A new class of p-harmonic functions

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Abstract

A linear operator $\mathcal{L}^n$ on the class S_{H_p} of p-harmonic functions is considered and by involving this operator, a new class $S_{H_p}(m, n, \alpha)$ of p-harmonic functions is defined and studied. We obtain a sufficient coefficient inequality for functions $F \in S_{H_p}$ to be sense preserving and univalent in $\mathbb{D}$ and to belong to the class $S_{H_p}(m, n, \alpha)$. It is shown that this sufficient coefficient inequality is necessary for functions belonging to a subclass $\overline{S_{H_p}}(m, n, \alpha)$. On using the necessary and sufficient coefficient inequality for the class $\overline{S_{H_p}}(m, n, \alpha)$, it is shown that this class is convex under linear combination. The extreme points for this class are also given.

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1 Introduction

A complex-valued function $f = u + iv$ is said to be a harmonic map in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ if u and v are real-valued harmonic functions in $\mathbb{D}$. A harmonic function f satisfy the Laplace equation $\Delta f = 0$, where

$$\Delta = \frac{4\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and it can also be expressed as $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. The Jacobian of the function $f = h + \overline{g}$ is given by $J_f = |f_z|^2 - |f_{\overline{z}}|^2$. According to the Lewy [3], every harmonic function $f = h + \overline{g}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $J_f > 0$ in $\mathbb{D}$. A complex-valued mapping F in a domain $\mathbb{D}$ is called *p-harmonic* (or *multi-harmonic* or *poly-harmonic*) if F satisfies the equation

$$\underbrace{\Delta ... \Delta}_{p} F = \Delta^p F = \Delta(\Delta^{p-1} F) = 0$$

for some $p \in \mathbb{N} = \{1, 2, 3, ...\}$. A function F which is p-harmonic in the unit disk $\mathbb{D}$ has the representation

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} f_{p-k+1}(z), \tag{1.1}$$

where $f_{p-k+1}(z)$ is harmonic in $\mathbb{D}$ for each $k \in \{1, ..., p\}$ (see [1]). Clearly, a p-harmonic map is harmonic.

Let S_{H_p} denotes a class of all p-harmonic mappings of the form (1.1) in the unit disk $\mathbb{D}$, where for each $k \in \{1, ..., p\}$

$$f_{p-k+1}(z) = h_{p-k+1}(z) + \overline{g_{p-k+1}(z)} \tag{1.2}$$

and

$$h_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j, \; g_{p-k+1}(z) = \sum_{j=1}^{\infty} b_{j,p-k+1} z^j$$
$$(a_{1,p} = 1, |b_{1,p}| < 1; z \in \mathbb{D}).$$

Note that for $p = 1$ and $p = 2$, the mapping of the form (1.1) is called, respectively a harmonic and a bi-harmonic map. The p-harmonic mappings have been studied by several researchers [5, 7, 8, 9] (see also [2, 6]).

We denote a subclass of S_{H_p} by $\overline{S_{H_p}}$ if functions therein are of the form

$$F(z) = 2z - \sum_{k=1}^{p} \sum_{j=1}^{\infty} |z|^{2(k-1)} \left(|a_{j,p-k+1}| z^j + |b_{j,p-k+1}| \overline{z}^j \right) \tag{1.3}$$

$$(a_{1,p} = 1, |b_{1,p}| < 1; z \in \mathbb{D})$$

which may also be given by

$$F(z) = z - \sum_{j=2}^{\infty} |a_{j,p}| z^j - \sum_{j=1}^{\infty} |b_{j,p}| \overline{z}^j - \sum_{k=2}^{p} \sum_{j=1}^{\infty} |z|^{2(k-1)} \left(|a_{j,p-k+1}| z^j + |b_{j,p-k+1}| \overline{z}^j \right). \tag{1.4}$$

Motivated with the operator defined by Li and Liu [4] (see also [9]), we consider an operator $\mathcal{L}^n : S_{H_p} \to S_{H_p}$ which is defined for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by

$$\mathcal{L}^0 F(z) = F(z), \mathcal{L}^1 F(z) = \mathcal{L} F(z) = [z(F(z))_z + \bar{z}(F(z))_{\bar{z}}]$$
$$\text{and } \mathcal{L}^n F(z) = \mathcal{L}^1(\mathcal{L}^{n-1} F(z)), n \in \mathbb{N}.$$

Associated with the operator $\mathcal{L}^n$, we define a class $S_{H_p}(m, n, \alpha)$ of functions $F \in S_{H_p}$ which satisfy the condition

$$Re \left\{ \frac{\mathcal{L}^m F(z)}{\mathcal{L}^n F(z)} \right\} > \alpha, \tag{1.5}$$

where $m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, 0 \leq \alpha < 1$ and at $z = 0$, $\frac{\mathcal{L}^m F(z)}{\mathcal{L}^n F(z)} = 1$.

