Some integral involving extended Bessel-Maitland function with Jacobi polynomial

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Abstract

Motivated by Ghayasuddin and Khan [1] introduced generalized Bessel-Maitland function. In the present paper, authors establish a new interesting integral formulas involving the Wright generalized Bessel-Maitland function with Jacobi polynomials, which are also expressed in terms of generalized hyper geometric function. Further, some special cases of our main results are also considered.

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1 Introduction

Bessel type functions have received an importance due to their frequent use in mathematics and physical applications. In physical applications, the modified Bessel type functions of the third kind appear as solutions of certain radial Schrodinger equations and as Dirichlet problems with boundary conditions on a wedge. Besides, they play an important role in diffraction and hydrodynamics problems and are the approximant in certain uniform asymptotic expansions as well.

In last decade, many authors (see, e.g., [1-19] have developed numerous integral formulas involving a variety of special functions. Also many integral formulas associated with the Bessel functions of several kinds have been presented (see, e.g., [1-8]). Those integrals involving Bessel-Maitland functions are not only of great interest to the pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering (see [10]). Several methods for evaluating infinite or finite integrals involving Bessel-Maitland functions have been known (see, e.g.,[1] and [18]). However, these methods usually work on a case-by-case basis.

The well known Mittag-Leffler function $E_\alpha(z)$ (which is the generalization of exponential function), occurs as the solution of fractional order differential and integral equation is defined by (see [8]):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.

A generalization of $E_\alpha(z)$ was introduced by Wiman [19] as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$, which is also known as Mittag-Leffler function or Wiman's function.

Afterwards, Prabhakar [9] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$.

In (2007), Shukla and Prajapati [12] introduced and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.4}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0,1) \bigcup \mathbb{N}$ and $(\gamma)_{qn} = \dfrac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$, denotes the generalized Pochhammer symbol.

A new generalized Mittag-Leffler function was defined by Salim [14] as:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n}, \tag{1.5}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0$.

Further, Salim and Faraj [13] introduced the following extension of Mittag-Leffler function:

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \tag{1.6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}, \min\{\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0\} > 0; p, q > 0$ and $q < \Re(\alpha) + p$.

Very recently, Ghayasuddin and Khan [1] introduced and investigate a new extension of Bessel-Maitland function as follows:

$$J^{\mu,q,p}_{\nu,\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(n\mu + \nu + 1)(\delta)_{pn}}, \tag{1.7}$$

where $\mu, \nu, \gamma, \delta \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

Equation (1.7) is a generalization of equation (1.1)-(1.6).

We investigate some special case of the generalized Bessel-Maitland function (1.7) by particular values to the parameters $\mu, \nu, \delta, \gamma, p, q$.

• On replacing ν by $\nu - 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.6) and (1.5).

• On setting $p = \delta = 1$ and replacing ν by $\nu - 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.4).

• On setting $p = q = \delta = 1$ and replacing ν by $\nu - 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.3).

• On setting $p = q = \delta = \gamma = 1$ and the replacing ν by $\nu - 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.2).

• On setting $p = q = \delta = \gamma = 1$ and $\nu = 0$, (1.7) reduces to the Mittag-Leffler function defined by (1.1).

2 Integral involving extended Bessel-Maitland function with Jacobi polynomial

The generalized Jacobi polynomial is defined by (see [18]):

$$P_n^{(\alpha,\beta,c,d)}(x) = \frac{(1+\alpha)_n}{\Gamma(n+1)} {}_3F_2 \left[\begin{matrix} -n, & (1+\alpha+\beta+n), & c; & \frac{1-x}{2} \\ (1+\alpha) & d, & ; & \end{matrix} \right], \tag{2.1}$$

where $d \in \mathbb{C} - Z^- \bigcup (0)$; $\alpha, n \in \mathbb{C} - Z^-$; $\beta \in \mathbb{C}$; $\Re(d - \beta - c) > 0$.

In dealing with Jacobi function, it is natural to make such use of our knowledge of the ${}_2F_1$ function (see [11]):

$$I_1 = \int_{-1}^{1} (\ 1-x)^{\lambda}(1+x)^{\delta} P_n^{(\alpha,\beta,c,d)}(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1+x)^h]dx \tag{2.2}$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}} \int_{-1}^{1} (\ 1-x)^{\lambda}(1+x)^{\delta+kh} P_n^{(\alpha,\beta,c,d)}(x)dx. \tag{2.3}$$

By using the formula given in (see [17]), we get

$$= \frac{2^{\lambda+\delta+1}(\alpha+1)_n\Gamma(\lambda+1)}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{\Gamma(\delta+kh+1)}{\Gamma(\lambda+\delta+hk+2)} J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^h]$$

$$\times {}_4F_3\left[\begin{array}{cccc} -n, & 1+\alpha+\beta+n, & c, & \lambda+1; \\ & & & 1 \\ \alpha+1, & d, & \lambda+\delta+hk+2; & \end{array}\right], \tag{2.4}$$

