Analysis of Chaotic Situation in Three Species Food Chain Model

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Abstract

We study of dynamics of a three species food chain model, where the growth rate of middle predator is reduced due to prey population is also consumed by top predator and growth rate of prey is also suppressed due to the same reason. The goal of our study is to demonstrate the presence of chaos in the class of ecological models. The model is analyzed in term of stability like as local and global stability at each equilibrium point including interior equilibrium point. To check the the validate of theoretical formulation by numerical simulation by considering the required set of parameters.

Subject class :30C45.

Keywords: Mathematical Model, Equilibrium Points, Local stability, Global stability.

1 Introduction

Ecological systems are characterized by interaction between species and their natural environment. An important type of interaction which effects population dynamics of all species is predation. In natural environment food chains are very important system.

The first model in the population biology was formulated by Mathus and later on modified for more realistic situation by Verhulst. Lotka-Volterra model or the prey-predator explained the origin of the cycle in biological population. The Lotka-Volterra model was subsequently modified by adding a logistic growth term for prey and a Holling type-II functional response for the predator [Holling 1959][7]. Such a model was studied in detail by Rosenzweig and MacArthur [6] as a more realistic representation of a predator-prey system.

In 1991, Hastings and Powell [5] studied the three species food chain model numerically and showed the possibility of chaotic dynamics. In 1994, Klebanoff and Hastings [4] showed the existence of chaotic attractors in biologically reasonable regions of parameter and state space. In 2003, Sze-Bi Hsu et al.[3] studied a three tropic level food chain model with ratio-dependent Michaelis-Menten type functional responses and showed that the model is rich in boundary dynamics and is capable of generating extinction dynamics. Their results indicated that the extinction dynamics and sensitivity to initial population levels are not only preserved, but also enriched in the three tropic level food chain model. In 2014, Deng[2] considered a ratio dependent predator-prey system with Holling type II functional response, two time delays and stage structure for the predator was investigated by applying the normal form theory and central manifold theorem. In 2018, Panday et al.[1] observed that fear can stabilize the system from chaos to stable focus through the period halving phenomenon and concluded that chaotic dynamics can be controlled by the fear factors.

The main purpose of the paper is to examine the survivability of all species in the three species food chain model. Here, we have considered a mathematical model where there is an interaction between the three species of the food chain model, in which, the number of prey is consumed by per middle predator per unit time, the number of middle predator is consumed by top predator per unit time and the number of prey is consumed by per top predator per unit time.

2 Mathematical Model

Consider a continuous three-species food chain model with realistic growth term for prey and Holling type-II functional response for the Middle predator-prey interaction and Top predator-prey interaction that is,

$$\tag{2.1} \begin{cases} \frac{dX}{dT} = R_0X(1-\frac{X}{K_0}) - \frac{\alpha XY}{B_1+X} - \frac{\beta XZ}{B_2+X}, \\ \frac{dY}{dT} = \alpha_1\frac{\alpha XY}{B_1+X} - \frac{\gamma YZ}{B_3+Y} - D_1Y, \\ \frac{dZ}{dT} = \beta_1\frac{\beta XZ}{B_2+X} + \gamma_1\frac{\gamma YZ}{B_3+Y} - D_2Z. \end{cases}$$

Here X, Y and Z are the densities of prey, middle predator and top predator population. In the absence of predator, prey population grows logistically with an intrinsic growth rate R_0 and carrying capacity K_0. α, β and γ are is the maximum consumption rate of middle predator on prey, top predator on prey and top predator on middle predator respectively. B_1, B_2 and B_3 are the half saturation level of middle predator on prey, top predator on prey and top predator on middle predator respectively. α_1, β_1 and γ_1 represent the conversion rate of middle predator on prey, the conversion rate of top predator on prey and the conversion rate of top predator on middle predator respectively. D_1 and D_2 are the natural death rate of middle and top predator respectively.