For a p-harmonic mapping F of the form (1.1), we have

$$\mathcal{L}^n F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} |z|^{2(k-1)} \{j + 2(k-1)\}^n \left(a_{j,p-k+1} z^j + \overline{b_{j,p-k+1} z^j} \right) \tag{1.6}$$

$$(a_{1,p} = 1, |b_{1,p}| < 1; z \in \mathbb{U}).$$

Denote $\overline{S_{H_p}}(m,n,\alpha) = S_{H_p}(m,n,\alpha) \cap \overline{S_{H_p}}$. The class $S_{H_p}(n+1,n,\alpha)$ reduces to the class denoted by $SH_p(n,\lambda,\alpha)$ studied by Yaşar and Yalçin [9] for special value $\lambda = 1$.

In this paper, a linear operator $\mathcal{L}^n$ for the class S_{H_p} of p-harmonic functions is considered and by involving this operator, a new class $S_{H_p}(m,n,\alpha)$ of p-harmonic functions is defined and studied. In our main results, we obtain a sufficient coefficient inequality for functions $F \in S_{H_p}$ to be sense preserving and univalent in $\mathbb{D}$ and belonging to the class $S_{H_p}(m,n,\alpha)$. It is shown that this sufficient coefficient inequality is necessary for functions belonging to a subclass $\overline{S_{H_p}}(m,n,\alpha)$ of the class $S_{H_p}(m,n,\alpha)$. On using the necessary and sufficient coefficient inequality for the class $\overline{S_{H_p}}(m,n,\alpha)$, it is shown that this class is convex under linear combination. The extreme points for this class are also given.

2 Main Results

In our first main result, we show first a necessary condition for functions to be univalent, sense-preserving and in the class $S_{H_p}(m,n,\alpha)$.

Theorem 2.1. *Let $F \in S_{H_p}$ be a p-harmonic function of the form (1.1) and let*

$$\sum_{k=1}^{p} \sum_{j=1}^{\infty} \left[\{j + 2(k-1)\}^m - \alpha \{j + 2(k-1)\}^n \right] \left(|a_{j,p-k+1}| + |b_{j,p-k+1}| \right) \leq 2(1-\alpha), \tag{2.1}$$

where $m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, 0 \leq \alpha < 1, a_{1,p} = 1, |b_{1,p}| < 1$. Then F is sense preserving and univalent in $\mathbb{D}$ and $F \in S_{H_p}(m,n,\alpha)$.

Proof. Suppose $z_1, z_2 \in \mathbb{D}$ $(z_1 \neq z_2)$, $|z_1| \leq |z_2| < 1$. Then by using (2.1), we get

$$|F(z_1) - F(z_2)| = \left| \sum_{k=1}^{p} \left\{ |z_1|^{2(k-1)} f_{p-k+1}(z_1) - |z_2|^{2(k-1)} f_{p-k+1}(z_2) \right\} \right|$$
$$\geq |z_1 - z_2| \left\{ 1 - \sum_{j=2}^{\infty} |a_{j,p}| \frac{|z_1^j - z_2^j|}{|z_1 - z_2|} - \sum_{j=1}^{\infty} |b_{j,p}| \frac{|z_1^j - z_2^j|}{|z_1 - z_2|} \right.$$
$$\left. - \sum_{k=2}^{p} \sum_{j=1}^{\infty} |z_2|^{2(k-1)} \frac{|z_1^j - z_2^j|}{|z_1 - z_2|} \left(|a_{j,p-k+1} + |b_{j,p-k+1}| \right) \right\}$$
$$> |z_1 - z_2| \left\{ 1 - \sum_{j=2}^{\infty} j\,|a_{j,p}| - \sum_{j=1}^{\infty} j|b_{j,p}| \right.$$
$$\left. - \sum_{k=2}^{p} \sum_{j=1}^{\infty} j \left(|a_{j,p-k+1}| + |b_{j,p-k+1}| \right) \right\}$$
$$\geq |z_1 - z_2| \left[2 - \sum_{k=1}^{p} \sum_{j=1}^{\infty} \frac{[\{j + 2(k-1)\}^m - \alpha \{j + 2(k-1)\}^n]}{1 - \alpha} \left(|a_{j,p-k+1}| + |b_{j,p-k+1}| \right) \right]$$
$$\geq 0$$

