Certain Subclasses of Sakaguchi Type Bi-Univalent Functions

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Abstract

In this paper some new subclasses of Sakaguchi type bi-univalent functions of complex order in the open unit disc $U = \{z :| z |< 1\}$ defined by quasi-subordination are introduced. The estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in these subclasses are obtained. The results present in this paper would generalise and improve the related works of several earlier authors.

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1 Introduction and Preliminaries

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $U = \{z :| z |< 1\}$. By S, we denote the class of functions $f(z) \in A$ and univalent in U.

Let us denote by B, the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the unit disc U and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in U.$$

A function $f \in S$ is said to be starlike with respect to symmetric points if it satisfy the inequality

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0(z \in U).$$

The class of starlike functions with respect to symmetric points is denoted by S_s^* and was introduced by Sakaguchi [15].

A function $f \in S$ is said to be convex with respect to symmetric points if it satisfy the inequality

$$Re\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0 (z \in U).$$

The class of convex functions with respect to symmetric points is denoted by K_s and was introduced by Das and Singh [5].

A function $f \in S$ is said to be α starlike functions with respect to symmetric points if it satisfy the inequality

$$Re\left((1-\alpha)\frac{zf'(z)}{f(z)-f(-z)} + \alpha\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

The class of α-starlike functions with respect to symmetric points is denoted by $M_s^*(\alpha)$ and was introduced by Das and Singh [5]. In particular $M_s^*(0) \equiv S_s^*$ and $M_s^*(1) \equiv K_s$.

By $M_s(\alpha)$, we denote a class of functions $f \in S$ which satisfy the following conditions:

$$Re\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)(f(z)-f(-z)) + \alpha z(f(z)-f(-z))'}\right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

Obviously $M_s(\alpha)$ is a subclass of the class of α-symmetric functions with respect to symmetric points and was introduced by Selvaraj and Vasanthi [16]. Particularly $M_s(0) \equiv S_s^*$ and $M_s(1) \equiv K_s$.

For $s, t \in C$ and $s \neq t$, $S(s,t)$ is the class of functions $f \in S$ with the following conditions:

$$Re\left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right) > 0 (z \in U).$$

The class $S(s,t)$ was introduced by Frasin [7]. Also $S(1,-1) \equiv S_s^*$.
For $s, t \in C$ and $s \neq t$, $T(s,t)$ is the class of functions $f \in S$ with the following conditions:

$$Re\left(\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right) > 0 (z \in U).$$

The class $S(s,t)$ was introduced by Frasin [7]. Also $T(1,-1) \equiv K_s$.

For $s, t \in C$ and $s \neq t$, by $M_s^*(\alpha, s, t)$ we denote the class of functions $f \in S$ which satisfy the following conditions:

$$Re\left((1-\alpha)\frac{(s-t)zf'(z)}{f(sz)-f(tz)} + \alpha\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}\right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

The class $M_s^*(\alpha, s, t)$ is a subclass of generalized Sakaguchi-type functions and was introduced by Singh [18].

Also the following observations are obvious:
(i) $M_s^*(\alpha, 1, -1) \equiv M_s^*(\alpha)$.
(ii) $M_s^*(0, s, t) \equiv S(s, t)$.
(iii) $M_s^*(1, s, t) \equiv T(s, t)$.

Again for $s, t \in C$ and $s \neq t$, by $M_s(\alpha, s, t)$ we denote the class of functions $f \in S$ which satisfy the following conditions:

$$Re\left(\frac{(s-t)[zf'(z) + \alpha z^2 f''(z)]}{(1-\alpha)(f(sz) - f(tz)) + \alpha z(f(sz) - f(tz))'}\right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

The class $M_s(\alpha, s, t)$ is a subclass of generalized Sakaguchi-type functions.
In particular
(i) $M_s(\alpha, 1, -1) \equiv M_s(\alpha)$.
(ii) $M_s(0, s, t) \equiv S(s, t)$.
(iii) $M_s(1, s, t) \equiv T(s, t)$.