We can rescale this model by introducing dimensionless variables $x = \frac{X}{K_0}$, $y = \frac{\alpha_1 Y}{K_0}$, $z = \frac{\beta_1 Z}{K_0}$ and $t = TR_0$. Then (2.1) becomes

$$\tag{2.2} \begin{cases} \frac{dx}{dt} = x(1-x) - a_1\frac{xy}{b_1+x} - a_3\frac{xz}{b_2+x}, \\ \frac{dy}{dt} = a_2\frac{xy}{b_1+x} - a_5\frac{yz}{b_3+y} - d_1y, \\ \frac{dz}{dt} = a_4\frac{xz}{b_2+x} + a_6\frac{yz}{b_3+y} - d_2z. \end{cases}$$

where $a_1 = \frac{\alpha}{R_0\alpha_1}$, $a_2 = \frac{\alpha\alpha_1^2}{R_0\alpha_1}$, $a_3 = \frac{\beta}{R_0\beta_1}$, $a_4 = \frac{\beta\beta_1^2}{R_0\beta_1}$, $a_5 = \frac{\gamma\alpha_1}{R_0\beta_1}$, $a_6 = \frac{\gamma\gamma_1\beta_1^2}{R_0\beta_1\alpha_1}$, $b_1 = \frac{B_1}{K_0}$, $b_2 = \frac{B_2}{K_0}$, $b_3 = \frac{B_3\alpha_1}{K_0}$, $d_1 = \frac{D_1}{R_0}$ and $d_2 = \frac{D_2}{R_0}$.

The set $\Omega = \{(x, y, z) : 0 \le x + y + z \le \frac{M}{\mu}\}$, where $M = \frac{(1+\mu)^2}{4}$ and $\mu = \min\{(d_1 + a_1 - a_2), (d_2 + a_3 - a_4 + a_5 - a_6)\}$ is a region of attraction for the system (2.1) and it attracts all solutions initiating in the interior of the positive octant.

3 Equilibrium Analysis

The above model (2.2) have five non-negative equilibrium points. They are listed as follows.
(1) The trivial equilibrium $E_0(0,0,0)$, always exists.
(2) $E_1(x_1,0,0)$, if $x_1 = 1$.
(3) $E_2(x_2,y_2,0)$, where x_2 and y_2 are given by: $x_2 = \frac{b_1 d_1}{a_2 - d_1}$, if $a_2 > d_1$ and $y_2 = \frac{a_2 b_1 (a_2 - (1+b_1)d_1)}{a_1(a_2-d_1)^2}$. ; if $\frac{a_2}{d_1} - b_1 > 1$.
(4) $E_3(x_3,0,z_3)$, where x_3 and z_3 are given by: $x_3 = \frac{b_2 d_2}{a_4 - d_2}$; if $a_4 > d_2$ and $z_3 = \frac{a_4 b_2 (a_4 - (1+b_2)d_2)}{a_3(a_4-d_2)^2}$. ; if $\frac{a_4}{d_2} - b_2 > 1$.
(5) To obtain the equilibrium point $E^*(x^*,y^*,z^*)$, we equate to zero the right hand sides of the system of equation (2.2) to zero which on solving gives,

$$x^*(1-x^*) - \frac{a_1 x^* y^*}{b_1 + x^*} - \frac{a_3 x^* z^*}{b_2 + x^*} = 0, \tag{3.1}$$

$$\frac{a_2 x^* y^*}{b_1 + x^*} - \frac{a_5 y^* z^*}{b_3 + y^*} - d_1 y^* = 0 \tag{3.2}$$

and

$$\frac{a_4 x^* z^*}{b_2 + x^*} + \frac{a_6 y^* z^*}{b_3 + y^*} - d_2 z^* = 0. \tag{3.3}$$

From these equations, we get

$$(x^*)^3 + A_1 (x^*)^2 + A_2 x^* + A_3 = 0 \tag{3.4}$$

where A_1, A_2, A_3 are constants depending on the variables a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, d_1 and d_2. If all A_i, $i = 1,2,3$ are positive then equation (3.4) does not have any positive roots thus, in this case, there is no equilibrium point E^*. Whereas, if any of the A_i, $i = 1,2,3$ is negative then (3.4) has atleast one positive root ensuring the existence of the equilibrium point E^*.

