Minimal Statistical Immersions of Statistical Manifolds.

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Abstract

In this paper we study certain properties of statistical immersions. A necessary condition is obtained for a statistical immersion to be an immersion with α-connections in the case of co-dimension one. Then necessary condition for a statistical immersion into a dually flat statistical manifold to be minimal in the case of immersion of co-dimension one is given. Also obtained condition for a statistical immersion to be minimal for statistical manifolds with α- connections.

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1 Introduction

Information geometry provides a geometric approach to statistical models [1], [2], [8], [9] . The information geometric tools are widely used in various fields such as statistics, information theory, neural network, statistical physics, neuroscience etc. Geometric properties of statistical manifolds defined by embedding functions are studied using concepts from affine differential geometry [5], [7],[10]. Geometry of statistical immersions of statistical manifold are studied in [6]. In [3] H Furuhata Studied affine immersion of statistical manifold. He has given a necessary and sufficient condition for a statistical structure to be realized as a minimal affine hyper-surface. Also he proved a necessary and sufficient condition for a statistical structure to be realized as a minimal centro-affine immersion of co-dimension two. In this paper we study minimal statistical immersions of statistical manifolds.

A pseudo - Riemannian Manifold $(\mathbf{M}, g)$ with affine connection ∇ is called a statistical manifold if ∇g is symmetric. For a statistical manifold $(\mathbf{M}, \nabla, g)$ the dual connection ∇^* is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{1.1}$$

for any $X, Y, Z \in \mathcal{X}(\mathbf{M})$, where $\mathcal{X}(\mathbf{M})$ denotes the set of all vector fields on $\mathbf{M}$. $(\mathbf{M}, \nabla^*, g)$ also a statistical manifold called the dual statistical manifold of $(\mathbf{M}, \nabla, g)$. Let R^∇ and R^{∇^*} be the curvature tensor of ∇ and ∇^* respectively, it follows from (1.1) that,

$$g(R^\nabla(X,Y)Z, W) = -g(Z, R^{\nabla^*}(X,Y)W) \tag{1.2}$$

for X, Y, Z and W in $\mathcal{X}(\mathbf{M})$. We say $(\mathbf{M}, \nabla, \nabla^*, g)$ has constant curvature k if

$$R^\nabla(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.3}$$

A statistical manifold with constant curvature $k = 0$ is called a flat statistical manifold and in that case $(\mathbf{M}, \nabla, \nabla^*, g)$ is called a dually flat statistical manifold. A parametric family of affine connections ∇^{α}, $\alpha \in \mathbb{R}$ on $\mathbf{M}$ are defined by

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^* \tag{1.4}$$

with

$$\nabla^{(1)} = \nabla,\ \nabla^{(-1)} = \nabla.$$

For $\alpha = 0$, we have the Levi-Civita connection with respect to g. That is,

$$\nabla^{(0)} = \frac{1}{2}(\nabla + \nabla^*)$$

and is also denoted by $\widehat{\nabla}$.

Definition 1.1. *Let $(\mathbf{M}, \nabla, \nabla^*, g)$ be an n- dimensional statistical manifold. Then the difference tensor $K(X,Y)$ is defined by*

$$K(X,Y) = \nabla^*_X Y - \nabla_Y X; \quad \forall\ X, Y \in \mathcal{X}(\mathbf{M}). \tag{1.5}$$

Remark 1.1. *From*

$$\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*),$$

$K(X,Y) = 2(\widehat{\nabla}_X Y - \nabla_Y X) = 2(\nabla^*_X Y - \widehat{\nabla}_Y X)$ *for every $X, Y \in \mathcal{X}(\mathbf{M})$. Also we can see that*

$$\nabla^{\alpha}_X Y = \widehat{\nabla}_X Y - \frac{\alpha}{2}K(X,Y)$$

with

$$\nabla_X Y = \widehat{\nabla}_X Y - \tfrac{1}{2}K(X,Y),\ \nabla^*_X Y = \widehat{\nabla}_X Y + \tfrac{1}{2}K(X,Y).$$

Definition 1.2. *Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold of dimension n. The Ricci curvature tensor field of ∇ is denoted by Ric^{∇}, defined as*

$$Ric^{\nabla}(Y,Z) = tr\{X \mapsto R^{\nabla}(X,Y)Z\} \tag{1.6}$$

and the scalar curvature with respect to g is denoted by $Scal^{(\nabla,g)}$, defined as

$$Scal^{(\nabla,g)} = tr_g\{(X,Y) \mapsto Ric^{\nabla}(X,Y)\}. \tag{1.7}$$

The curvature tensor $R^{\nabla^{\alpha}}$ for the α connection ∇^{α} satisfies

$$\begin{aligned} R^{\nabla^{(\alpha)}}(X,Y)Z &= \frac{1+\alpha}{2}R^{\nabla}(X,Y)Z + \frac{1-\alpha}{2}R^{\nabla^*}(X,Y)Z \\ &\quad + \frac{1-\alpha^2}{4}(K(Y,K(X,Z)) - K(X,K(Y,Z))) \end{aligned}$$

