On $(\epsilon)-$Lorentzian trans-Sasakian manifolds

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Abstract

In this paper we study the trans-Sasakian structure on a manifold with $(\epsilon)-$Lorentzian metric and give an example of such manifolds. Also conformally flat $(\epsilon)-$Lorentzian trans-Sasakian manifolds and Weyl semi-symmetric $(\epsilon)-$Lorentzian trans-Sasakian manifolds have been studied .

Keywords: $(\epsilon)-$Lorentzian trans-Sasakian manifold, $\eta-$Einstein manifold, conformally flat manifold, Weyl semi-symmetric..

1 Introduction

Let $M(\phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold [2]. Then the product $\widetilde{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\widetilde{M}, J, G)$ dictates the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g) and gives different structures on M like Sasakian structure, quasi-Sasakian structure, Kenmotsu structure and others ([2], [3], [11]) that have been studied several authors.

In the classification of Gray and Hervella [9] of the almost Hermitian manifolds there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformally Kahler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on M is trans-Sasakian [3] if $(M \times R, J, G)$ belongs to the class W_4 where J is the almost complex structure on $M \times R$ defined by

$$J(X\ ,\ f\frac{d}{dt}) = \phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is any vector field X on M, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [3].

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{1.1}$$

where α, β are smooth functions defined on M.

In 1985, J. A. Oubina [15] studied a new class of almost contact metric manifolds known as trans-Sasakian manifolds which generalizes both α-Sasakian and β-Kenmotsu manifolds. Sasakian, α-Sasakian, Kenmotsu, β-Kenmotsu manifolds are particular cases of trans-Sasakian manifolds of type (α, β). Nearly trans-Sasakian manifolds was introduced by C. Gherghe [4]. In [16], Prasad, Shukla and Tripathi have studied some special type of trans-Sasakian manifolds.

In Riemannian geometry, we study manifolds with a metric which is positive definite. Since manifolds with indefinite metric have significant use in Physics, it is interesting to study such manifolds equipped with different structures. In [1], A. Bejancu and K.L. Duggal introduced the notion of (ϵ)-Sasakian manifolds with indefinite metric. In [23], Xu Xufeng and Chao Xixaoli proved that (ϵ)-Sasakian manifold is a hypersurface of an indifinite Kahlerian manifold. Recently, in 2009, U.C. De and Avijit Sarkar [5] introduced and studied the notion of (ϵ)-Kenmotsu manifolds with indefinite metric.

A quasi-Conformal curvature tensor was introduced by Yano and Sawaki [21]. A $(2n+1)-$dimensional Riemannian manifold (M, g) is quasi-Conformally flat if $\check{C} = 0$,where $\check{C}$ is the quasi-Conformal curvature tensor, defined as

$$\begin{aligned}\check{C}(X,Y)Z &= aR(X,Y)Z + b[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y] - \frac{r}{(2n-1)}\{\frac{a}{2n} + 2b\}[g(Y,Z)X \\ &\quad -g(X,Z)Y],\end{aligned} \tag{1.2}$$

where a, b are constants and R, S, Q and r are the Riemannian curvature tensor, the Ricci-tensor, the Ricci operator and the scalar curvature tensor of the manifold respectively. If $a = 1$ and $b = -\frac{1}{2n-1}$, then $\check{C}$ becomes a conformal curvature C, given by

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[g(Y,Z)QX - g(X,Z)QY \\ &\quad +S(Y,Z)X - S(X,Z)Y] + \frac{r}{2n(2n-1)}[g(Y,Z)X \\ &\quad -g(X,Z)Y].\end{aligned} \tag{1.3}$$

If M is conformally flat, dimension $2n+1$, $(n > 1)$ then $C = 0$ in equation (1.3) and we have

$$\begin{aligned}R(X,Y)Z &= \frac{1}{(2n-1)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y] - \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.4}$$

for all vector fields X, Y, Z on M.

Let M be a differentiable manifold. If M has a Lorentzian metric g, that is a symmetric non-degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1-dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric.

Therefore, it is very natural and interesting idea to define both a trans-Sasakian structure and a Lorentzian metric on an odd dimensional manifold.

If $\alpha = 0$ and $\beta \in \mathbb{R}$, (the set of real numbers), then the manifold reduces to a Lorentzian β-Kenmotsu manifold studied by Funda Yaliniz ,Yildiz and Turan [19]. If $\beta = 0$ and $\alpha \in \mathbb{R}$, then the manifold reduces to a Lorentzian α-Sasakian manifold studied by Yildiz , Turan and Murathan [20]. If $\alpha = 0$ and $\beta = 1$,then the manifold reduces to a Lorentzian Kenmotsu manifold introduced by Mihai, Oiaga and Rosca [14]. Further more, if $\beta = 0$ and $\alpha = 1$.then the manifold reduces to a Lorentzian Sasakian manifold . Also Lorentzian para contact manifolds were introduced by Matsumoto [13].