4 Stability

4.1 Local Stability Analysis

The local stability of the system (2.2) around each of the equilibrium point is obtained by computing the Jacobian matrix $V(E)$, where $E(x,y,z)$ is an equilibrium point. The Jacobian matrix corresponding to the system (2.2) is given below,

$$J(E) = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

where $P_{11} = 1-2x-\frac{a_1 y}{(b_1+x)}+\frac{a_1 xy}{(b_1+x)^2}-\frac{a_3 z}{(b_2+x)}+\frac{a_3 xz}{(b_2+x)^2}$, $P_{12} = -\frac{a_1 x}{(b_1+x)}$, $P_{13} = -\frac{a_3 x}{(b_2+x)}$, $P_{21} = \frac{a_2 y}{(b_1+x)}-\frac{a_2 xy}{(b_1+x)^2}$, $P_{22} = \frac{a_2 x}{(b_1+x)}-\frac{a_5 z}{(b_3+y)}+\frac{a_5 yz}{(b_3+y)^2}-d_1$, $P_{23} = -\frac{a_5 y}{(b_3+y)}$, $P_{31} = \frac{a_4 z}{(b_2+x)}-\frac{a_4 xz}{(b_2+x)^2}$, $P_{32} = \frac{a_6 z}{(b_3+y)}-\frac{a_6 yz}{(b_3+y)^2}$ and $P_{33} = \frac{a_4 x}{(b_2+x)}+\frac{a_6 y}{(b_4+y)}-d_2$.

The stability conditions for the equilibrium points E_0, E_1, E_2, E_3 & E^* are stated in the following cases:

Case-I: The Jacobian matrix $J(E_0)$ around trivial equilibrium point $E_0(0,0,0)$ is given by

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

and the eigenvalues of the Jacobian matrix are 1, $-d_1$, $-d_2$. Since one of the eigenvalues of $J(E_0)$ at equilibrium point $E_0(0,0,0)$ is positive. Hence the trivial equilibrium point $E_0(0,0,0)$, is always unstable.

Case-II: The Jacobian matrix $J(E_1)$ around equilibrium point $E_1(1,0,0)$ is given by

$$J(E_1) = \begin{bmatrix} -1 & \frac{-a_1}{(b_1+1)} & \frac{-a_3}{(b_2+1)} \\ 0 & \frac{a_2}{(b_1+1)}-d_1 & 0 \\ 0 & 0 & \frac{a_4}{(b_2+1)}-d_2 \end{bmatrix}$$

and the eigenvalues of the Jacobian matrix $J(E_1)$ at the equilibrium point $E_1(1,0,0)$ are -1, $\frac{a_2}{(b_1+1)}-d_1$, $\frac{a_4}{(b_2+1)}-d_2$. Hence the equilibrium point $E_1(1,0,0)$, is locally asymptotically stable if $\frac{a_2}{(b_1+1)d_1} < 1$ and $\frac{a_4}{(b_2+1)d_2} < 1$.

Case-III: The Jacobian matrix $J(E_2)$ around equilibrium point $E_2(x_2,y_2,0)$ is given by

$$J(E_2) = \begin{bmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ 0 & 0 & P_7 \end{bmatrix}$$

and the corresponding characteristic equation is

$$(P_7-\lambda)(\lambda^2-(P_1+P_5)\lambda+P_1P_5-P_2P_4) = 0 \tag{4.1}$$

where $P_1 = 1-2x_2-\frac{a_1 y_2}{(b_1+x_2)}+\frac{a_1 x_2 y_2}{(b_1+x_2)^2}$, $P_2 = -\frac{a_1 x_2}{(b_1+x_2)}$, $P_3 = -\frac{a_3 x_2}{(b_2+x_2)}$, $P_4 = \frac{a_2 y_2}{(b_1+x_2)}-\frac{a_2 x_2 y_2}{(b_1+x_2)^2}$, $P_5 = \frac{a_2 x_2}{(b_1+x_2)}-d_1$, $P_6 = -\frac{a_5 y_2}{(b_3+y_2)}$ and $P_7 = \frac{a_4 x_2}{(b_2+x_2)}+\frac{a_6 y_2}{(b_3+y_2)}-d_2$.

Now all the eigenvalues of the characteristic equation are negative or having negative real parts if $P_7 < 0$, $(P_1+P_5) < 0$ and $(P_1P_5-P_2P_4) > 0$. Hence under these conditions the equilibrium point $E_2(x_2,y_2,0)$ is locally asymptotically stable.

