Uniqueness of Entire functions sharing one value IM

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Abstract

In this paper, we study the behavior of uniqueness of entire functions of differential polynomials sharing one value IM. The results of this paper improves the previous results of R.S. Dyavanal and R.v. Desai[8], X.Shi[11], X.Qi and J. Dou[13], J.Zhang[14] and many others.

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1 Introduction:

In this paper, by meromorphic functions we will always mean meromorphic in the complex plane C. We use the standard notations of the Nevanlinna's theory of meromorphic functions such as $T(r,f)$,$m(r,f)$,$N(r,f)$ and $S(r,f)$ as explained in [1],[2]etc. In addition, for a meromorphic function $f(z)$ we use $\delta(a,f)$ to denote the Nevanlinna deficiency of $a \in C \cup \{\infty\}$,where $\delta(a,f) = \limsup_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)}$.
Let f and g be two non-constant meromorphic functions and a is a point in C. We say that f and g share the point a CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. similarly we say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities.Let k be an integer. We denote by $N_{k)}(r,\frac{1}{f-a})$ the counting function for zeros of $f-a$ with multiplicity not greater than k and by $\overline{N}_{k)}(r,\frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Now we set, $N_k(r,\frac{1}{f-a}) = \overline{N}(r,\frac{1}{f-a}) + \overline{N}_{(2}(r,\frac{1}{f-a}) + ... + \overline{N}_{(k}(r,\frac{1}{f-a})$.
In the recent years many authors paid their attention to the developement of difference analogue of Nevanlinna theory and a few number of papers have been published on the value distribution and uniqueness of differences and difference operator of Nevanlinna theory.

In 2010, J.Zhang[14] proved an interesting result, whicch ststes that–

Theorem A: Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and n is an integer. If $n \geq 7$, then $f(z)^n(f(z)-1)f(z+c)$ and $g(z)^n(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) = g(z)$.

In 2013, X.Qi and J.Dou[13] obtained another result replacing CM by IM in the above mentioned theorem–
Theorem B: Let f and g be transcendental entire functions with finite order, let c be a non-zero complex constant, and let $n \geq 16$ be an integer. If $f(z)^n(f(z)-1)f(z+c)$ and $g(z)^n(g(z)-1)g(z+c)$ share a IM, where a is a non-zero small function to f and g, then $f(z) = g(z)$.

In 2014, R.S.Dyavanal and R.V. Desai[8] proved the following theorem–
Theorem C: Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose thst c is a non-zero complex constant. $k \geq 1$, $n \geq k+6$, if $f^n(z)(f(z)-1)^k f(z+c)$ and $g^n(z)(g(z)-1)^k g(z+c)$ share $\alpha(z)$ CM, then $f(z) = tg(z)$, where $t^k = 1$.

In the same paper R.S.Dyavanal and R.V. Desai[8] posed some open questions.
In this paper, we give a positive answer to the second question by replacing CM by IM. Though X.Qi and J. Dou[13] proved the result in one way, but we take more general polynomials to prove it and also we prove in a different way.
Here,we prove the following results which improve Theorem A, Theorem B, Theorem C and many more results.

2 Main Results:

Theorem 2.1: Let f and g be transcendental entire functions with finite order. Suppose that c is a non-zero complex constant and k, n be two positive integers such that $n \geq 16$ if $k = 1$ and $n \geq 19 - k$ if $k \geq 2$. If $f(z)^n(f(z)-1)^k f(z+c)$ and $g(z)^n(g(z)-1)^k g(z+c)$ share 1 IM, then $f(z) = tg(z)$ where $t^k = 1$.

Theorem 2.2: Let f and g be transcendental entire functions with finite order. Suppose that c is a non-zero complex constant and m, $n > 4m + 9$ be two positive integers. If $f(z)^n(f(z)^m - 1)f(z+c)$ and $g(z)^n(g(z)^m - 1)g(z+c)$ share 1 IM, then $f(z) = tg(z)$ where $t^k = 1$.

3 Lemmas:

To prove the theorems we need some results which are given below in the form of lemmas.

Lemma 3.1[17]: Let f and g be two non constant entire functions. If f and g share 1 IM then one of the following cases holds:
(i)$T(r,f) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + 2\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + S(r,f) + S(r,g)$
the same inequality holding for $T(r,g)$.
(ii) $f = g$
(iii) $fg = 1$
Lemma 3.2[18]: Let $f(z)$ be a meromorphic function of finite order and c be a non-zero complex constant. Then
$T(r, f(z+c)) = T(r,f) + S(r,f)$

$N(r, f(z+c)) = N(r, f) + S(r, f)$, $N(r, \frac{1}{f(z+c)}) = N(r, \frac{1}{f}) + S(r, f)$

Lemma 3.3[8]: Let $f(z)$ be a transcendental entire function of finite order and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that c is a non zero complex constant and $n \geq 2$ and $k \geq 1$ and $F(z) = \frac{f^n(z)(f(z)-1)^k f(z+c)}{\alpha(z)}$. Then $T(r, F) = (n+k+1)T(r, f) + S(r, f)$.