In this paper, we introduce (ϵ)-Lorentzian trans-Sasakian manifolds with indefinite metric, which appear as a natural generalization of both (ϵ)-Lorentzian Sasakian and (ϵ)-Lorentzian Kenmostsu manifolds. This paper is organised as follows:

Section 1, is introductry. Section 2 contains necessary details about (ϵ)-Lorentzian trans-Sasakian manifold. Further in section 3, existance of (ϵ)-Lorentzian trans-Sasakian manifold is shown by an example. Some basic results regarding to such type of manifolds are also given in section 2. In Section 4, we introduce the conformally flat (ϵ)-Lorentzian trans-Sasakian manifolds. In section 5, we study Weyl-semi symmetric (ϵ)-Lorentzian trans-Sasakian manifolds.

2 Preliminaries

A differentiable manifold M of dimension $(2n+1)$ is called a (ϵ)-Lorentzian almost contact manifold if it admits a $(1,1)$ tensor field ϕ, a contra-variant vector field ξ, a co-variant vector field η and the ϵ-Lorentzian metric g which satisfy

$$\phi^2 = -I - \eta \otimes \xi, \quad \eta(\xi) = -1, \ \phi\xi = 0, \tag{2.1}$$

$$g(\xi,\xi) = -\epsilon, \qquad \eta(X) = \epsilon g(X,\xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \qquad g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

for all vector fields X, Y on M, where ϵ is 1 or -1 according as ξ is space like or light like vector fields, here

$$d\eta(X,Y) = g(X, \phi Y), \quad \text{for all} \quad X, Y \in \Gamma(TM). \tag{2.4}$$

An (ϵ)-Lorentzian almost contact metric manifold is called an (ϵ)-Lorentzian trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi - \epsilon\eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X\}, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$, where ∇ is Levi-Civita connection of semi-Riemannian metric g and α and β are smooth functions on M.

From equations (2.1), (2.2), (2.3) and (2.5), we have

$$\nabla_X \xi = \epsilon\{\alpha\phi X - \beta(X + \eta(X)\xi)\}, \tag{2.6}$$

where ∇ denotes the operator of co-variant differentiation with respect to the Lorentzian metric g. On (ϵ)-Lorentzian trans-Sasakian manifold η is closed.

Definition 2.1. *An (ϵ)-Lorentzian trans-Sasakian manifold is called an η-Einstein manifold if its Ricci tensor S satisfies the condition*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.7}$$

where a and b are scalar functions and X, Y are any vector fields on M.

Definition 2.2. *An (ϵ)-Lorentzian trans-Sasakian manifold is called a manifold of quasi-constant curvature if its curvature tensor R of type (0,4) satisfies*

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad - g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)],\end{aligned}$$

where $\widetilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1,3)$, p, q are scalar functions and ρ is a unit vector field defined by

$$g(X,\rho) = T(X). \tag{2.8}$$

Definition 2.3. *An (ϵ)-Lorentzian trans-Sasakian manifold is called a manifold Weyl-semi symmetric if it satisfies the condition $R.C = 0$, where R denotes the curvature tensor and C is the Weyl-conformal curvature tensor.*

3 Some Basic Results

Lemma 3.1. *Let M be an (ϵ)-Lorentzian trans-Sasakian manifold, we have*

$$(\nabla_X \eta)Y = -\beta\{g(X,Y) + \epsilon\eta(X)\eta(Y)\} - \alpha g(\phi X, Y). \tag{3.1}$$

Proof. By using (2.2), we have

$$\begin{aligned}(\nabla_X \eta)Y &= \nabla_X \eta(Y) - \eta(\nabla_X Y), \\ &= \nabla_X(\epsilon g(Y,\xi)) - \epsilon g(\nabla_X Y, \xi), \\ &= \epsilon g(Y, \nabla_X \xi).\end{aligned} \tag{3.2}$$

By using the equation (2.6) in (3.2), we obtain (3.1). □

Lemma 3.2. *Let M be an (ϵ)-Lorentzian trans Sasakian manifold, we have*

$$\begin{aligned}R(X,Y)\xi &= (\alpha^2 - \beta^2)\{\eta(X)Y - \eta(Y)X\} + 2\alpha\beta \\ &\quad \{\eta(X)\phi Y - \eta(Y)\phi X\} + \epsilon\{(X\alpha)\phi Y \\ &\quad -(Y\alpha)\phi X + (X\beta)\phi^2 Y - (Y\beta)\phi^2 X\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}R(\xi,X)Y &= (\alpha^2 - \beta^2)\{\eta(Y)X - \epsilon g(X,Y)\xi\} \\ &\quad -2\alpha\beta\{\eta(Y)\phi X - \epsilon g(\phi X, Y)\xi\} \\ &\quad +\epsilon\{-(Y\alpha)\phi X + g(\phi X, Y)(grad\alpha) \\ &\quad +(Y\beta)\phi^2 X - g(\phi^2 X, Y)(grad\beta)\},\end{aligned} \tag{3.4}$$

$$R(\xi, X)\xi = \{(\alpha^2 - \beta^2) + \epsilon(\xi\beta\}\phi^2 X, \tag{3.5}$$

$$2\alpha\beta - \epsilon(\xi\alpha) = 0, \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$*, where* R *is the curvature tensor.*