The inverse functions of the functions in the class S may not be defined on the entire unit disc U although the functions in the class S are invertible. However using Koebe-one quarter theorem [6] it is obvious that the image of U under every function $f \in S$ contains a disc of radius $\frac{1}{4}$. Hence every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z (z \in U)$$

and

$$f(f^{-1}(w)) = w \left(\mid w \mid < r_0(f) : r_0(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

By Σ, we denote the class of bi-univalent functions in U defined by (1).
Consider two functions f and g analytic in U. We say that f is subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in B$ for which $f(z) = g(u(z))$. This result is known as principle of subordination.
Robertson [14] introduced the concept of quasi-subordination in 1970. If f and ϕ are analytic functions, then we say that f is quasi-subordinate to ϕ (symbolically $f \prec_q \phi$) if there exists analytic functions k and ω with $|k(z)| \leq 1$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{f(z)}{k(z)} \prec \phi(z),$$

or it is equivalent to

$$f(z) = k(z)\phi(\omega(z)). \tag{1.3}$$

In particular for $k(z) = 1$, $f(z) = \phi(\omega(z))$, so that $f(z) \prec \phi(z)$ in U. It is obvious to see that the quasi-subordination is a generalization of the usual subordination. The work on quasi-subordination is quite extensive which finds interesting dimensions in some recent investigations [1,9,11,13].
Lewin [10] discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [4] investigated certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functins and bi-convex functions and obtained estimates on the initial coefficients. Also the subclasses of bi-univalent Sakaguchi-type functions were studied by various authors [2,3,8,17,19].
The earlier work on bi-univalent Sakaguchi type functions and quasi-subordination motivate us to define the following subclasses:
Also we assume that $\phi(z)$ is analytic in U with $\phi(0) = 1$ and let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + ...(B_1 \in R^+) \tag{1.4}$$

and

$$k(z) = A_0 + A_1 z + A_2 z^2 + ...(|k(z)| \leq 1, z \in U). \tag{1.5}$$

To avoid repetition, throughout the paper we assume that $s, t \in C$, $s \neq t$, $0 \leq \alpha \leq 1$ and $z \in U$.

Definition 1.1. A function $f \in \Sigma$ given by (1) is said to be in the class $M_s(\alpha, \gamma, \phi)$ if it satisfy the following conditions:

$$\frac{1}{\gamma}\left[\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} - 1\right] \prec_q (\phi(z) - 1) \tag{1.6}$$

and

$$\frac{1}{\gamma}\left[\frac{2wh'(w) + 2\alpha w^2 h''(w)}{(1-\alpha)(h(w) - h(-w)) + \alpha w(h(w) - h(-w))'} - 1\right] \prec_q (\phi(w) - 1) \tag{1.7}$$

where $h = f^{-1}$ and $z, w \in U$.
In particular,
(i) $M_s(\alpha, 1, \phi) \equiv M_s(\alpha, \phi)$.
(ii) $M_s(0, 1, \phi) \equiv S_s^*(\phi)$.
(iii) $M_s(1, 1, \phi) \equiv K_s(\phi)$.

Definition 1.2. A function $f \in \Sigma$ given by (1) is said to be in the class $M_s^*(\alpha, \gamma, \psi; s, t)$ if it satisfy the following conditions:

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{(s-t)zf'(z)}{f(sz) - f(tz)} + \alpha\frac{(s-t)(zf'(z))'}{(f(sz) - f(tz))'} - 1\right] \prec_q (\psi(z) - 1) \tag{1.8}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{(s-t)wv'(w)}{v(sw) - v(tw)} + \alpha\frac{(s-t)(wv'(w))'}{(v(sw) - v(tw))'} - 1\right] \prec_q (\psi(z) - 1), \tag{1.9}$$

where $v = f^{-1}$ and $z, w \in U$.
Particularly,
(i) $M_s^*(\alpha, 1, \psi; s, t) \equiv M_s^*(\alpha, \psi; s, t)$.
(ii) $M_s^*(0, 1, \psi; s, t) \equiv S(\psi; s, t)$.
(iii) $M_s^*(1, 1, \psi; s, t) \equiv T(\psi; s, t)$.
(iv) $M_s^*(\alpha, 1, \psi; 1, -1) \equiv M_s^*(\alpha, \psi)$.
(v) $M_s^*(0, 1, \psi; 1, -1) \equiv S_s^*(\psi)$.
(vi) $M_s^*(1, 1, \psi; 1, -1) \equiv K_s(\psi)$.