Case-IV: The Jacobian matrix $J(E_3)$ around equilibrium point $E_3(x_3,0,z_3)$ is given by

$$J(E_3) = \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ 0 & Q_4 & 0 \\ Q_5 & Q_6 & Q_7 \end{bmatrix}$$

and the corresponding characteristic equation is

$$(Q_4-\lambda)(\lambda^2-(Q_1+Q_7)\lambda+Q_1Q_7-Q_3Q_5)=0 \tag{4.2}$$

where $Q_1=1-2x_3-\frac{a_3z_3}{(b_2+x_3)}+\frac{a_3x_3z_3}{(b_2+x_3)^2}$, $Q_2=-\frac{a_1x_3}{(b_1+x_3)}$, $Q_3=-\frac{a_3x_3}{(b_2+x_3)}$, $Q_4=\frac{a_2x_3}{(b_1+x_3)}-\frac{a_5z_3}{b_3}-d_1$, $Q_5=\frac{a_4z_3}{(b_2+x_3)}-\frac{a_4x_3z_3}{(b_2+x_3)^2}$, $Q_6=\frac{a_6z_3}{b_3}$ and $Q_7=\frac{a_4x_3}{(b_2+x_3)}-d_2$.
Now all the eigenvalues of the characteristic equation are negative or having negative real parts if $Q_4<0$, $(Q_1+Q_7)<0$ and $(Q_1Q_7-Q_3Q_5)>0$. Thus under these assumptions the equilibrium point $E_2(x_3,0,z_3)$ is locally asymptotically stable.

Case-V: The Jacobian matrix $J(E^*)$ around the interior equilibrium point $E^*(x^*,y^*,z^*)$ is given by

$$J(E)=\begin{bmatrix} M_1 & M_2 & M_3\\ M_4 & M_5 & M_6\\ M_7 & M_8 & M_9\end{bmatrix}$$

and the corresponding characteristic equation is

$$\lambda^3+\lambda^2\sigma_1+\lambda\sigma_2+\sigma_3=0 \tag{4.3}$$

where $\sigma_1=-(M_1+M_5+M_9)$, $\sigma_2=(M_1M_5+M_1M_9+M_5M_9-M_2M_4-M_3M_7-M_8M_6)$ and $\sigma_3=M_1M_8M_6+M_2M_4M_9+M_3M_7M_5-M_2M_7M_6-M_3M_4M_8-M_1M_5M_9$.
with $M_1=1-2x^*-\frac{a_1y^*}{(b_1+x^*)}+\frac{a_1x^*y^*}{(b_1+x^*)^2}-\frac{a_3z^*}{(b_2+x^*)}+\frac{a_3x^*z^*}{(b_2+x^*)^2}$, $M_2=-\frac{a_1x^*}{(b_1+x^*)}$, $M_3=-\frac{a_3x^*}{(b_2+x^*)}$, $M_4=\frac{a_2y^*}{(b_1+x^*)}-\frac{a_2x^*y^*}{(b_1+x^*)^2}$, $M_5=\frac{a_2x^*}{(b_1+x^*)}-\frac{a_5z^*}{(b_3+y^*)}+\frac{a_5y^*z^*}{(b_3+y^*)^2}-d_1$, $M_6=-\frac{a_5y^*}{(b_3+y^*)}$, $M_7=\frac{a_4z^*}{(b_2+x^*)}-\frac{a_4x^*z^*}{(b_2+x^*)^2}$, $M_8=\frac{a_6z^*}{(b_3+y^*)}-\frac{a_6y^*z^*}{(b_3+y^*)^2}$ and $M_9=\frac{a_4x^*}{(b_2+x^*)}+\frac{a_6y^*}{(b_4+y^*)}-d_2$.

Therefore, the interior equilibrium will be stable if the coefficients of the characteristic equation satisfy the Routh-Hurwitz criterion, i.e. $\sigma_1>0$, $\sigma_3>0$ and $\sigma_1\sigma_2-\sigma_3>0$.

4.2 Global Stability Analysis

Theorem 4.1. *(a) The equilibrium point $E_1(x_1,0,0)$, is globally asymptotically stable if $d_1>(a_1+a_2)$ and $d_2>(a_3+a_4+a_6-a_5)$.*
(b) The equilibrium point $E_2(x_2,y_2,0)$, is globally asymptotically stable if $a_1y_2<(b_1+x)(b_1+x_2)$.
(c) The equilibrium point $E_3(x_3,0,z_3)$, is globally asymptotically stable if $a_3z_3<(b_2+x)(b_2+x_3)$.
(d) The interior equilibrium point $E^*(x^*,y^*,z^*)$, is globally asymptotically stable if $\frac{a_1y^*}{(b_1+x^*)(b_1+x)}+\frac{a_3z^*}{(b_2+x^*)(b_2+x)}<1$.