Proof. We have

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{3.7}$$

From equations (2.1), (2.6), put in (3.7), we get

$$\begin{aligned} R(X, Y)\xi \;&=\; (\alpha^2 - \beta^2)\{\eta(X)Y - \eta(Y)X\} \\ &\quad +2\alpha\beta\{\eta(X)\phi Y - \eta(Y)\phi X\} \\ &\quad +\epsilon\{(X\alpha)\phi Y - (Y\alpha)\phi X \\ &\quad +(X\beta)\phi^2 Y - (Y\beta)\phi^2 X\}. \end{aligned} \tag{3.8}$$

Putting $X = \xi$, we have

$$\begin{aligned} R(\xi, Y)\xi &= (\alpha^2 - \beta^2)\{-Y - \eta(Y)\xi\} + 2\alpha\beta\{-\phi Y\} + \epsilon\{\xi\alpha\}\phi Y + \{\xi\beta\}\phi^2 Y \\ &= -\{(\alpha^2 - \beta^2) + \epsilon\xi\beta\}(Y + \eta(Y)\xi) - \{2\alpha\beta - \epsilon(\xi\alpha)\}\phi Y \\ &= \{(\alpha^2 - \beta^2) + \epsilon\xi\beta\}\phi^2 Y - \{2\alpha\beta - \epsilon(\xi\alpha)\}\phi Y, \end{aligned}$$

$$R(\xi, Y)\xi = \{(\alpha^2 - \beta^2) + \epsilon\xi\beta\}\phi^2 Y - \{2\alpha\beta - \epsilon(\xi\alpha)\}\phi Y, \tag{3.9}$$

By using equation (3.3), we get

$$\begin{aligned} g(R(\xi, Y)X, Z) \;&=\; g(R(X, Z)\xi, Y), \\ R(\xi, X)Y \;&=\; (\alpha^2 - \beta^2)\{\eta(Y)X - \epsilon g(X, Y)\xi\} \\ &\quad -2\alpha\beta\{\eta(Y)\phi X - \epsilon g(\phi X, Y)\xi\} \\ &\quad +\epsilon\{-Y\alpha)\phi X + g(\phi X, Y)(grad\alpha) \\ &\quad +(Y\beta)(\phi^2 X - g(\phi^2 X, Y)(grad\beta)\}. \end{aligned} \tag{3.10}$$

Now putting $Y = X$ in equation (3.9), we get

$$R(\xi, X)\xi = \{(\alpha^2 - \beta^2) + \epsilon(\xi\beta)\}\phi^2 X - \{2\alpha\beta - \epsilon(\xi\alpha)\}\phi X. \tag{3.11}$$

From equation (3.9) and (3.11), we get the equation (3.5) and (3.6). □

Lemma 3.3. *Let* M *be a* $(2n+1)$*-dimensional* (ϵ)*-Lorentzian trans-Sasakian manifold. Then we have*

$$\begin{aligned} \eta(R(X, Y)Z) \;&=\; \epsilon(\alpha^2 - \beta^2)\{g(X, Z)\eta(Y) - \eta(X)g(Z, Y)\} \\ &\quad +2\alpha\beta\epsilon\{g(\phi Z, Y)\eta(X) + g(\phi X, Z)\eta(Y)\} \\ &\quad -\{(X\alpha)g(\phi Y, Z) - (Y\alpha)g(\phi X, Z)\} \\ &\quad +\{(X\beta)g(\phi^2 Y, Z) - (Y\beta)g(\phi^2 X, Z)\}, \end{aligned} \tag{3.12}$$

$$\eta(R(X, Y)\xi) = 0. \tag{3.13}$$

Proof. We have from equation (3.3), we get

$$\begin{aligned}\eta(R(X,Y)Z) &= \epsilon(g(R(X,Y)Z,\xi) \\ &= -\epsilon(g(R(X,Y)\xi,Z) \\ &= -\epsilon(\alpha^2-\beta^2)\{\eta(X)g(Z,Y)-g(X,Z)\epsilon g(\xi,Y)\} \\ &\quad +2\alpha\beta\epsilon\{g(\phi Z,Y)\eta(X)+g(\phi X,Z)\eta(Y)\} \\ &\quad -\{(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)\} \\ &\quad +\{(X\beta)g(\phi^2Y,Z)-(Y\beta)g(\phi^2X,Z)\},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\eta(R(X,Y)Z) &= \epsilon(\alpha^2-\beta^2)\{g(X,Z)\eta(Y)-\eta(X)g(Z,Y)\} \\ &\quad +2\alpha\beta\epsilon\{g(\phi Z,Y)\eta(X)+g(\phi X,Z)\eta(Y)\} \\ &\quad -\{(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)\} \\ &\quad +\{(X\beta)g(\phi^2Y,Z)-(Y\beta)g(\phi^2X,Z)\}.\end{aligned}$$