which proves that each mapping in $S_{H_p}(m, n, \alpha)$ is univalent. Now to show F is sense-preserving, we need to show that the Jacobian:

$$J_F(z) = \left(\left| \frac{\partial}{\partial z} F(z) \right| - \left| \frac{\partial}{\partial \overline{z}} F(z) \right| \right) \left(\left| \frac{\partial}{\partial z} F(z) \right| + \left| \frac{\partial}{\partial \overline{z}} F(z) \right| \right) > 0,$$

We get for $z \neq 0$,

$$\begin{aligned}
&\left|\frac{\partial}{\partial z}F(z)\right| - \left|\frac{\partial}{\partial \overline{z}}F(z)\right| \\
&= \left|1 + \sum_{j=2}^{\infty} j a_{j,p} z^{j-1} + \sum_{k=2}^{p}\sum_{j=1}^{\infty} |z|^{2(k-1)}\,(j+k-1)a_{j,p-k+1} z^{j-1}\right. \\
&\left. + \sum_{k=1}^{p} \frac{|z|^{2(k-1)}}{z} \sum_{j=1}^{\infty} (k-1)\,\bar{b}_{j,p-k+1}\overline{z}^{j}\right| \\
&- \left|\sum_{k=1}^{p} \frac{|z|^{2(k-1)}}{\overline{z}} \sum_{j=1}^{\infty} (k-1)\,a_{j,p-k+1} z^{j} + \sum_{k=1}^{p}\sum_{j=1}^{\infty} |z|^{2(k-1)}\,(j+k-1)\bar{b}_{j,p-k+1}\overline{z}^{j-1}\right|. \\
&> 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \{j+2(k-1)\}\,(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \\
&\geq 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \frac{[\{j+2(k-1)\}^{m} - \alpha\,\{j+2(k-1)\}^{n}]}{1-\alpha}\,(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \\
&\geq 0
\end{aligned}$$

by using the inequality (2.1). This proves that the mapping F is sense preserving. Now to show $F \in S_{H_p}(m,n,\alpha)$, we need to show the condition (1.5) which is trivial at $z = 0$. Let $0 \neq z \in \mathbb{D}$. Then, to show the condition (1.5) we let

$$\frac{\mathcal{L}^{m}F(z) - \alpha\mathcal{L}^{n}F(z)}{\mathcal{L}^{n}F(z)} = (1-\alpha)\,\frac{1+w(z)}{1-w(z)}$$

and then, we only need to show

$$|w(z)| = \left|\frac{\mathcal{L}^{m}F(z) - \mathcal{L}^{n}F(z)}{\mathcal{L}^{m}F(z) + (1-2\alpha)\,\mathcal{L}^{n}F(z)}\right| < 1.$$

Using (1.6), we get

$$\begin{aligned}
&|w(z)| \\
= &\left|\frac{\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}|z|^{2(k-1)}\left[\{j+2(k-1)\}^m-\{j+2(k-1)\}^n\right]\left(a_{j,p-k+1}z^j+\overline{b_{j,p-k+1}z^j}\right)}{\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}|z|^{2(k-1)}\left[\{j+2(k-1)\}^m+(1-2\alpha)\{j+2(k-1)\}^n\right]\left(a_{j,p-k+1}z^j+\overline{b_{j,p-k+1}z^j}\right)}\right| \\
< &\frac{\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}\left[\{j+2(k-1)\}^m-\{j+2(k-1)\}^n\right](|a_{j,p-k+1}|+|b_{j,p-k+1}|)}{4(1-\alpha)-\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}\left[\{j+2(k-1)\}^m+(1-2\alpha)\{j+2(k-1)\}^n\right](|a_{j,p-k+1}|+|b_{j,p-k+1}|)} \\
\leq &\ 1
\end{aligned}$$

by using the inequality (2.1). This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $F \in \overline{S_{H_p}}$ be of the form (1.3). Then $F \in \overline{S_{H_p}}(m,n,\alpha)$ if and only if (2.1) holds.*