Definition 1.3. A function $f \in \Sigma$ given by (1) is said to be in the class $M_s(\alpha, \gamma, \eta; s, t)$ if it satisfy the following conditions:

$$(1.10) \qquad \frac{1}{\gamma}\left[\frac{(s-t)(zf'(z)+\alpha z^2 f''(z))}{(1-\alpha)(f(sz)-f(tz))+\alpha z(f(sz)-f(tz))'} - 1\right] \prec_q (\eta(z)-1)$$

and

$$(1.11) \qquad \frac{1}{\gamma}\left[\frac{(s-t)(wg'(w)+\alpha w^2 g''(w))}{(1-\alpha)(g(sw)-g(tw))+\alpha w(g(sw)-g(tw))'} - 1\right] \prec_q (\eta(w)-1)$$

where $g = f^{-1}$ and $z, w \in U$.

The following observations are obvious:
(i) $M_s(\alpha, 1, \eta; s, t) \equiv M_s(\alpha, \eta; s, t)$.
(ii) $M_s(0, 1, \eta; s, t) \equiv S(\eta; s, t)$.
(iii) $M_s(1, 1, \eta; s, t) \equiv T(\eta; s, t)$.
(iv) $M_s(\alpha, 1, \eta; 1, -1) \equiv M_s(\alpha, \eta)$.
(v) $M_s(0, 1, \eta; 1, -1) \equiv S_s^*(\eta)$.
(vi) $M_s(1, 1, \eta; 1, -1) \equiv K_s(\eta)$.

For deriving our main results, we need the following lemma:

Lemma 1.4.[12] If $p \in P$ be family of all functions p analytic in U for which $Re[p(z)] > 0$
and have the form $p(z) = 1 + p_1 z + p_2 z^2 + ...$ for $z \in U$, then $|p_n| \leq 2$ for each n.

2 Coefficient bounds for the function class $M_s(\alpha, \gamma, \phi)$

Theorem 2.1. If $f \in M_s(\alpha, \gamma, \phi)$, then

$$(2.1) \qquad |a_2| \leq min.\left[\frac{|A_0\gamma|B_1}{2(1+\alpha)}, \sqrt{\frac{|A_0\gamma|(B_1+|B_2|)}{2(1+2\alpha)}}\right]$$

and

$$(2.2) \qquad |a_3| \leq \frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{2(1+2\alpha)}|\gamma|.$$

Proof. As $f \in M_s(\alpha, \gamma, \phi)$, so by Definition 1.1, using the concept of quasi-subordination, there exists Schwarz functions $r(z)$ and $s(z)$ and analytic function $k(z)$ such that

$$\frac{1}{\gamma}\left[\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)(f(z)-f(-z))+\alpha z(f(z)-f(-z))'}-1\right]=k(z)(\phi(r(z)-1) \tag{2.3}$$

and

$$\frac{1}{\gamma}\left[\frac{2wh'(w)+2\alpha w^2 h''(w)}{(1-\alpha)(h(w)-h(-w))+\alpha w(h(w)-h(-w))'}-1\right]=k(w)(\phi(s(w))-1) \tag{2.4}$$

where $r(z) = 1 + r_1 z + r_2 z^2 + ...$ and $s(w) = 1 + s_1 w + s_2 w^2 + ...$

Define the functions $p(z)$ and $q(z)$ by

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1 z+(c_2-\frac{c_1^2}{2})z^2+...\right] \tag{2.5}$$

and

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left[d_1 z+(d_2-\frac{d_1^2}{2})z^2+...\right]. \tag{2.6}$$