Putting $Z=\xi$, we get

$$\eta(R(X,Y)\xi)=0.$$

□

Let M be a $(2n+1)$ dimensional ϵ-Lorentzian trans-Sasakian manifold. The Ricci tensor S and scalar curvature r of M is defined by

$$S(X,Y)=\sum_{i=1}^{2n+1}\epsilon_i g(R(e_i,X)Y,e_i),$$

$$r=\sum_{i=1}^{2n+1}\epsilon_i g(e_i,e_i),$$

where $\{e_i,e_2.....e_{2n},e_{2n+1}=\xi\}$ is orthonormal basis field and $\epsilon_i=g(e_ie_i)$.

Lemma 3.4. *Let M be a $(2n+1)$ dimensional ϵ-Lorentzian trans-Sasakian manifold. Then*

$$\begin{aligned}S(X,\xi) &= \{-2n(\alpha^2-\beta^2)+\epsilon(\xi\beta)\}\eta(X) \\ &\quad +\epsilon(\phi X)\alpha-\epsilon(2n-1)(X\beta),\end{aligned} \tag{3.15}$$

$$\begin{aligned}Q\xi &= \epsilon[\{-2n(\alpha^2-\beta^2)+\epsilon\xi\beta\}\xi-\phi(grad\alpha) \\ &\quad -(2n-1)(grad\beta),\end{aligned} \tag{3.16}$$

where S is the Ricci curvature and Q is the Ricci operator given by

$$S(X,Y)=g(QX,Y). \tag{3.17}$$

Proof. Let M be an $(2n+1)$-dimensional (ϵ)-Lorentzian trans-Sasakian manifold. Then the Ricci tensor S of the manifold M is defined by

$$S(X,Y)=\sum_{i=1}^{2n+1}\epsilon_i g(R(e_i,X)\xi,e_i),$$

$$\begin{aligned}(g(R(X,Y)\xi,Z) &= (\alpha^2-\beta^2)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}\\ &+2\alpha\beta\{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z\}\\ &+\epsilon\{(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)\}\\ &+\{(X\beta)g(\phi^2 Y,Z)-(Y\beta)g(\phi^2 X,Z)\}.\end{aligned} \tag{3.18}$$

Multiply both side by $\epsilon_i\{\epsilon_i=g(e_ie_i)=1$ if $1\leq i\leq 2n, \epsilon_{2n+1}=\xi\}$, and then putting $X=Z=e_i$ in equation 3.18 and taking summation over $i=1$ to $i=2n+1$, where $\{e_i,e_2.....e_{2n,}e_{2n+1}=\xi\}$ is an orthonormal basis of tangent space at each point of the manifold

$$\begin{aligned}\sum_{i=1}^{2n+1}\epsilon_i(g(R(e_i,Y)\xi,e_i) &= (\alpha^2-\beta^2)\left\{\sum_{i=1}^{2n+1}\epsilon_i g(e_i,\xi)g(Y,e_i)\epsilon_i-\eta(Y)\sum_{i=1}^{2n+1}\epsilon_i g(e_i,e_i)\}\right\}\\ &+2\alpha\beta\left\{\sum_{i=1}^{2n+1}\epsilon_i g(e_i,\xi)g(\phi Y,e_i)\epsilon_i-\eta(Y)\sum_{i=1}^{2n+1}\epsilon_i g(\phi e_i,e_i)\}\right\}\\ &+\epsilon\left\{\sum_{i=1}^{2n+1}\epsilon_i g(grad\alpha,e_i)g(\phi Y,e_i)-(Y\alpha)\sum_{i=1}^{2n+1}\epsilon_i g(\phi e_i,e_i)\}\right\}\\ &+\sum_{i=1}^{2n+1}\epsilon_i g(grad\beta,e_i)g(\phi^2 Y,e_i)-(Y\beta)\sum_{i=1}^{2n+1}\epsilon_i g(\phi^2 e_i,e_i)\},\end{aligned}$$

replacing Y by X, we get

$$S(X,\xi)=\{-2n(\alpha^2-\beta^2)+\epsilon(\xi\beta)\}\eta(X)+\epsilon(\phi X)\alpha-\epsilon(2n-1)(X\beta). \tag{3.19}$$

and hence from (3.17), we get (3.16). □

Lemma 3.5. *If in a $(2n+1)$-dimensional (ϵ)-Lorentzian trans-Sasakian manifold of type (α,β), if we consider $\phi(grad\alpha)=-(2n-1)grad\beta$ then*

$$\begin{gathered}(2n-1)(Y\beta)-(\phi Y)\alpha=0,\\ \xi\beta=0.\end{gathered} \tag{3.20}$$

Proof. We know that

$$\begin{aligned}X\beta &= g(X,grad\beta)\\ &= g(X,\frac{\phi(grad\alpha)}{2n-1}),\\ X\beta &= \frac{1}{2n-1}g(\phi X,grad\alpha),\end{aligned}$$