Proof. (a) To prove the global stability of the equilibrium point$E_1(x_1,0,0)$, we construct a Lyapunov function as follows:

$$L_1=\frac{1}{2}(x-1)^2+\frac{1}{2}y^2+\frac{1}{2}z^2+y+z.$$

The derivative of the above equation is

$$\frac{dL_1}{dt}=(x-1)\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}+\frac{dy}{dt}+\frac{dz}{dt},$$

$$\frac{dL_1}{dt} \leq -x(1-x)^2 - \frac{a_1x^2y}{b_1+x} - \frac{a_3x^2z}{b_2+x} - \frac{a_5y^2z}{b_3+y} + (a_1+a_2-d_1)y + (a_3+a_4+a_6-a_5-d_2)z$$
$$+(a_2-d_1)y^2 + (a_4+a_6-d_2)z^2.$$

So $\frac{dL_1}{dt} \leq 0$ if $d_1 > (a_1+a_2)$, $d_2 > (a_3+a_4+a_6-a_5)$ and $\frac{dL_1}{dt} = 0$, when $E_1(x_1,0,0) = (1,0,0)$. Hence E_1 is globally asymptotically stable.

(b) To prove the global stability of the equilibrium point $E_2(x_2,y_2,0)$, we construct a Lyapunov function as follows:

$$L_2 = \left(x - x_2 - x_2 ln\frac{x}{x_2}\right) + \alpha\left(y - y_2 - y_2 ln\frac{y}{y_2}\right).$$

The derivative of the above equation is

$$\frac{dL_2}{dt} = \left(1 - \frac{x_2}{x}\right)\frac{dx}{dt} + \alpha\left(1 - \frac{y_2}{y}\right)\frac{dy}{dt}.$$

$$\frac{dL_2}{dt} = -(x-x_2)^2 + \frac{(\alpha a_2 b_1 - a_1 b_1 - a_1 x_2)(x-x_2)(y-y_2)}{(b_1+x)(b_1+x_2)} + \frac{a_1(x-x_2)^2 y_2}{(b_1+x)(b_1+x_2)}.$$

Now if we choose $\alpha = \frac{a_1(b_1+x_2)}{a_2 b_1}$, then the above equation reduces to

$$\frac{dL_2}{dt} \leq -(x-x_2)^2\left(1 - \frac{a_1 y_2}{(b_1+x)(b_1+x_2)}\right).$$

So $\frac{dL_2}{dt} \leq 0$ if $\frac{a_1y_2}{(b_1+x)(b_1+x_2)} < 1$, i.e. $a_1 y_2 < (b_1+x)(b_1+x_2)$ and $\frac{dL_2}{dt} = 0$ when $(x,y,z) = (x_2,y_2,0)$. Hence E_2 is globally asymptotically stable.

(c) To prove the global stability of the equilibrium point $E_3(x_3,0,z_3)$, we construct a Lyapunov function as follows:

$$L_3 = \left(x - x_3 - x_3 ln\frac{x}{x_3}\right) + \beta\left(z - z_3 - z_3 ln\frac{z}{z_3}\right).$$

The derivative of the above equation is

$$\frac{dL_3}{dt} = \left(1 - \frac{x_3}{x}\right)\frac{dx}{dt} + \beta\left(1 - \frac{z_3}{z}\right)\frac{dz}{dt}.$$

$$\frac{dL_3}{dt} = -(x-x_3)^2 - \frac{(\beta a_4 b_2 - a_3 b_2 - a_3 x_3)(x-x_3)(z-z_3)}{(b_2+x)(b_2+x_3)} + \frac{a_3 z_3 (x-x_3)^2}{(b_2+x)(b_2+x_3)}.$$

Now if we choose $\beta = \frac{a_3(b_2+x_3)}{a_4 b_2}$, then eq. reduces to

$$\frac{dL_3}{dt} \leq -(x-x_3)^2\left(1 - \frac{a_3 z_3}{(b_2+x)(b_2+x_3)}\right).$$

So $\frac{dL_3}{dt} \leq 0$ if $\frac{a_3 z_3}{(b_2+x)(b_2+x_3)} < 1$, i.e. $a_3 z_3 < (b_2+x)(b_2+x_3)$ and $\frac{dL_3}{dt} = 0$ when $(x,y,z) = (x_3, 0, z_3)$. Hence E_3 is globally asymptotically stable.