Proof. The class $\overline{S_{H_p}}(m,n,\alpha) \subset S_{H_p}(m,n,\alpha)$, the "if part" is proved in Theorem 2.1. For "only if" part, we consider functions $F \in \overline{S_{H_p}}$ of the form (1.3) satisfying the class condition (1.5) for all $z \in \mathbb{D}$. Choosing a real positive $z \to 1^-$, the class condition (1.5) is given by

$$\begin{aligned}
&\frac{2(1-\alpha)-\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}\left[\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n\right](|a_{j,p-k+1}|+|b_{j,p-k+1}|)}{2-\sum\limits_{k=1}^{p}\sum\limits_{j=1}^{\infty}\{j+2(k-1)\}^n(|a_{j,p-k+1}|+|b_{j,p-k+1}|)} \\
&> 0
\end{aligned}$$

which easily proves the inequality (1.1). □

Theorem 2.3. *The family $\overline{S_{H_p}}(m,n,\alpha)$ is closed under convex combinations.*

Proof. Suppose for $i = 1, 2 \ldots$, $F^i \in \overline{S_{H_p}}(m,n,\alpha)$ be of the form

$$F^i(z) = 2z - \sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}\left(\left|a^i_{j,p-k+1}\right|z^j + |b^j_{j,p-k+1}|\overline{z}^j\right), a^i_{1,p} = 1, \left|b^i_{1,p}\right| < 1.$$

Then by Theorem 2.2 we have

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\frac{\left[\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n\right]}{2(1-\alpha)}\left(|a^i_{j,p-k+1}|+|b^i_{j,p-k+1}|\right) \leq 1. \tag{2.2}$$

For $t_i \in [0,1]$, with $\sum\limits_{i=1}^{\infty} t_i = 1$, the convex combination of $F^i(z)$ may be written as

$$\sum_{i=1}^{\infty} t_i F^i(z) = 2z - \sum_{k=1}^{p}\sum_{j=1}^{\infty} |z|^{2(k-1)} \sum_{i=1}^{\infty} t_i \left(\left|a^i_{j,p-k+1}\right| z^j + |b^i_{j,p-k+1}| \overline{z}^j \right).$$

Using (2.2), we get

$$\begin{aligned} & \sum_{k=1}^{p}\sum_{j=1}^{\infty} \frac{\left[\{j+2(k-1)\}^m - \alpha\{j+2(k-1)\}^n\right]}{2(1-\alpha)} \sum_{i=1}^{\infty} t_i \left(|a^i_{j,p-k+1}| + |b^i_{j,p-k+1}| \right) \\ = \; & \sum_{i=1}^{\infty} t_i \sum_{k=1}^{p}\sum_{j=1}^{\infty} \frac{\left[\{j+2(k-1)\}^m - \alpha\{j+2(k-1)\}^n\right]}{2(1-\alpha)} \left(|a^i_{j,p-k+1}| + |b^i_{j,p-k+1}| \right) \\ \leq \; & \sum_{i=1}^{\infty} t_i = 1 \end{aligned}$$

which by Theorem 2.2 proves that $\sum_{i=1}^{\infty} t_i F^i \in \overline{S_{H_p}}(m,n,\alpha)$. □

Remark 2.1. *The inequality (2.1) may also be given by*

$$\begin{aligned} & \sum_{j=2}^{\infty} (j^m - \alpha j^n) |a_{j,p}| + \sum_{j=1}^{\infty} (j^m - \alpha j^n) |b_{j,p}| \\ & + \sum_{k=2}^{p}\sum_{j=1}^{\infty} \left[\{j+2(k-1)\}^m - \alpha\{j+2(k-1)\}^n\right] \left(|a_{j,p-k+1}| + |b_{j,p-k+1}| \right) \\ \text{(2.3)} \quad \leq \; & 1-\alpha \end{aligned}$$

or

$$\begin{aligned} & \sum_{k=1}^{p} \left\{ (2k-1)^m - \alpha (2k-1)^n \right\} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}| \right) \\ & + \sum_{k=1}^{p}\sum_{j=2}^{\infty} \left[\{j+2(k-1)\}^m - \alpha\{j+2(k-1)\}^n\right] \left(|a_{j,p-k+1}| + |b_{j,p-k+1}| \right) \\ \text{(2.4)} \quad \leq \; & 2(1-\alpha). \end{aligned}$$

