2\lambda^2 + \beta^2 - 2\lambda\beta - \lambda)} \ .$$

Now by applying Lemma 1.1 for p_2 and q_2, we have

$$| a_2 | \leq \sqrt{\frac{2(1-\gamma)}{\beta^2 + 2\lambda^2 - 2\lambda\beta - \lambda}}.$$

Which is the bound on $|a_2|$ as given in (3.3).
Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$(4\beta - 6\lambda + 2)a_2^2 + (6\lambda - 4\beta - 2)a_3 = (1-\gamma)(p_2 - q_2).$$

$$a_3 = \frac{(1-\gamma)(p_2 - q_2)}{2(3\lambda - 2\beta - 1)} + a_2^2.$$

From (3.11), we get $p_1^2 = q_1^2$ and also using (3.12) we have

$$a_3 = \frac{(1-\gamma)(p_2 - q_2)}{2(3\lambda - 2\beta - 1)} + \frac{(1-\gamma)^2 p_1^2}{(2\lambda - \beta - 1)^2}$$

Again applying Lemma 1.1 for p_1, p_2 and q_2 we get

$$| a_3 | \leq \frac{2(1-\gamma)}{(3\lambda - 2\beta - 1)} + \frac{4(1-\gamma)^2}{(2\lambda - \beta - 1)^2}.$$

This is bound on $| a_3 |$ as asserted in (3.4). Which established the Theorem 3.1.
Remark. By specializing the parameters we get the results which were established by [8] and [13].
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