(d)To prove the global stability of the equilibrium point $E^*(x^*, y^*, z^*)$, we construct a Lyapunov function as follows:

$$L_4 = \left(x - x^* - x^* ln\frac{x}{x^*}\right) + \gamma\left(y - y^* - y^* ln\frac{y}{y^*}\right) + \delta\left(z - z^* - z^* ln\frac{z}{z^*}\right).$$

The derivative of the above equation is

$$\frac{dL_4}{dt} = \left(1 - \frac{x^*}{x}\right)\frac{dx}{dt} + \gamma\left(1 - \frac{y^*}{y}\right)\frac{dy}{dt} + \delta\left(1 - \frac{z^*}{z}\right)\frac{dz}{dt},$$

$$\begin{aligned}\frac{dL_4}{dt} &= -(x-x^*)^2 + \frac{((\gamma a_2 - a_1)b_1 - a_1 x^*)(y-y^*)(x-x^*)}{(b_1+x^*)(b_1+x)} + \frac{a_1(x-x^*)^2 y^*}{(b_1+x^*)(b_1+x)} \\ &+ \frac{((\delta a_4 - a_3)b_2 - a_3 x^*)(x-x^*)(z-z^*)}{(b_2+x^*)(b_2+x)} + \frac{a_3(x-x^*)^2 z^*}{(b_2+x^*)(b_2+x)} \\ &+ \frac{(-\gamma a_5 b_3 + \gamma a_5 z^* + \delta a_6 b_3)(y-y^*)(z-z^*)}{(b_3+y)(b_3+y^*)} - \frac{\gamma a_5 y^*(z-z^*)^2}{(b_3+y)(b_3+y^*)}.\end{aligned}$$

Now if we choose $(\gamma a_2 - a_1)b_1 - a_1 x^* = 0$, $(\delta a_4 - a_3)b_2 - a_3 x^* = 0$ & $-\gamma a_5 b_3 + \gamma a_5 z^* + \delta a_6 b_3 = 0$ then the the above equation reduces to

$$\frac{dL_4}{dt} \leq -(x-x^*)^2\left(1 - \frac{a_1 y^*}{(b_1+x^*)(b_1+x)} - \frac{a_3 z^*}{(b_2+x^*)(b_2+x)}\right) - \frac{\gamma a_5 y^*(z-z^*)^2}{(b_3+y)(b_3+y^*)}.$$

So $\frac{dL_4}{dt} \leq 0$ if $\frac{a_1 y^*}{(b_1+x^*)(b_1+x)} + \frac{a_3 z^*}{(b_2+x^*)(b_2+x)} < 1$ and $\frac{dL_4}{dt} = 0$ when $(x,y,z) = (x^*, y^*, z^*)$. Hence E^* is globally asymptotically stable.

□

5 Numerical Simulation

In this section, a numerical verification is provided for the existence and the stability properties of the equilibrium point E^*. The model is simulated using the different set of parameter values. To check the feasibility of our analysis regarding stability conditions, we have conducted some numerical computation using MATHEMATICA by choosing the values of the parameters in model (2.2) as $a_1 = 0.6$, $a_2 = 0.1$, $a_3 = 2.0$, $a_4 = 0.08$, $a_5 = 2.31$, $a_6 = 2.0$, $b_1 = 0.099$, $b_2 = 0.95$, $b_3 = 2.0$, $d_1 = 0.00137$, $d_2 = 0.1905$.

For the above set of values of the parameters, the equilibrium point E^* is given as $x^* = 0.79358$, $y^* = 0.166951$, $z^* = 0.0821176$.

The eigenvalues of the Variational matrix $V(E*)$ corresponding to the equilibrium E^* for the model system (2.2) are $-0.650929, -0.533451, -0.910288$. We note that every eigenvalue of variational matrix is negative. Hence the equilibrium point E^* is asymptotically stable for these values of parameters.

On varying the values of the parameter a_4, the conversion rate of the top predator on prey, and fixing the values of the other parameters as $a_1 = 0.6$, $a_2 = 0.1$, $a_3 = 2.0$, $a_5 = 2.31$, $a_6 = 2.0$, $b_1 = 0.099$, $b_2 = 0.95$, $b_3 = 2.0$, $d_1 = 0.00137$, $d_2 = 0.1905$ one can see from Figure 1 that when $a_4 = 0.08$, the population of all the three species prey, middle predator and the top predator is steady and there is an equilibrium among them. The figure 2 shows a limit cycle between the top and the middle predator, when $a_4 = 0.08$ which indicates that there is a smooth interaction among the prey and the predator.