Lemma 3.4 : Let $f(z)$, n, k and c be same as in lemma3.3, and $F(z) = f^n(z)(f^k(z) - 1)f(z+c)$. Then $T(r, f) = (n+m+1)T(r, f) + S(r, f)$.

Lemma 3.5[19]: Let $s > 0$ and t are relatively prime integers and let c be a complex number such that $c^s = 1$. Then there exist one and only one common zero of $w^s - 1$ and $w^t - c$.

Proof of the theorem 2.1 :
Let, $F(z) = f^n(z)(f(z)-1)^k f(z+c)$ and $G(z) = g^n(z)(g(z)-1)^k g(z+c)$.
Then F and G share 1 IM. By lemma 3.1, we consider the following three cases:
CaseI.

$$\begin{aligned} T(r, F) &\leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + o(logr) \\ &\leq N_2(r, \frac{1}{f^n}) + N_2(r, \frac{1}{(f(z)-1)^k}) + N_2(r, \frac{1}{f(z+c)}) + N_2(r, \frac{1}{g^n}) \\ &+ N_2(r, \frac{1}{(g(z)-1)^k}) + N_2(r, \frac{1}{g(z+c)}) + 2[\overline{N}(r, \frac{1}{f^n}) + \overline{N}(r, \frac{1}{(f(z)-1)^k}) \\ &+ \overline{N}(r, \frac{1}{f(z+c)})] + \overline{N}(r, \frac{1}{g^n}) + \overline{N}(r, \frac{1}{(g(z)-1)^k}) + \overline{N}(r, \frac{1}{g(z+c)}) + O(logr) \\ &\leq 8T(r, f) + 5T(r, g) + min\{k, 2\}T(r, f) + min\{k, 2\}T(r, g) + o(logr) \end{aligned}$$

Using lemma 3.3, the above inequality becomes-

$$(3.1)\ (n+k+1)T(r, f) \leq (8 + min\{k, 2\})T(r, f) + (5 + min\{k, 2\})T(r, g) + O(logr)$$

Similarly we get,

$$(3.2)\ (n+k+1)T(r, g) \leq (8 + min\{k, 2\})T(r, g) + (5 + min\{k, 2\})T(r, f) + O(logr)$$

Combining (2) and (3), we have,
$(n+k+1)[T(r, f) + T(r, g)] \leq (13 + 2min\{k, 2\})[T(r, f) + T(r, g)] + O(logr)$
If $k = 1$ then, $n\{T(r, f) + T(r, g)\} \leq 13\{T(r, f) + T(r, g)\} + o(logr)$
which contradicts $n \geq 14$.
If $k \geq 2$, then $(n+k)\{T(r, f) + T(r, g)\} \leq 16\{T(r, f) + T(r, g)\} + o(logr)$
which also contradicts $n \geq 17 - k$.

CaseII. $F(z).G(z) = 1$
$i.e., f^n(z)(f(z)-1)^k f(z+c).g^n(g(z)-1)^k g(z+c) = 1$

Since R.H.S.=1, then $N(r,\frac{1}{f}) = S(r,f)$ and $N(r,\frac{1}{f-1}) = S(r,f)$ and similarly for $g(z)$. Then $\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3$ which is not possible.

CaseIII. $F(z)=G(z)$
i.e.,$f^n(z)(f(z)-1)^k f(z+c) = g^n(g(z)-1)^k g(z+c)$
The assertion now follows in [8,p.3423].
Hence the result.

Proof of the theorem 2.2 :
Let $F(z)=f^n(f^m-1)f(z+c)$ and $G=g^n(g^m-1)g(z+c)$
Then F and G share 1 IM. By lemma1, we consider the following three cases :

CaseI.

$$\begin{aligned}
T(r,F) &\leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + O(logr) \\
&= N_2(r,\frac{1}{f^n(f^m-1)f(z+c)}) + N_2(r,\frac{1}{g^n(g^m-1)g(z+c)}) \\
&+ 2\overline{N}(r,\frac{1}{f^n(f^m-1)f(z+c)}) + \overline{N}(r,\frac{1}{g^n(g^m-1)g(z+c)}) + O(logr) \\
&\leq N_2(r,\frac{1}{f^n}) + N_2(r,\frac{1}{f^m-1}) + N_2(r,\frac{1}{f(z+c)}) + N_2(r,\frac{1}{g^n}) \\
&+ N_2(r,\frac{1}{g^m-1}) + N_2(r,\frac{1}{g(z+c)}) + 2[\overline{N}(r,\frac{1}{f^n}) + \overline{N}(r,\frac{1}{f^m-1}) \\
&+ \overline{N}(r,\frac{1}{f(z+c)})] + \overline{N}(r,\frac{1}{g^n}) + \overline{N}(r,\frac{1}{g^m-1}) + \overline{N}(r,\frac{1}{g(z+c)}) + O(logr) \\
&\leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{f^m-1}) + N_2(r,\frac{1}{f(z+c)}) + N_2(r,\frac{1}{g}) \\
&+ N_2(r,\frac{1}{g^m-1}) + N_2(r,\frac{1}{g(z+c)}) + 2[\overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f^m-1}) \\
&+ \overline{N}(r,\frac{1}{f(z+c)})] + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g^m-1}) + \overline{N}(r,\frac{1}{g(z+c)}) + O(logr) \\
&\leq T(r,f) + mT(r,f) + T(r,f) + T(r,g) + mT(r,g) + T(r,g) \\
&+ 2[T(r,f) + mT(r,f) + T(r,f)] + T(r,g) + mT(r,g) + T(r,g) + O(logr)
\end{aligned}$$