Under the local co-ordinates we have

$$\begin{aligned} \nabla_{\partial_i}\partial_j &= \sum_{\ell} \Gamma_{ij}^{\ell}\partial_{\ell} \\ R^{\nabla}(\partial_i,\partial_j)\partial_{\ell} &= \sum_{k} R_{\ell ij}^{k}\partial_k \end{aligned}$$

with

$$R_{\ell ij}^{k} = \partial_i\Gamma_{ij}^{k} - \partial_j\Gamma_{\ell i}^{k} + \sum_{m}(\Gamma_{mi}^{k}\Gamma_{ij}^{m} - \Gamma_{\ell i}^{m}\Gamma_{mj}^{k})$$

and

$$K(\partial_i,\partial_j) = \sum_k K_{ij}^{k}\partial_k = \sum_k(\Gamma_{ij}^{*k} - \Gamma_{ij}^{k})\partial_k$$

with

$$\begin{aligned} K(\partial_i, K(\partial_j,\partial_{\ell})) &= K(\partial_i, \sum_m K_{j\ell}^{m}\partial_m) \\ &= \sum_m K_{j\ell}^{m} K(\partial_i,\partial_m) \\ &= \sum_{m,k} K_{j\ell}^{m} K_{im}^{k}\partial_k \end{aligned}$$

Also

$$R_{\ell ij}^{(\alpha)k} = \frac{1+\alpha}{2}R_{\ell ij}^{k} + \frac{1-\alpha}{2}R_{\ell ij}^{*k} + \frac{1-\alpha^2}{4}(\sum_m K_{i\ell}^{m}K_{jm}^{k} - \sum_m K_{im}^{k}K_{j\ell}^{m}).$$

The Ricci tensor $R_{ij}^{\alpha} = \sum_k R_{ikj}^{(\alpha)k}$ gives

$$R_{ij}^{(\alpha)} = \frac{1+\alpha}{2}R_{ij} + \frac{1-\alpha}{2}R_{ij}^{*} + \frac{1-\alpha^2}{4}(\sum_{m,k} K_{ik}^{m}K_{jm}^{k} - \sum_{m,k} K_{ij}^{m}K_{km}^{k})$$

and the scalar curvature $Scal^{(\nabla,h)} = \sum_{j\ell} h^{j\ell}R_{\ell,j}^{\nabla}$.

2 Minimal affine immersions

In this section we study about affine immersions and minimal affine immersions of statistical manifolds

Definition 2.1. *Let* $\mathbf{M}$ *be an* n*- dimensional manifold, an immersion* $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ *is called affine immersion if there exist a transversal vector field* ξ *on* $\mathbf{M}$ *such that*

$$T_{f(x)}(\mathbb{R}^{n+1}) = f_*(T_x\mathbf{M}) \bigoplus Span\{\xi_x\} \tag{2.1}$$

for $x \in \mathbf{M}$ *and*

$$D_X f_* Y = f_*(\nabla_X Y) + h(X,Y)\xi \tag{2.2}$$

Where D *and* ∇ *are affine connections on* $\mathbb{R}^{n+1}$ *and* $\mathbf{M}$ *respectively and* h *is called second fundamental form.*
The affine shape operator S *and transversal connection form* τ *are given by*

$$D_X \xi = -f_*(SX) + \tau(X)\xi. \tag{2.3}$$

Definition 2.2. *Let* $\mathbf{M}$ *be an* n*- dimensional manifold and* $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ *be an affine immersion of co-dimension one,* f *is called non-degenerate if* h *is non-degenerate. That is* $h(X,Y) = 0$ *for all vector field* Y *on* $\mathbf{M}$ *implies* $X = 0$*.*

Definition 2.3. *An affine immersion* $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ *is called equiaffine if* $D_X\xi$ *has only tangential part. That is the transversal connection form* $\tau = 0$*.*

Fundamental Equations

Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ be an affine immersion of co-dimension one, ξ be an arbitrary transversal vector field on $\mathbf{M}$, D be a standard connection on $\mathbb{R}^{n+1}$. Let ∇ be the induced connection on $\mathbf{M}$ and $R(X,Y)Z$ denoted the curvature of $\mathbf{M}$ with respect to ∇. Since $(\mathbb{R}^{n+1}, D)$ is flat we have the following fundamental equations.

