Application of Sumudu Transform to Fractional Integro-Differential Equations Involving Generalized R-Function

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Abstract

In this paper, we applied Sumudu transform to investigate the solution of a Volterra type fractional integro-differential equation involving generalized R-function. Some specific cases are also mentioned.

Keywords: Sumudu transform, Volterra type Integro-differential equation, Riemann-Liouville Fractional Integral and Derivative, Generalized R-function of Lorenzo-Hartley.

1 Introduction

With the arrival of computers and advanced softwares, the differential equations played an essential role in all aspects of applied mathematics and gained greater importance. Numerous integral based transforms such as Laplace, Mellin, Fourier, Hankel etc. are applied to solve differential equations dealing with the engineering problems. In 1993, Watugala [7] proposed the Sumudu transform. Unit and scale preserving properties of Sumudu transform has a marvelous potential of applicability in the field of engineering mathematics and applied sciences. Weerakoon [11] gave the inverse Sumudu transform. We can find further details about it from Belgacem et al [5, 6], Loonker and Banerji [4] and many others.

The aim of this paper is to present some of the significant characteristics of Sumudu transform and a simple alternative derivation of the solution of Volterra type fractional integro-differential equation. Many authors have demonstrated the solution of fractional-integro-differential equation, including Barrett [8], Kilbas, Saigo, and Saxena [1], Ross and Sachdeva [2], Saxena [9], Jain and Tomar [10] and others.

2 Preliminaries

2.1 Sumudu Transform:

Over the set of function

$$A = \{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)\}, \tag{2.1}$$

the Sumudu transform for the function of exponential order defined [7] by

$$G(u) = S[f(t) : u] = \begin{cases} \int_0^\infty e^{-t} f(ut)dt, & 0 \leq u < \tau_2, \\ \int_0^\infty e^{-t} f(ut)dt, & -\tau_1 \leq u < 0. \end{cases} \tag{2.2}$$

The Sumudu transform, in other words can be written as

$$G(u) = S[f(t) : u] = \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} f(t)dt, \; u \in (-\tau_1, \tau_2). \tag{2.3}$$

The convolution of Sumudu Transform is

$$S(f * g)(\tau) = sS[f(\tau)]S[g(\tau)] = sF(s)G(s), for \, \Re(s) > 0. \tag{2.4}$$

The Sumudu transform inversion formula is given by

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\frac{t}{u}} G(u)du. \tag{2.5}$$

2.2 Generalized R-function:

The generalized R-function is defined by, Lorenzo-Hartley [3] as:

$$G_{q,\nu,\gamma}(a, t) = \sum_{j=0}^{\infty} \frac{(\gamma)_j (a)^j t^{(\gamma+j)q-\nu-1}}{\Gamma(j+1)\Gamma(\gamma+j)q-\nu}, \; \Re(q\gamma - \nu) > 0, \tag{2.6}$$

Which is a particular case of $\gamma = 1$, reduces it to a well-known R-function.

2.3 Fractional Integral and Derivative of Riemann-Liouville

The α order fractional integral of Riemann-Liouville is defined as

$${}_0D_\tau^{-\alpha} h(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - t)^{\alpha-1} f(t)dt, where \; \Re(\alpha) > 0. \tag{2.7}$$

Similarly, the η order fractional derivative of Riemann-Liouville is defined as

$${}_0D_\tau^{\eta}[h(\tau)] = \frac{1}{\Gamma(n-\eta)} \frac{d^n}{d\tau^n} \int_0^\tau (\tau - t)^{n-\eta-1} f(t)dt, where \; \Re(\eta) > 0. \tag{2.8}$$

The Sumudu transform of fractional derivative is defined as [2, 11]

$$S[{}_0D_\tau^{\eta}(h(\tau); s)] = s^{-\eta} H(s) - \Big[\sum_{r=1}^{n} s^{-r} {}_0D_\tau^{\eta-r} h(\tau)\Big]_{\tau=0}. \tag{2.9}$$

2.4 Sumudu Transform of Generalized R-function

Sumudu Transform of Generalized R-function is given by

$$S[G_{q,\nu,\gamma}(a,t)] = \int_0^\infty e^{-t} G_{q,\nu,\gamma}(a,st)dt, \tag{2.10}$$

$$S[G_{q,\nu,\gamma}(a,t)] = s^{\gamma q-\nu-1}[1-(as^q)]^{-\gamma}. \tag{2.11}$$

3 Solution of Fractional Integro-Differential Equation

Different physical phenomenon such as diffusion can be modeled in terms of integro-differential equation. These can be elongated to fractional integral equation by replacing their fractional counterparts with ordinary integral. Considering integro-differential equation of Volterra type involving generalized R-function and can be solved by using Sumudu transform.