Remark 2.2. *The coefficient inequality (2.4) for $\alpha = 0$ and for $m = 1, n = 0$, was considered as a class condition for p-harmonic functions by Qiao and Wang in [8].*

Theorem 2.4. *Let $F \in \overline{S_{H_p}}(m,n,\alpha)$. Then for $|z| = r\ (<1)$*

$$
\begin{aligned}
|F(z)| \leq{}& (1+|b_{1,p}|)\, r \\
&+ \left(\frac{2(1-\alpha)}{2^m - 2^n\alpha} - \sum_{k=1}^{p} \frac{\{(2k-1)^m - \alpha\,(2k-1)^n\}}{2^m - 2^n\alpha} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}|\right) \right) r^2 \\
&+ \left(\sum_{k=2}^{p} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}|\right) \right) r^3 \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
|F(z)| \geq{}& (1-|b_{1,p}|)\, r \\
&- \left(\frac{2(1-\alpha)}{2^m - 2^n\alpha} - \sum_{k=1}^{p} \frac{\{(2k-1)^m - \alpha\,(2k-1)^n\}}{2^m - 2^n\alpha} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}|\right) \right) r^2 \\
&- \left(\sum_{k=2}^{p} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}|\right) \right) r^3. \qquad (2.6)
\end{aligned}
$$

Proof. Let $F \in \overline{S_{H_p}}(m,n,\alpha)$ be of the form (1.3). Then for $|z| = r\ (<1)$

$$
\begin{aligned}
|F(z)| ={}& \Bigg| z - |b_{1,p}|\,\overline{z} - \sum_{k=2}^{p} |z|^{2(k-1)} \left(|a_{1,p-k+1}|\, z + |b_{1,p-k+1}|\, \overline{z}\right) \\
&- \sum_{k=1}^{p} \sum_{j=2}^{\infty} |z|^{2(k-1)} \left(|a_{j,p-k+1}|\, z^j + |b_{j,p-k+1}|\, \overline{z}^j\right) \Bigg| \\
\leq{}& (1+|b_{1,p}|)\, r + \left(\sum_{k=2}^{p} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}|\right) \right) r^3 \\
&+ \left(\sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(|a_{j,p-k+1}| + |b_{j,p-k+1}|\right) \right) r^2 \\
\leq{}& (1+|b_{1,p}|)\, r + \left(\sum_{k=2}^{p} \left(|a_{1,p-k+1}| + |b_{1,p-k+1}|\right) \right) r^3 \\
&+ \frac{r^2}{2^m - 2^n\alpha} \sum_{k=1}^{p} \sum_{j=2}^{\infty} \left[\{j+2(k-1)\}^m - \alpha\,\{j+2(k-1)\}^n\right] \left(|a_{j,p-k+1}| + |b_{j,p-k+1}|\right)
\end{aligned}
$$

which by coefficient inequality (2.4) proves the result (2.5). Similarly, we can prove (2.6). □

Theorem 2.5. *Let* $F \in \overline{S_{H_p}}$*. Then* $F \in \overline{S_{H_p}}(m, n, \alpha)$ *if and only if*

$$F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} \{x_{j,p-k+1} h_{j,p-k+1}(z) + y_{j,p-k+1} g_{j,p-k+1}(z)\}, \tag{2.7}$$

where

$$\begin{aligned} h_{1,p}(z) &= z, \ h_{j,p}(z) = z - \frac{1-\alpha}{j^m - \alpha j^n} z^j \quad (j \geq 2), \\ g_{j,p}(z) &= z - \frac{1-\alpha}{j^m - \alpha j^n} \overline{z}^j \quad (j \geq 1), \\ h_{j,p-k+1}(z) &= z - |z|^{2(k-1)} \frac{1-\alpha}{\{j+2(k-1)\}^m - \alpha \{j+2(k-1)\}^n} z^j, \\ g_{j,p-k+1}(z) &= z - |z|^{2(k-1)} \frac{1-\alpha}{\{j+2(k-1)\}^m - \alpha \{j+2(k-1)\}^n} \overline{z}^j \end{aligned} \tag{2.8}$$

for $2 \leq k \leq p, j \geq 1$ *and*

$$\sum_{k=1}^{p} \sum_{j=1}^{\infty} (x_{j,p-k+1} + y_{j,p-k+1}) = 1 \ \ (x_{j,p-k+1} \geq 0, y_{j,p-k+1} \geq 0).$$

In particular, the extreme points of $\overline{S_{H_p}}(m, n, \alpha)$ *are* $\{h_{j,p-k+1}\}$ *and* $\{g_{j,p-k+1}\}$.