Provided
(i) $\mu,\nu,\gamma,\omega,\in\mathbb{C}$; $\Re(\mu)\geq 0, \Re(\nu)\geq -1, \Re(\gamma)\geq 0, \Re(\omega)\geq 0$; $p,q>0$ and $p<\Re(\alpha+1)$.
(ii)$\Re(\lambda)>-1, \alpha>-1$ and $\beta>-1$.

$$I_2=\int_{-1}^{1}(1-x)^{\lambda}(\ 1+x)^{\delta}P_n^{(\alpha,\beta,c,d)}(x)P_m^{(\rho,\sigma,e,f)}(x)J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1-x)^h]dx \tag{2.5}$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}}\int_{-1}^{1}(\ 1-x)^{\lambda+kh}(1+x)^{\delta}P_n^{(\alpha,\beta,c,d)}(x)P_m^{(\rho,\sigma,e,f)}(x)dx. \tag{2.6}$$

Using (2.1) in above expression, we get

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}}\frac{(1+\rho)_m}{m!}\sum_{k=0}^{\infty}\frac{(-m)_k(1+\rho+\sigma+m)_k(e)_k}{(1+\rho)_k 2^k k!(f)_k}$$

$$\times\int_{-1}^{1}(\ 1-x)^{\lambda+kh+k}(1+x)^{\delta}P_n^{(\alpha,\beta,c,d)}(x)dx. \tag{2.7}$$

Again using (2.1) in (2.7), we get

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}}\frac{(1+\rho)_m(1+\alpha)_n)}{m!n!}$$

$$\times\sum_{k=0}^{\infty}\frac{(-m)_k(1+\rho+\sigma+m)_k(e)_k}{(1+\rho)_k(f)_k 2^k k!}\sum_{l=0}^{\infty}\frac{(-n)_l(1+\alpha+\beta+n)_l(c)_l}{(1+\alpha)_l(d)_l 2^l l!}\int_{-1}^{1}(\ 1-x)^{\lambda+kh+k+l}(1+x)^{\delta}dx. \tag{2.8}$$

Using the formula

$$\int_{-1}^{1}(\ 1-x)^{n+\alpha}(1+x)^{n+\beta}dx=2^{2n+\alpha+\beta+1}B(1+\alpha+n,1+\beta+n). \tag{2.9}$$

Here (2.8) becomes,

$$I_2=\frac{2^{\lambda+\delta+1}\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m!n!}\sum_{k=0}^{\infty}\frac{(-m)_k(1+\rho+\sigma+m)_k(e)_k}{\Gamma(1+\rho+k)(f)_k k!}\sum_{l=0}^{\infty}\frac{(-n)_l(1+\alpha+\beta+n)_l(c)_l}{\Gamma(1+\alpha+l)(d)_l l!}$$

$$\times J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^h]B(1+\lambda+kh+k+l,1+\delta). \tag{2.10}$$

Provided
(i) $\mu,\nu,\gamma,\delta,\omega \in \mathbb{C}$; $\Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\omega) \geq 0, \Re(\delta) \geq 0, p, q > 0$ and $q < \Re(\alpha + p)$.
(ii) $\Re(\beta) \geq -1, h$ and λ are positive numbers.

$$I_3 = \int_{-1}^{1} (1-x)^{\lambda}(1+x)^{\delta} P_n^{(\alpha,\beta,c,d)}(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1-x)^h(1+x)^t]dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}} \int_{-1}^{1} (1-x)^{\lambda+hk}(1+x)^{\delta+tk} P_n^{(\alpha,\beta,c,d)}(x)dx. \tag{2.11}$$

Now, by using (2.1) in (2.11), we get

$$I_3 = \frac{2^{\lambda+\delta+1}(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k(c)_k}{(1+\alpha)_k(k!)(d)_k}$$

$$\times J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^{h+t}] B(1+\lambda+kh+k, 1+\delta+tk). \tag{2.12}$$

Provided
(i) $\mu,\nu,\gamma,\delta,\omega \in \mathbb{C}$; $p, q \in \mathbb{C}$ and $q < \Re(\alpha + p)$.
(ii) $\Re(\alpha) > -1$, and $\Re(\beta) > -1$.

$$I_4 = \int_{-1}^{1} (1-x)^{\lambda}(1+x)^{\delta} P_n^{(\alpha,\beta,c,d)}(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1-x)^h(1+x)^{-t}]dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}} \int_{-1}^{1} (1-x)^{\lambda+kh}(1+x)^{\delta-tk} P_n^{(\alpha,\beta,c,d)}(x)dx. \tag{2.13}$$

Now, by using (2.1) in (2.13), we obtain

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k+\nu+1)(\omega)_{pk}} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k(c)_k}{(1+\alpha)_k 2^k k!(d)_k}$$

$$\times \int_{-1}^{1} (1-x)^{\lambda+kh+k}(1+x)^{\delta-kt}dx. \tag{2.14}$$

Using (2.9) in (2.14), we get

$$I_4 = \frac{2^{\lambda+\delta+1}(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k(c)_k}{(1+\alpha)_k(k!)(d)_k}$$

$$\times J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^{h-t}] B(1+\lambda+kh+k, 1+\delta-tk). \tag{2.15}$$