Using (16) and (17) in (14) and (15) respectively, it yields

$$\frac{1}{\gamma}\left[\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)(f(z)-f(-z))+\alpha z(f(z)-f(-z))'}-1\right]=k(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right] \tag{2.7}$$

and

$$\frac{1}{\gamma}\left[\frac{2wh'(w)+2\alpha w^2 h''(w)}{(1-\alpha)(h(w)-h(-w))+\alpha w(h(w)-h(-w))'}-1\right]=k(w)\left[\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]. \tag{2.8}$$

But

$$\frac{1}{\gamma}\left[\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)(f(z)-f(-z))+\alpha z(f(z)-f(-z))'}-1\right]=\frac{1}{\gamma}\left[(1+\alpha)2a_2 z+2(1+2\alpha)a_3 z^2+...\right] \tag{2.9}$$

and

$$\frac{1}{\gamma}\left[\frac{2wh'(w)+2\alpha w^2 h''(w)}{(1-\alpha)(h(w)-h(-w))+\alpha w(h(w)-h(-w))'}-1\right]=\frac{1}{\gamma}[-2(1+\alpha)a_2 w+2(1+2\alpha)(2a_2^2-a_3)w^2+...]. \tag{2.10}$$

Again using (4) and (5) in (16) and (17) respectively, we get

$$k(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right]=\frac{1}{2}A_0B_1c_1z+\left[\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2c_1^2}{4}\right]z^2+... \tag{2.11}$$

and
(2.12)
$$k(w)\left[\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]=\frac{1}{2}A_0B_1d_1w+\left[\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2d_1^2}{4}\right]w^2+...$$

Using (20) and (22) in (18) and equating the coefficients of z and z^2, we get

$$\frac{(1+\alpha)}{\gamma}2a_2=\frac{1}{2}A_0B_1c_1 \tag{2.13}$$

and

$$\frac{2(1+2\alpha)a_3}{\gamma}=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2. \tag{2.14}$$

Again using (21) and (23) in (19) and equating the coefficients of w and w^2, we get

$$-\frac{(1+\alpha)}{\gamma}2a_2=\frac{1}{2}A_0B_1d_1 \tag{2.15}$$

and

$$\frac{2(1+2\alpha)(2a_2^2-a_3)}{\gamma}=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2. \tag{2.16}$$

From (24) and (26), it is clear that

$$c_1=-d_1 \tag{2.17}$$

and

$$a_2=\frac{A_0B_1c_1\gamma}{4(1+\alpha)}=-\frac{A_0B_1d_1\gamma}{4(1+\alpha)}. \tag{2.18}$$

Therefore on applying triangle inequality and using Lemma 1.4, (29) yields

$$|a_2|\leq\frac{|A_0\gamma|B_1}{2(1+\alpha)}. \tag{2.19}$$

Adding (25) and (27), it yields

$$\frac{4(1+2\alpha)}{\gamma}a_2^2=\frac{1}{2}A_0B_1(c_2-\frac{c_1^2}{2})+\frac{1}{2}A_0B_1(d_2-\frac{d_1^2}{2})+\frac{A_0B_2}{4}(c_1^2+d_1^2). \tag{2.20}$$

Using Lemma 1.4 and on applying triangle inequality in (31), we obtain

$$|a_2|^2\leq\frac{|A_0\gamma|(B_1+|B_2|)}{2(1+2\alpha)}. \tag{2.21}$$

So, the result (12) can be easily obtained from (30) and (32).

From (25), it yields that

$$|a_3| \leq \frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{2(1+2\alpha)}|\gamma|. \tag{2.22}$$

Using (26) along with Lemma 1.4 and triangle inequality in (27), it gives

$$|a_3| \leq \frac{|A_0|^2 B_1^2|\gamma|^2}{2(1+\alpha)^2}+\frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{2(1+2\alpha)}|\gamma|. \tag{2.23}$$

Now subtracting (27) from (25), we obtain

$$a_3 = a_2^2+\frac{A_1B_1(c_1-d_1)+A_0B_1(c_2-d_2)}{8(1+2\alpha)}\gamma. \tag{2.24}$$

Applying triangle inequality and using Lemma 1.4 and (32) in (35), it yields

$$|a_3| \leq \frac{|A_0|(2B_1+|B_2|)+|A_1|B_1}{2(1+2\alpha)}|\gamma|. \tag{2.25}$$

Since R.H.S. of (34) and (36) is greater than that of (33), so result (13) is obvious. On applying Lemma 1.4 in (36), the result (11) is obvious.