Proof. Let F be expressed by (2.7). Then on using (2.8), we obtain

$$\begin{aligned} F(z) &= \sum_{k=1}^{p} \sum_{j=1}^{\infty} \{x_{j,p-k+1} h_{j,p-k+1}(z) + y_{j,p-k+1} g_{j,p-k+1}(z)\} \\ &= z - \sum_{j=2}^{\infty} \frac{1-\alpha}{j^m - \alpha j^n} x_{j,p} z^j - \sum_{j=1}^{\infty} \frac{1-\alpha}{j^m - \alpha j^n} y_{j,p} \overline{z}^j \\ &\quad - \sum_{k=2}^{p} \sum_{j=1}^{\infty} |z|^{2(k-1)} \frac{1-\alpha}{\{j+2(k-1)\}^m - \alpha \{j+2(k-1)\}^n} \\ &\quad \times \left(x_{j,p-k+1} z^j + y_{j,p-k+1} \bar{z}^j\right) \end{aligned}$$

which satisfy the coefficient inequality (2.3)

$$\begin{aligned}
& \sum_{j=2}^{\infty} \frac{j^m-\alpha j^n}{1-\alpha} \times \frac{1-\alpha}{j^m-\alpha j^n} x_{j,p} \\
& + \sum_{j=1}^{\infty} \frac{j^m-\alpha j^n}{1-\alpha} \times \frac{1-\alpha}{j^m-\alpha j^n} y_{j,p} \\
& + \sum_{k=2}^{p} \sum_{j=1}^{\infty} \frac{\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n}{1-\alpha} \\
& \times \frac{1-\alpha}{\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n} (x_{j,p-k+1}+y_{j,p-k+1}) \\
= \; & 1-x_{1,p} \leq 1.
\end{aligned}$$

So $F \in \overline{S_{H_p}}(m,n,\alpha)$. Conversely, let $F \in \overline{S_{H_p}}(m,n,\alpha)$ be of the form (1.4), then by the inequality (2.3)

$$\begin{aligned}
|a_{j,p}| &\leq \frac{1-\alpha}{j^m-\alpha j^n} \quad (j \geq 2), \quad |b_{j,p}| \leq \frac{1-\alpha}{j^m-\alpha j^n} \quad (j \geq 1), \\
|a_{j,p-k+1}| &\leq \frac{1-\alpha}{\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n} \quad (j \geq 1, 2 \leq k \leq p), \\
|b_{j,p-k+1}| &\leq \frac{1-\alpha}{[(j+2(k-1))^{m-n}-\alpha][j+2(k-1)]^n} \quad (j \geq 1, 2 \leq k \leq p).
\end{aligned}$$

Set

$$\begin{aligned}
x_{j,p} &= \frac{j^m-j^n\alpha}{1-\alpha}|a_{j,p}| \quad (j \geq 2), \quad y_{j,p} = \frac{j^m-j^n\alpha}{1-\alpha}|b_{j,p}| \quad (j \geq 1), \\
x_{j,p-k+1} &= \frac{\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n}{1-\alpha}|a_{j,p-k+1}| \quad (j \geq 1, 2 \leq k \leq p), \\
y_{j,p-k+1} &= \frac{\{j+2(k-1)\}^m-\alpha\{j+2(k-1)\}^n}{1-\alpha}|b_{j,p-k+1}| \quad (j \geq 1, 2 \leq k \leq p)
\end{aligned}$$

and

$$x_{1,p} = 1 - \sum_{j=2}^{\infty} x_{j,p} - \sum_{j=1}^{\infty} y_{j,p} - \sum_{k=2}^{p}\sum_{j=1}^{\infty}(x_{j,p-k+1}+y_{j,p-k+1}) \geq 0.$$

Then, can easily be proved that $F(z)$ is given by (2.7). □

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