On putting $Y = X$ in equation (3.20), we get

$$(2n-1)(X\beta) - (\phi X)\alpha = 0. \tag{3.21}$$

By using equation (3.15), (3.16) and (3.21), we get

$$S(X,\xi) = [-2n(\alpha^2 - \beta^2) + \epsilon(\xi\beta)]\eta(X) + \epsilon(\phi X)\alpha - \epsilon(2n-1)(X\beta), \tag{3.22}$$

$$Q\xi = \epsilon[\{-2n(\alpha^2 - \beta^2) + \epsilon(\xi\beta)\}\xi - \phi(grad\alpha) - (2n-1)(grad\beta)]. \tag{3.23}$$

If $\phi(grad\alpha) = -(2n-1)(grad\beta)$, then form equation (3.20), (3.22) and (3.23), we get

$$S(X,\xi) = -2n[(\alpha^2 - \beta^2) + \epsilon(\xi\beta)]\eta(X), \tag{3.24}$$

$$Q\xi = -2n\epsilon[(\alpha^2 - \beta^2) + \epsilon(\xi\beta)]\xi, \tag{3.25}$$

$$S(\xi,\xi) = -2n[(\alpha^2 - \beta^2) + \epsilon(\xi\beta)]. \tag{3.26}$$

□

Now we shall give an example of (ϵ)-Lorentzian trans-Sasakian manifold.

Let us consider a 3-dimensional manifold $M = \{(x, y.z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$.
Let $e_1 = e^z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$, $e_2 = e^z\frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$, which are linearly independent vector fields at each point of M. Define a semi-Riemannian metric g on M as

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -\epsilon$$

where $\epsilon = \pm 1$
Let η be the 1-form defined by $\eta(Z) = \epsilon g(Z, e_3)$, for any Z$\in \Gamma(TM)$. Let ϕ be a tensor field of type $(1, 1)$ defined by $\phi e_1 = -e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$. Then by using linearity of ϕ and g, we have

$$\eta(e_3) = -1, \quad \phi^2 Z = -Z - \eta(Z)e_3, \quad g(\phi Z, \phi U) = g(Z, U) + \epsilon\eta(Z)\eta(U),$$
$$\text{for any } Z, U \in \Gamma(TM).$$

If we take $e_3 = \xi$, $(\phi, \xi, \eta, g, \epsilon)$ defines an (ϵ)-Lorentzian almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of type (1,3), then we have

$$[e_1, e_2] = ye^z e_2 - e^{2z} e_3, \; [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2,$$

Taking $e_3 = \xi$ and using Koszul's formula for the Levi-Civita connection with respect to g, we obtain

$$\nabla_{e_1} e_3 = -e_1 - \frac{1}{2}\epsilon e^{2z} e_2, \quad \nabla_{e_2} e_3 = -e_2 + \frac{1}{2}\epsilon e^{2z} e_i, \quad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = y e^z e_2 - \frac{1}{2} e^{2z} e_3, \quad \nabla_{e_2} e_2 = -\epsilon e_3, \; \nabla_{e_3} e_2 = \frac{1}{2}\epsilon e^{2z} e_1,$$

$$\nabla_{e_1} e_1 = -\epsilon e_3, \quad \nabla_{e_2} e_1 = -\frac{1}{2}\epsilon e^{2z} e \quad \nabla_{e_3} e_1 = -\frac{1}{2}\epsilon e^{2z} e_2.$$

Now for $\xi = e_3$, above results satisfy

$$\nabla_X \xi = \epsilon\{\alpha\phi X - \beta(X + \eta(X)\xi)\},$$

From the above it can be easily see that (ϕ, ξ, η, g) is an (ϵ)-Lorentzian trans-Sasakian structure on M, with $\alpha = -\frac{1}{2}e^{2z} \neq 0$ and $\beta = \epsilon \neq 0$. Consequently $M(\phi, \xi, \eta, g, \epsilon)$ is a 3-dimensional (ϵ)-Lorentzian trans-Sasakian manifold. Here α, β satisfy the equation (3.6).

4 Conformally Flat (ϵ)-Lorentzian Trans-Sasakian Manifolds

In this section, we consider conformally flat (ϵ)-Lorentzian trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g), (n > 1)$.The conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.1}$$

where R, S, Q and r are the curvature tensor, the Ricci-tensor, the Ricci operator and the scalar curvature tensor of the Riemannian manifold respectively. If the manifold is conformally flat i.e. $C = 0$, then from (4.1), we have

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.2}$$

Now, taking scalar product on both side of above equation with W, we have

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{1}{(2n-1)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\quad +g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)] \\ &\quad -\frac{r}{2n(2n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4.3}$$

On putting $W=\xi$, the equation (4.3), we get

$$\begin{aligned}(4.4)\qquad \epsilon\eta(R(X,Y)Z) &= \frac{1}{(2n-1)}[\epsilon S(Y,Z)\eta(X)-\epsilon S(X,Z)\eta(Y) \\ &+g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)] \\ &-\frac{r\epsilon}{2n(2n-1)}[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)].\end{aligned}$$