Theorem 3.1. *Let $q, \nu, \gamma, k, \eta \in \mathbb{C}$, $0 \leq \tau \leq 1$ and $\Re(q) > 0, \Re(\nu) > 0, \Re(q-\nu) > 0$. Then the Cauchy problem for the Volterra type fractional integro- differential equation*

$$_0D_\tau^\alpha[h(\tau)] = \eta f(\tau) + k \int_0^\tau G_{q,\nu,\gamma}(a,t)h(\tau-t)dt \tag{3.1}$$

along with the initial conditions $_0D_\tau^{\alpha-k}[h(\tau)] = a_k$, $k = 1,\ldots,n = -[-\Re(\eta)]$, $-1 < \eta \leq n$, $n \in \mathbb{N}$. Where $a_1,\ldots,a_k$ are prescribed constants and unique solution to the Cauchy problem (3.1) provided by

$$h(\tau) = \sum_{k=1}^n a_k \Omega_k(\tau) + \eta \int_0^\tau \theta(\tau-t)f(t)dt \tag{3.2}$$

where

$$\Omega_k(\tau) = \sum_{m=0}^\infty k^m G_{q,k+(\nu-\alpha)m-\alpha-1,\gamma m}(a,\tau) \tag{3.3}$$

and

$$\theta(\tau-t) = \sum_{m=0}^\infty k^m G_{q,(\nu-\alpha)m-\alpha-1,\gamma m}(a,\tau-t). \tag{3.4}$$

Proof. Applying Sumudu transform on equation (3.1), we have

$$s^{-\alpha}H(s) - \sum_{k=1}^n s^{-k} D_\tau^{\alpha-k} h(\tau)|_{\tau=0} = \eta F(s) + ksH(s)\frac{s^{\gamma q-\nu-1}}{[1-as^q]^\gamma}$$

$$H(s) = \sum_{k=1}^n s^{\alpha-k} a_k \sum_{m=0}^\infty k^m \frac{s^{(\gamma q-\nu+\alpha)m}}{[1-as^q]^{\gamma m}} + \eta s^\alpha F(s) \sum_{m=0}^\infty k^m \frac{s^{(\gamma q-\nu+\alpha)m}}{[1-as^q]^{\gamma m}} \tag{3.5}$$

Inverting the Sumudu transform, we have

(3.6)
$$s^{-1}[H(s)]=\sum_{k=1}^{n}a_k s^{-1}\Big[\sum_{k=0}^{\infty}k^m\frac{s^{(\gamma q-\nu+\alpha)m+\alpha-k+1-1}}{[1-as^q]^{\gamma m}}\Big]+\eta s^{-1}\Big[F(s)\sum_{m=0}^{\infty}k^m\frac{s^{(\gamma q-\nu+\alpha)m+\alpha+1-1}}{[1-as^q]^{\gamma m}}\Big]$$

Now applying the Sumudu transform theorem of Convolution

$$h(\tau)=\sum_{k=1}^{n}a_k\Omega_k(\tau)+\eta\int_0^{\tau}\theta(\tau-t)f(t)dt \tag{3.7}$$

where $\Omega_k(\tau)$ and $\theta(\tau-t)$ is given respectively by equation (3.3) and (3.4). □

Theorem 3.2. *Let $q,\nu,\gamma,k,\eta\in\mathbb{C}$, $0\le\tau\le1$ and $\Re(q)>0,\Re(\nu)>0,\Re(q-\nu)>0$. Then the Volterra type integral equation*

$${}_0D_{\tau}^{-\alpha}[h(\tau)]=\eta f(\tau)+k\int_0^{\tau}G_{q,\nu,\gamma}(a,t)h(\tau-t)dt \tag{3.8}$$

has a solution $h(\tau)=\eta\int_0^{\tau}\theta(\tau-t)f(t)dt$ where

$$\theta(\tau-t)=\sum_{m=0}^{\infty}k^mG_{q,(\nu+\alpha)m+\alpha-1,\gamma m}(a,\tau-t). \tag{3.9}$$

Proof. Applying Sumudu transform on equation (3.8), we have

$$s^{\alpha}H(s)=ksH(s)\frac{s^{\gamma q-\nu-1}}{[1-as^q]^{\gamma}}+\eta F(s)$$
$$H(s)=\eta s^{-\alpha}F(s)\sum_{m=0}^{\infty}k^m\frac{s^{(\gamma q-\nu-\alpha)m}}{[1-as^q]^{\gamma m}} \tag{3.10}$$

Inverting the Sumudu transform, we have

$$s^{-1}\Big[H(s)\Big]=s^{-1}\Big[\eta s^{-\alpha}F(s)\sum_{m=0}^{\infty}k^m\frac{s^{(\gamma q-\nu-\alpha)m}}{[1-as^q]^{\gamma m}}\Big] \tag{3.11}$$

Now applying the Sumudu transform theorem of Convolution, we get

$$h(\tau)=\eta\int_0^{\tau}\theta(\tau-t)f(t)dt, \tag{3.12}$$

where $\theta(\tau-t)$ is given by (3.11). □

4 Conclusion:

In the present paper we have utilized an interesting and efficient transform to obtain solution of a fractional integro-differential equation. The results derived in this paper are expected to apply to a wide range of areas including mathematical, engineering, physical and chemical sciences.

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