Provided
(i) $\mu, \nu, \gamma, \delta, \omega \in \mathbb{C}$; $p, q > 0$ and $q < \Re(\alpha + p)$.
(ii) $\Re(\alpha) > -1$, and $\Re(\beta) > -1$.

$$I_5 = \int_{-1}^{1} (\ 1 - x)^{\lambda} (1 + x)^{\delta} P_n^{(\alpha,\beta,c,d)}(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1 + x)^{-h}] dx$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k + \nu + 1)(\omega)_{pk}} \int_{-1}^{1} (\ 1 - x)^{\lambda} (1 + x)^{\delta - hk} P_n^{(\alpha,\beta,c,d)}(x) dx. \tag{2.16}$$

Now, by using (2.1) in (2.16), we get

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(-z)^k}{\Gamma(\mu k + \nu + 1)(\omega)_{pk}} \frac{(1 + \alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \alpha + \beta + n)_k (c)_k}{(1 + \alpha)_k 2^k k! (d)_k}$$

$$\times \int_{-1}^{1} (\ 1 - x)^{\lambda + k} (1 + x)^{\delta - kh} dx. \tag{2.17}$$

Using(2.9) in (2.17), we get

$$I_5 = \frac{2^{\lambda + \delta + 1}(1 + \alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \alpha + \beta + n)_k (c)_k}{(1 + \alpha)_k (k!) (d)_k}$$

$$\times J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^{-h}] B(1 + \lambda + k, 1 + \delta - hk). \tag{2.18}$$

Provided
(i) $\mu, \nu, \gamma, \delta, \omega \in \mathbb{C}$; $p, q > 0$ and $q < \Re(\alpha + p)$.
(ii) $\Re(\alpha) > -1$, and $\Re(\beta) > -1$.

3 Special Cases

(i). On setting $\alpha = \beta = c = d = \delta = 0$ and replacing λ by $\lambda - 1$ then integral I_1 transforms into the following integral involving Legendre polynomial (see [11], [17]):

$$I_6 = \int_{-1}^{1} (\ 1 - x)^{\lambda - 1} P_n(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1 + x)^h] dx$$

$$= 2^{\lambda} \Gamma(\lambda) \sum_{k=0}^{\infty} \frac{\Gamma(kh + 1)}{\Gamma(\lambda + hk + 1)} J_{\nu,\gamma,\omega}^{\mu,q,p}[z 2^h] \times {}_3F_2 \left[\begin{matrix} -n, & n + 1, & \lambda; & \\ & & & 1 \\ 1, & \lambda + hk + 1; & & \end{matrix} \right]. \tag{3.1}$$

(ii). On setting $\alpha = \beta = \rho = c = d = e = f = \delta = \sigma = 0$ and replacing λ by $\lambda - 1$ then integral I_2 transforms into the following integral involving Legendre polynomials (see [11], [17]):

$$I_7 = \int_{-1}^{1} (1-x)^{\lambda-1} P_n(x) P_m(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1-x)^h]dx$$
$$= 2^{\lambda} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-m)_k(-n)_l(1+m)_k(1+n)_l}{(k!)^2(l!)^2} J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^h] B(\lambda+kh+k+l,1). \tag{3.2}$$

(iii). On setting $\alpha = \beta = c = d = \delta = 0$ and replacing λ by $\lambda - 1$ then integral I_3 transforms into the following integral involving Legendre polynomial (see [11], [17]):

$$I_8 = \int_{-1}^{1} (\ 1-x)^{\lambda-1} P_n(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1-x)^h(1+x)^t]dx$$
$$= 2^{\lambda} \sum_{k=0}^{\infty} \frac{(-n)_k(1+n)_k}{(k!)^2} J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^{h+t}] B(\lambda+kh+k, 1+tk). \tag{3.3}$$

(iv). On setting $\alpha = \beta = c = d = \delta = 0$ and replacing λ by $\lambda - 1$ then integral I_4 transforms into the following integral involving Legendre polynomial (see [11], [17]):

$$I_9 = \int_{-1}^{1} (\ 1-x)^{\lambda-1} P_n(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1-x)^h(1+x)^{-t}]dx$$
$$= 2^{\lambda} \sum_{k=0}^{\infty} \frac{(-n)_k(1+n)_k}{(k!)^2} J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^{h-t}] B(\lambda+kh+k, 1-tk). \tag{3.4}$$

(v). On setting $\alpha = \beta = c = d = \delta = 0$ and replacing λ by $\lambda - 1$ then integral I_5 transforms into the following integral involving Legendre polynomial (see [11], [17]):

$$I_{10} = \int_{-1}^{1} (\ 1-x)^{\lambda-1} P_n(x) J_{\nu,\gamma,\omega}^{\mu,q,p}[z(1+x)^{-h}]dx$$
$$= 2^{\lambda} \sum_{k=0}^{\infty} \frac{(-n)_k(1+n)_k}{(k!)^2} J_{\nu,\gamma,\omega}^{\mu,q,p}[z(2)^{-h}] B(\lambda+k, 1-hk). \tag{3.5}$$

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