Replacing Y by ξ in above equation and using equations (3.12), (3.24), we get

$$\begin{aligned}(4.5)\qquad S(X,Z) &= [2n\epsilon(\alpha^2-\beta^2)-(\xi\beta)-(2n-1)(\xi\beta)+\frac{r}{2n} \\ &+(2n-1)(\alpha^2-\beta^2)-(2n-1)(\xi\beta)]g(X,Z) \\ &+[4n(\alpha^2-\beta^2)-2\epsilon(\xi\beta)+\frac{r\epsilon}{2n}+(2n-1) \\ &(\alpha^2-\beta^2)\epsilon-(2n-1)(\xi\beta)]\eta(X)\eta(Z) \\ &-\epsilon\{(\phi Z)\alpha\}\eta(X)+(2n-1)\epsilon(Z\beta)\eta(X) \\ &-\epsilon\{(\phi X)\alpha\}\eta(Z)+(2n-1)\epsilon(\alpha\beta)\eta(Z)\end{aligned}$$

If $\phi(grad\alpha)=-(2n-1)(grad\beta)$.Then $(\xi\beta)=0,(2n-1)(X\beta)-(\phi X)\alpha=0$, and $(2n-1)(Z\beta)-(\phi Z)\alpha=0$.

So equation (4.5) becomes

$$\begin{aligned}S(X,Z) &= [2n\epsilon(\alpha^2-\beta^2)+\frac{r}{2n}+(2n-1)(\alpha^2-\beta^2)]g(X,Z) \\ &+[4n(\alpha^2-\beta^2)\frac{r\epsilon}{2n}+(2n-1)(\alpha^2-\beta^2)]\eta(X)\eta(Z).\end{aligned}$$

Hence we have the following theorem

Theorem 4.1. *A conformally flat (ϵ)-Lorentzian trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ is an η-Einstein manifold if $\phi(grad\alpha)=-(2n-1)(grad\beta)$ Thus in view of definition is an η-Einstein manifold.*

Theorem 4.2. *A $(2n+1)$-dimensional $(n>1)$ conformally flat (ϵ)-Lorentzian trans-Sasakian manifold is of quasi-constant curvature, if $(2n-1)(grad\beta)-\phi(grad\alpha)=(2n-1)(\xi\beta)\xi$.*

Corollary 4.1. *A conformally flat (ϵ)-Lorentzian β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ is an η-Einstein manifold.*

5 Weyl Semi-Symmetric (ϵ)-Lorentzian trans-Sasakian Manifolds

In this section, we study Weyl semi− symmetric (ϵ)-Lorentzian trans-Sasakian manifolds.

An (ϵ)-Lorentzian trans-Sasakian manifolds is said to be Weyl semi-symmetric if $R.C=0$, where $R(X,Y)$ is the curvature operator and $C(X,Y)Z$ is Weyl conformal curvature tensor.

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.1}$$

Taking scalar product with ξ on both side of equation (5.1), we get

$$g(C(X,Y)Z,\xi) = g(R(X,Y)Z,\xi) - \frac{1}{(2n-1)}[S(Y,Z)g(X,\xi) - S(X,Z)g(Y,\xi) + g(Y,Z)g(QX,\xi) - g(X,Z)g(QY,\xi)] + \frac{r}{2n(2n-1)}[g(Y,Z)g(X,\xi) - g(X,Z)g(Y,\xi)]. \tag{5.2}$$

From the equation (2.2), we get

$$\epsilon\eta(C(X,Y)Z) = \epsilon\eta(R(X,Y)Z) - \frac{1}{(2n-1)}[\epsilon S(Y,Z)\eta(X) - \epsilon S(X,Z)\eta(Y) + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)] + \frac{r\epsilon}{2n(2n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{5.3}$$

Putting $Z = \xi$ in the equation (5.3), we have

$$\epsilon\eta(C(X,Y)\xi) = 0. \tag{5.4}$$

On putting $X = Y$, the equation (5.3), gives equation

$$\epsilon\eta(C(X,X)Z) = \epsilon\eta(R(X,X)Z) = 0. \tag{5.5}$$

Again putting $X = \xi$ in equation (5.3), we get

$$\epsilon\eta(C(\xi,Y)Z) = \epsilon\eta(R(\xi,Y)Z) - \frac{1}{(2n-1)}[-\epsilon S(Y,Z) - \epsilon S(\xi,Z)\eta(Y) + g(Y,Z)S(\xi,\xi) - \epsilon S(Y,\xi)\eta(Z)] + \frac{r\epsilon}{2n(2n-1)}[-g(Y,Z) - \eta(Z)\eta(Y)]. \tag{5.6}$$