- Gauss: $R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$
- Codazzi for h: $(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z)$
- Codazzi for S: $(\nabla_Y S)(X) - \tau(Y)SX = (\nabla_X S)(Y) - \tau(X)SY$
- Ricci: $d\tau(X,Y) = h(SY,X) - h(Y,SX)$

Definition 2.4. *Let* $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ *be an affine immersion of co-dimension one,* ξ *be an arbitrary transversal vector field on* $\mathbf{M}$*. Let* ∇ *be the induced connection on* $\mathbf{M}$ *with respect to* ξ*, if* f *is non-degenerate and equiaffine from the fundamental equations we can see that*

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) \tag{2.4}$$

That is (M, ∇, h) *becomes a statistical manifold. We call* (M, ∇, h) *the statistical manifold realized in* $\mathbb{R}^{n+1}$

Definition 2.5. *Two statistical manifolds* $(\mathbf{M}, \nabla, h)$ *and* $(\mathbf{M}, \tilde{\nabla}, h)$ *are said to be dual-projectively equivalent if there exist a* $1-$*from* τ *on* $\mathbf{M}$ *such that*

$$\tilde{\nabla}_X Y = \nabla_X Y - h(X,Y)\tau^{\#},$$

Where $\tau^{\#}$ *is a metrical dual vector field defined by* $h(\tau^{\#}, X) = \tau(X)$*.*

We say that $(\mathbf{M}, \nabla, h)$ is dual-projectively flat if ∇ is dual-projectively equivalent to some flat affine connection in a neighbourhood of an arbitrary point of $\mathbf{M}$.

Definition 2.6. *An affine immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ is said to be minimal if the affine mean curvature $H = \frac{1}{n} trS$ vanishes identically.*

Theorem 2.1. *[3] Let $\mathbf{M}$ be a simply- connected, oriented $C^{\infty}-$manifold of dimension n, and (∇, h) a statistical structure on $\mathbf{M}$. A necessary and sufficient condition for (∇, h) to be induced by a minimal affine immersion of $\mathbf{M}$ into $\mathbb{R}^{n+1}$ is the following:*

1. $\nabla Vol_h = 0$
2. (∇, h) *is dual-projectively flat.*
3. *The scalar curvature of (∇, h) vanishes identically.*

Where Vol_h is the volume form determined by h.

2.1 Centro-affine Immersion of Co-dimension Two

Let $\mathbf{M}$ be an n- dimensional manifold and $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ be an immersion. Let D be the standard flat affine connection of $\mathbb{R}^{n+2}$ and η be the radial vector field of $\mathbb{R}^{n+2} \setminus \{0\}$. That is $\eta = \sum_{i=1}^{n+2} x^i \frac{\partial}{\partial x^i}$, where $\{x^1, ..., x^{n+2}\}$ be the affine co-ordinate system.

Definition 2.7. *An immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ is called centro affine immersion of co-dimension two if there exist a vector field ξ along f such that at each point $\xi \in \mathbf{M}$, the tangent space*

$$T_{f(x)}(\mathbb{R}^{n+2}) = f_*(T_x\mathbf{M}) \bigoplus Span\{\xi_x\} \bigoplus Span\{\eta_{f(x)}\} \tag{2.5}$$

and

$$D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi + T(X, Y)\eta \tag{2.6}$$

Where ∇ be the affine connection on $\mathbf{M}$.

Now for a vector field ξ

$$D_X \xi = -f_*(SX) + \tau(X)\xi + \mu(X)\eta. \tag{2.7}$$

This equation is called Weingarten formula. Here T, h are symmetric $(0, 2)$- tensor fields, μ, τ are 1- forms and S is a $(1, 1)$ tensor field on $\mathbf{M}$. Since the connection D is flat we have the following fundamental equations

- Gauss: $R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY - T(Y, Z)X + T(X, Z)Y$
- Codazzi for h: $(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z)$
- Codazzi for T: $(\nabla_X T)(Y, Z) + \mu(X)h(Y, Z) = (\nabla_Y T)(X, Z) + \mu(Y)h(X, Z)$
- Codazzi for S: $(\nabla_Y S)(X) - \tau(Y)SX + \mu(X)Y = (\nabla_X S)(Y) - \tau(X)SY + \mu(Y)X$
- Ricci: $h(SY, X) - h(Y, SX) = (\nabla_X \tau)(Y) - \nabla_Y \tau)(X)$

- Ricci: $T(SY,X) - T(Y,SX) = (\nabla_X\mu)(Y) - \nabla_Y\mu)(X) + \tau(Y)\mu(X) - \tau(X)\mu(Y)$

The objects ∇, T, h, S, τ and μ depends on the choice of ξ.