For $\gamma=1$, Theorem 2.1 gives the following result:
Corollary 2.2. If $f(z)\in M_s(\alpha,\phi)$, then

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{2(1+\alpha)},\sqrt{\frac{|A_0|(B_1+|B_2|)}{2(1+2\alpha)}}\right]$$

and

$$|a_3| \leq \frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{2(1+2\alpha)}.$$

For $\alpha=0$ and $\gamma=1$, Theorem 2.1 gives the following result:
Corollary 2.3. If $f(z)\in S_s^*(\phi)$, then

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{2},\sqrt{\frac{|A_0|(B_1+|B_2|)}{2}}\right]$$

and

$$|a_3| \leq \frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{2}.$$

For $\alpha=1$ and $\gamma=1$, Theorem 2.1 gives the following result:
Corollary 2.4. If $f(z)\in K_s(\phi)$, then

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{4},\sqrt{\frac{|A_0|(B_1+|B_2|)}{6}}\right]$$

and

$$|a_3| \leq \frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{6}.$$

3 Coefficient bounds for the function class $M_s^*(\alpha,\gamma,\psi;s,t)$

Theorem 3.1. If $f\in M_s^*(\alpha,\gamma,\psi;s,t)$, then

$$|a_2|\leq min.\left[\left[\frac{|A_0\gamma|B_1}{|(1+\alpha)(2-s-t)|}\right],\sqrt{\frac{|A_0\gamma|(B_1+|B_2|)}{|(1+2\alpha)(3-s^2-t^2-st)-(1+3\alpha)(s+t)(2-s-t)|}}\right]$$

and

$$|a_3|\leq min.\Big[\frac{|A_0\gamma|(B_1+|B_2|)}{|(1+2\alpha)(3-s^2-t^2-st)-(1+3\alpha)(s+t)(2-s-t)|}+\frac{(|A_1\gamma|+|A_0\gamma|)B_1)}{|(1+2\alpha)(3-s^2-t^2-st)|},$$
$$\frac{|\gamma|}{|(1+2\alpha)(3-s^2-t^2-st)|}\left[|\gamma||\frac{(1+3\alpha)(s+t)}{(1+\alpha)^2(2-s-t)}|B_1^2|A_0|^2+|A_0|(B_1+|B_2|)+|A_1|B_1\right]\Big].$$

Proof. As $f\in M_s^*(\alpha,\gamma,\psi;s,t)$, so by applying the concept of quasi-subordination in Definition 1.2 and following the procedure of Theorem 2.1, it yields

$$\text{(3.1)}\qquad \frac{1}{\gamma}\left[(1-\alpha)\frac{(s-t)zf'(z)}{f(sz)-f(tz)}+\alpha\frac{(s-t)(zf'(z))'}{(f(sz)-f(tz))'}-1\right]=k(z)\left[\psi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right]$$

and

$$\text{(3.2)}\quad \frac{1}{\gamma}\left[(1-\alpha)\frac{(s-t)wv'(w)}{v(sw)-v(tw)}+\alpha\frac{(s-t)(wv'(w))'}{(v(sw)-v(tw))'}-1\right]=k(w)\left[\psi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right].$$

where $k(z)$ is an analytic function.

On expanding (37) and (38) and equating coefficients of z, z^2 in (37) and w, w^2 in (38) and using the procedure of Theorem 2.1 along with Lemma 1.4, the proof of Theorem 3.1 is obvious.