From the equation (2.1), (2.2), (3.12), (3.22) and (3.26), we get

$$\begin{aligned}\epsilon\eta(C(\xi,Y)Z) &= \epsilon(\alpha^2-\beta^2)\{\epsilon\eta(Y)\eta(Z)+g(Y,Z)\} \\ &\quad -\frac{1}{(2n-1)}\{-\epsilon S(Y,Z)-\epsilon(-2n(\alpha^2-\beta^2) \\ &\quad -\epsilon(\xi\beta)\}\eta(Z)\eta(Y)-g(Y,Z)\{2n(\alpha^2-\beta^2)-\epsilon(\xi\beta)\} \\ &\quad -\epsilon[(-2n(\alpha^2-\beta^2)-\epsilon(\xi\beta)]\eta(Y)\eta(Z) \\ &\quad +\frac{r\epsilon}{2n(2n-1)}[-g(Y,Z)-\eta(Z)\eta(Y)].\end{aligned} \tag{5.7}$$

$$\begin{aligned}\epsilon\eta(C(\xi,Y)Z) &= \frac{1}{(2n-1)}\{\epsilon(2n-1)(\alpha^2-\beta^2) \\ &\quad -2n\epsilon(\xi\beta)-\frac{r\epsilon}{2n}\}g(Y,Z) \\ &\quad +\{(2n-1)(4n+\epsilon)(\alpha^2-\beta^2) \\ &\quad -4n(2n-1)(\xi\beta)-\frac{r\epsilon}{2n}\}\eta(Y)\eta(Z).\end{aligned} \tag{5.8}$$

If the manifold is Weyl semi-symmetric, $R.C=0$ then we have

$$\begin{aligned}&R(X,Y)C(U,V)W-C(R(X,Y)U,V)W \\ &-C(U,R(X,Y)V)W-C(U,V)R(X,Y)W=0.\end{aligned} \tag{5.9}$$

In equation (5.9), taking scalar product with ξ on both side putting $X=\xi$, we have

$$\begin{aligned}&\epsilon g(R(\xi,Y)C(U,V)W,\xi)-\epsilon g(C(R(\xi,Y)U,V)W,\xi) \\ &-\epsilon g(C(U,R(\xi,Y)V)W,\xi)-\epsilon g(C(U,V)R(\xi,Y)W,\xi)=0.\end{aligned} \tag{5.10}$$

From the equation (3.12), we have

$$\begin{aligned}&(\alpha^2-\beta^2)[g(C(U,V)W,Y)+\epsilon\eta(Y)\eta(C(U,V)W)] \\ &-(\alpha^2-\beta^2)[g(Y,U)\eta(C(\xi,V)W)+\epsilon\eta(U)\eta(C(Y,V)W)] \\ &-(\alpha^2-\beta^2)[g(V,Y)\eta(C(U,\xi)W)+\epsilon\eta(V)\eta(C(U,Y)W)] \\ &-(\alpha^2-\beta^2)[g(W,Y)\eta(C(U,V)\xi)+\epsilon\eta(W)\eta(C(U,V)Y)]=0.\end{aligned} \tag{5.11}$$

From the above equation can be written as, where $\widetilde{C}(U,V,W,Y)=g(C(U,V)W,Y)$.

$$\begin{aligned}&\widetilde{C}(U,V,W,Y)+\epsilon\eta(Y)\eta(C(U,V)W) \\ &-g(Y,U)\eta(C(\xi,V)W)-\epsilon\eta(W)\eta(C(Y,V)W) \\ &-g(V,Y)\eta(C(U,\xi)W)-\epsilon\eta(V)\eta(C(U,Y)W) \\ &-g(W,Y)\eta(C(U,V)\xi)-\epsilon\eta(W)\eta(C(U,V)Y)=0.\end{aligned} \tag{5.12}$$

On putting $Y = U$ in equation (5.12) and using (5.4), (5.5), we have

$$\widetilde{C}(U,V,W,U) - g(U,U)\eta(C(\xi,V)W) \\ -g(U,V)\eta(C(U,\xi)W) - \epsilon\eta(W)\eta(C(U,V)U = 0. \tag{5.13}$$

Putting $U = e_i$ and taking summation over $i, 1 \leq i \leq 2n+1$, where $\{e_i\}$ is orthonormal basis of the tangent space at each point of the manifold, we have

$$\sum_{i=1}^{2n+1}[\widetilde{C}(e_i,V,W,e_i)] - \sum_{i=1}^{2n+1} g(e_i,e_i)\eta(C(\xi,V)W) \\ -\sum_{i=1}^{2n+1}[g(e_i,V)\epsilon g(C(e_i,\xi)W,\xi] - \sum_{i=1}^{2n+1} g(W,\xi)g(C(e_i,V)e_i,\xi) = 0. \tag{5.14}$$

$$\begin{aligned} \sum_{i=1}^{2n+1}\epsilon_i g(V,e_i)\epsilon g(C(e_i,\xi)W,\xi) &= \epsilon g(C(\xi,V)W,\xi) \\ &= \epsilon^2\eta(C(\xi,V)W) \\ &= \eta(C(\xi,V)W). \end{aligned} \tag{5.15}$$