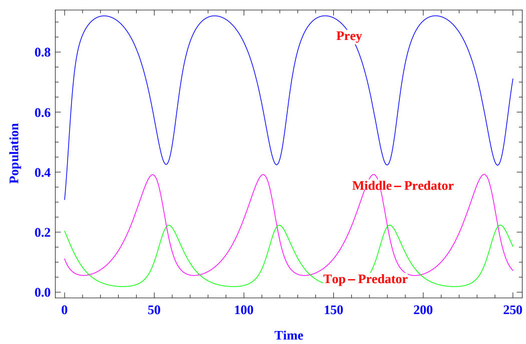

Fig. 1: At $a_4 = 0.08$, variation among populations with respect to time.

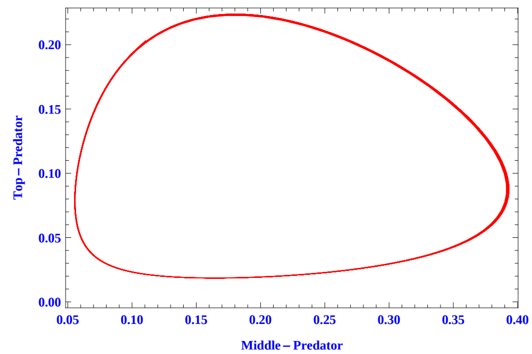

Fig. 2: Limit Cycle between Middle-Predator and Top-Predator.

But, as soon as the value of a_4 is increased form 0.08 to say 0.18 there is a disturbance, as shown in figure 3, in the population of prey and it decreases which in turn also affects the population of the middle and top predator which also decreases continuously. On further increasing the value of a_4 the prey is almost exhausted and the population of middle predator starts to reduce rapidly than that of the top predator. If the value of a_4 is increased upto 0.28 a chaotic situation is seen in figure 4 between the middle predator and the top predator showing the survival struggle for the middle predator as it has now become the only prey for the top predator.

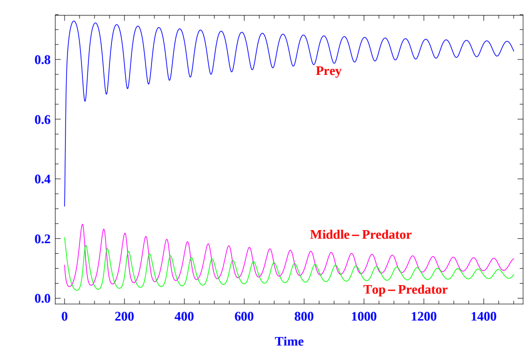

Fig. 3: At $a_4 = 0.18$, variation among populations with respect to time.

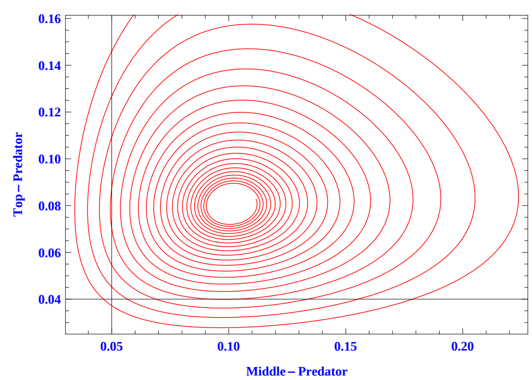

Fig. 4: Variation between Middle-Predator and Top-Predator at $a_4 = 0.28$.

Figure 5, shows a strange attractor obtained when a_4 is taken as 0.08 and the value of half saturation rate of top predator on prey b_2 is taken as 0.395 and that of top predator on middle predator b_3 is taken as 1.1.

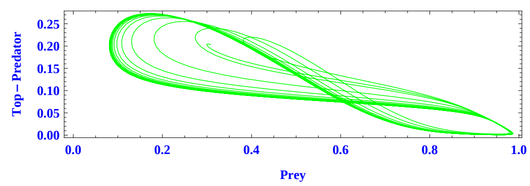

Fig. 5: Variation in Prey and Top-Predator at $a_4 = 0.08$, $b_2 = 0.395$ and $b_3 = 1.1$

6 Conclusion

It is very clear from the numerical calculations and the figures that as soon as the conversion rate of the top predator on prey crosses a certain limit then the equilibrium among the three species gets disturbed and if the conversion rate is further increased then the survivability of the middle predator becomes difficult and its population decreases.

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