For $s=1$, $t=-1$ and $\gamma=1$, Theorem 3.1 gives the following result due to Issar et al.[8]:

Corollary 3.2. If $f(z)\in M_s^*(\alpha,\psi)$, then

$$|a_2|\leq min.\left[\frac{|A_0|B_1}{2(1+\alpha)},\sqrt{\frac{|A_0|(B_1+|B_2|)}{2(1+2\alpha)}}\right]$$

and

$$|a_3|\leq\left[\frac{|A_0|(B_1+|B_2|)}{2(1+2\alpha)}+\frac{(|A_1|+|A_0|)B_1)}{2(1+2\alpha)}\right].$$

For $\alpha=0$ and $\gamma=1$, Theorem 3.1 gives the following result:

Corollary 3.3. If $f(z)\in S(\psi;s,t)$, then

$$|a_2|\leq min.\left[\left[\frac{|A_0|B_1}{|(2-s-t)|}\right],\sqrt{\frac{|A_0|(B_1+|B_2|)}{|(3-s^2-t^2-st)-(s+t)(2-s-t)|}}\right]$$

and

$$|a_3| \leq min.\Big[\frac{|A_0|(B_1+|B_2|)}{|(3-s^2-t^2-st)-(s+t)(2-s-t)|}+\frac{(|A_1|+|A_0|)B_1)}{|(3-s^2-t^2-st)|},\\ \frac{1}{|(3-s^2-t^2-st)|}\left[\left|\frac{(s+t)}{(2-s-t)}\right|B_1^2|A_0|^2+|A_0|(B_1+|B_2|)+|A_1|B_1\right]\Big].$$

For $\alpha = 1$ and $\gamma = 1$, Theorem 3.1 gives the following result:
Corollary 3.4. If $f(z) \in T(\psi; s, t)$, then

$$|a_2| \leq min.\left[\left[\frac{|A_0|B_1}{|2(2-s-t)|}\right], \sqrt{\frac{|A_0|(B_1+|B_2|)}{|3(3-s^2-t^2-st)-4(s+t)(2-s-t)|}}\right]$$

and

$$|a_3| \leq min.\Big[\frac{|A_0|(B_1+|B_2|)}{|3(3-s^2-t^2-st)-4(s+t)(2-s-t)|}+\frac{(|A_1|+|A_0|)B_1)}{|3(3-s^2-t^2-st)|},\\ \frac{1}{|3(3-s^2-t^2-st)|}\left[\left|\frac{(s+t)}{(2-s-t)}\right|B_1^2|A_0|^2+|A_0|(B_1+|B_2|)+|A_1|B_1\right]\Big].$$

4 Coefficient bounds for the function class $M_s(\alpha, \gamma, \eta; s, t)$

Theorem 4.1. If $f \in M_s(\alpha, \gamma, \eta; s, t)$, then

$$|a_2| \leq min.\left[\left[\frac{|A_0\gamma|B_1}{|(1+\alpha)(2-s-t)|}\right], \sqrt{\frac{|A_0\gamma|(B_1+|B_2|)}{|(1+2\alpha)(3-s^2-t^2-st)-(1+\alpha)^2(s+t)(2-s-t)|}}\right]$$

and

$$|a_3| \leq min.\Big[\frac{|s+t||A_0|^2|\gamma|^2B_1^2}{|(1+2\alpha)(3-s^2-t^2-st)(2-s-t)|}+\frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{|(1+2\alpha)(3-s^2-t^2-st)|}|\gamma|,\\ \left[\frac{|A_0|(B_1+|B_2|)}{|(1+2\alpha)(3-s^2-t^2-st)-(1+\alpha)^2(s+t)(2-s-t)|}+\frac{(|A_1|+|A_0|)B_1}{|(1+2\alpha)(3-s^2-t^2-st)|}\right]|\gamma|\Big].$$

Proof. Proceeding as in Theorem 2.1 and Theorem 3.1, Theorem 4.1 can be easily proved.

For $s = 1$, $t = -1$ and $\gamma = 1$, Theorem 4.1 gives the following result:
Corollary 4.2. If $f(z) \in M_s(\alpha, \eta)$, then

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{2(1+\alpha)}, \sqrt{\frac{|A_0|(B_1+|B_2|)}{2(1+2\alpha)}}\right]$$

and

$$|a_3| \leq \left[\frac{|A_0|(B_1+|B_2|)+|A_1|B_1}{2(1+2\alpha)}\right].$$

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