using lemma 3 , we have from the above,

$$(n+m+1)T(r,f) \leq (3m+6)T(r,f) + (2m+4)T(r,g) + o(logr) \tag{3.3}$$

Similarly we get,

$$(n+m+1)T(r,g) \leq (3m+6)T(r,g) + (2m+4)T(r,f) + o(logr) \tag{3.4}$$

Combining (2.1) and (2.2), we have -

$$(n+m+1)[T(r,f)+T(r,g)] \leq (5m+10)[T(r,f)+T(r,g)]+o(logr)$$

which contradicts that $n > 4m+9$.

Case II: $FG = 1$

$$i.e., f^n(f^m-1)f(z+c).g^n(g^m-1)g(z+c) = 1 \tag{3.5}$$

Therefore $N(r, \frac{1}{f}) = S(r,f)$ and similarly for $g(z)$. Now let $c_1, c_2, ..., c_k$ be the roots of $f^m - 1$ of order $l_1, l_2, ..., l_k$ such that $l_1 + l_2 + ... + l_k = m$.
So (5) can be written as,
$f^n(f-c_1)^{l_1}(f-c_2)^{l_2}..........(f-c_k)^{l_k}f(z+c)g^n(g^m-1)g(z+c) = 1$
Now, for each $c_i(i = 1, 2, ...k)$, we have
$N(r, \frac{1}{f-c_i}) = S(r,f)$ and similarly for $g(z)$.
Then for each $c_i(i = 1, 2, ...k)$,
$\delta(0,f) + \delta(c_i), f) + \delta(\infty, f) = 3$, which is not possible.

Case III: $F = G$

$$i.e., f^n(f^m-1)f(z+c) = g^n(g^m-1)g(z+c) \tag{3.6}$$

Let $h = \frac{f}{g}$. If h is a constant, then put $f = gh$ into (6), we have-
$g^nh^n(g^mh^m-1)hg(z+c) = g^n(g^m-1)g(z+c)$
i.e.,$(h^{n+m}g^m - h^n)h = g^m - 1$
i.e.,$h^{n+m+1}g^m - g^m = h^{n+1} - 1$

$$\begin{aligned} i.e., g^m &= \frac{1-h^{n+1}}{1-h^{m+n+1}} \\ &= \frac{(h-p_1).......(h-p_{n+1})}{(h-q_1)........(h-q_{n+m+1})} \end{aligned}$$

where $p_i(i = 1, 2, ..., n+1)$ are distinct roots of the equation $s^{n+1} = 1$ and $q_i(i = 1, ..., n+m+1)$ are distinct roots of the equation $s^{n+m+1} = 1$.
Let $d = GCD(n+m+1, n+1)$, then $n+m+1 = ud$, $n+1 = vd$ where u,v are co-prime integers and $u > v$, thus $m = (u-v)d$ which implies $d \leq m$. By lemma 4, there exist one and only one common zero of $w^u - 1$ and $w^v - 1$ namely $w = 1$. Therefore, there exist atleast $n+1$ of $q_i(i = 1, 2, ..., n+m+1)$ different from $p_i(i = 1, 2, ..., n+1)$. Let us suppose that $b_i(i = 1, 2, ..., n+1)$ are different from $a_i(i = 1, 2, ..., n+1)$, then all zeros of $h - q_i(i = 1, 2, ..., n+1)$ have order of atleast m.

Now applying the second fundamental theorem to h, we have-

$$\begin{aligned}(n-1)T(r,h) &\leq \sum_{i=1}^{n+1}\overline{N}(r,\frac{1}{h-b_i})+S(r,h)\\ &\leq \frac{1}{2}\sum_{i=1}^{n+1}N(r,\frac{1}{h-b_i})+S(r,h)\\ &\leq \frac{n+1}{2}T(r,h)+S(r,h)\end{aligned}$$

As $n > 4m+9$, thus we get a contradiction. Hence h is a constant. Thus we have $h^{n+m+1}=1$ and $h^{n+1}=1$.
Therefore $f=tg$ for some constant t such that $t^d=1$ where $d=GCD(n+m+1,n+1)$. Hence the proof.

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