Now, by using (3.24) and (3.26) in (5.3), we get

$$\begin{aligned} \sum_{i=1}^{2n+1}\epsilon\eta(C(e_i,V)e_i) &= \frac{1}{2n-1}[2n\epsilon(2n-1)(\alpha^2-\beta^2) \\ &\quad -2n\epsilon(2n+1)(\alpha^2-\beta^2-\epsilon(\xi\beta)) \\ &\quad -r(1-\epsilon)]\eta(V), \end{aligned} \tag{5.16}$$

$$\begin{aligned} \sum_{i=1}^{2n+1} g(e_i,V)\eta(C(e_i,\xi)W) &= \sum_{i=1}^{2n+1} g(e_i,V)\epsilon g(C(e_i,\xi)W,\xi) \\ &= \sum_{i=1}^{2n+1} -\epsilon g(C(\xi,V)W) \\ &= -\eta(C(\xi,V)W), \end{aligned} \tag{5.17}$$

$$\sum_{i=1}^{2n+1}\widetilde{C}(e_i,V,W,e_i) - (2n+\epsilon-1)\eta(C(\xi,V)W) \\ -\sum_{i=1}^{2n+1}[g(e_i,V)\eta(C(e_i,\xi)W) + \epsilon\eta(W)\eta(C(e_i,V)e_i)] = 0, \tag{5.18}$$

$$\begin{aligned}\epsilon\eta(C(\xi,V)W) &= \frac{1}{(2n-1)}[\epsilon(2n-1)(\alpha^2-\beta^2)-2n(\xi\beta)\\ &-\frac{r\epsilon}{2n}]g(V,W)+\{(2n-1)(4n+\epsilon)(\alpha^2-\beta^2)\\ &-4n(2n-1)(\xi\beta)-\frac{r\epsilon}{2n}\}\eta(V)\eta(W)+\epsilon S(V,W).\end{aligned} \tag{5.19}$$

From equation (4.1), we have

$$\begin{aligned}g(C(X,Y)Z,U) &= g(R(X,Y)Z,U)-\frac{1}{(2n-1)}[S(Y,Z)g(X,U)\\ &-S(X,Z)g(Y,U)+g(Y,Z)S(X,U)-g(X,Z)S(Y,U)]\\ &+\frac{r}{2n(2n-1)}[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)].\end{aligned} \tag{5.20}$$

Putting $X=U=e_i$ and $Y=V$, $Z=W$ in equation (5.20) and taking summation over $i, 1\leq i\leq 2n+1$, and using the equation (3.24), (3.26), we get

$$\begin{aligned}\sum_{i=1}^{2n+1} g(C(e_i,Y)Z,e_i) &= \frac{1}{2n-1}[(2n-1)\epsilon S(Y,Z)\\ &+(2n-1)(\alpha^2-\beta^2)-\frac{r}{2n}]g(Y,Z)\\ &-\frac{\epsilon}{2n-1}[(2n-1)(\alpha^2-\beta^2)\\ &-4n(\alpha^2-\beta^2-\epsilon(\xi\beta)+\frac{r}{2n}]\eta(Y)\eta(Z),\end{aligned} \tag{5.21}$$

$$\begin{aligned}\sum_{i=1}^{2n+1}(\widetilde{C}(e_i,V)W,e_i) &= (2n+\epsilon-1)\eta(C(\xi,V)W)\\ &+\sum_{i=1}^{2n+1}[g(e_i,V)\eta(C(e_i,\xi)W)\\ &-\epsilon\eta(W)\eta(C(e_i,V)e_i)]=0.\end{aligned} \tag{5.22}$$

Now, using the equation (5.16), (5.19), and (5.22), in equation (5.14), we get

$$\begin{aligned}S(V,W) &= \epsilon[(\alpha^2-\beta^2)-\epsilon(2n+\epsilon)(\alpha^2-\beta^2-2n(\xi\beta))\\ &-\frac{(2n+\epsilon)r}{2n}]g(V,W)-[(1+\epsilon)(4n+\epsilon)(\alpha^2-\beta^2)\\ &-\{(\alpha^2-\beta^2)-(4n+1)(\xi\beta)\}-\frac{(\epsilon+1)r}{2n}]n(V)\eta(W).\end{aligned} \tag{5.23}$$

Then the above manifold is an η-Einstein manifold.

Theorem 5.1. *A Weyl semi-symmetric* $(2n+1)$*-dimensional* (ϵ)*-Lorentzian trans-Sasakian manifold is an* η*-Einstein manifold, if and only if* $(2n-1)grad\beta - \phi(grad\alpha) = (2n-1)(\xi\beta)\xi$.

Corollary 5.1. *A* $(2n+1)$*-dimensional* $(n > 1)$ *Weyl-semi symmetric* (ϵ)*-Lorentzian trans-Sasakian manifold is of quasi-constant curvature,* $(2n-1)(grad\beta) = \phi(grad\alpha)$.

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