Remark 2.1. *We say f is non-degenerate if h is non-degenerate and (f,ξ) is called equiaffine immersion if τ vanishes identically.*

Definition 2.8. *Let (f,ξ) be a non-degenerate equiaffine centroaffine immersion of co-dimension two. Then from the Codazzi equation we get ∇h is symmetric, then the triplet $(\mathbf{M},\nabla,h)$ is a statistical manifold. In that case we say $(\mathbf{M},\nabla,h)$ is a statistical manifold realized in $\mathbb{R}^{n+2}$.*

Definition 2.9. *Two statistical manifolds $(\mathbf{M},\nabla,h)$ and $(\mathbf{M},\tilde{\nabla},\tilde{h})$ are said to be conformally-projectively equivalent if there exist two positive functions ϕ and ψ on $\mathbf{M}$ such that*

$$\begin{aligned}\tilde{h}(X,Y) &= \phi\psi h(X,Y),\\ h(\tilde{\nabla}_X Y,Z) &= h(\nabla_X Y,Z) - d(log\phi)(Z)h(X,Y)\\ &+ d(log\psi)(X)h(Y,Z) + d(log\psi)(Y)h(X,Z).\end{aligned}$$

A statistical manifold $(\mathbf{M},\nabla,h)$ is said to be conformally-projectively flat if it is conformally-projectively equivalent to a flat statistical manifold in a neighbourhood of an arbitrary point of $\mathbf{M}$.

Definition 2.10. *A centro affine immersion $f:\mathbf{M}\longrightarrow\mathbb{R}^{n+2}$ is said to be minimal if the affine mean curvature $H=\frac{1}{n}trS$ vanishes identically*

Theorem 2.2. *[3] Let $\mathbf{M}$ be a simply- connected, oriented $C^\infty-$manifold of dimension n, and (∇,h) a statistical structure on $\mathbf{M}$. A necessary and sufficient condition for (∇,h) to be induced by a minimal centro affine immersion of $\mathbf{M}$ into $\mathbb{R}^{n+2}$ is the following:*

1. $\nabla Vol_h = 0$
2. *(∇,h) is Conformally-projectively flat.*
3. *The scalar curvature of (∇,h) vanishes identically .*

Where Vol_h is the volume form determined by h.

3 Statistical Immersions

In this section we give a necessary condition for the existence of a statistical immersion $f:(\mathbf{M},\nabla^{(\alpha)},g)\longrightarrow(\tilde{\mathbf{M}},\tilde{\nabla}^{(\alpha)},\tilde{g})$. when $f:(\mathbf{M},\nabla,g)\longrightarrow(\tilde{\mathbf{M}},\tilde{\nabla},\tilde{g})$ is a non-degenerate equiaffine statistical immersion of co-dimension one.

Definition 3.1. *Let $(\mathbf{M},\nabla,g)$ and $(\tilde{\mathbf{M}},\tilde{\nabla},\tilde{g})$ be two statistical manifolds. An immersion $f:\mathbf{M}\longrightarrow\tilde{\mathbf{M}}$ is called statistical immersion if*

$$g = f^*\tilde{g} \tag{3.1}$$

$$g(\nabla_X Y,Z) = \tilde{g}(\tilde{\nabla}_X f_*Y, f_*Z) \tag{3.2}$$

for any $X,Y,Z\in\mathcal{X}(\mathbf{M})$.

Let $(\mathbf{M}, g)$ be an n- dimensional statistical manifold with an affine connection ∇ and $(\tilde{\mathbf{M}}, \tilde{g})$ be an $(n+1)$- dimensional statistical manifold with an affine connection $\tilde{\nabla}$. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a statistical immersion of co-dimension one with unit normal vector field ξ of f, then we can write for each $p \in \mathbf{M}$

$$T_{f(p)}(\tilde{\mathbf{M}}) = f_*(T_p(\mathbf{M})) + span\{\xi_p\} \tag{3.3}$$

Also we have the Gauss and Weingarten formulae

1. $\tilde{\nabla}_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi$
2. $\tilde{\nabla}_X \xi = -f_*(SX) + \tau(X)\xi$
3. $\tilde{\nabla}^*_X f_* Y = f_*(\nabla^*_X Y) + h^*(X, Y)\xi$
4. $\tilde{\nabla}^*_X \xi = -f_*(S^* X) + \tau^*(X)\xi$

for $X, Y \in \mathcal{X}(\mathbf{M})$, where $\tilde{\nabla}^*$ is the dual connection of $\tilde{\nabla}$ with respect to $\tilde{g}$, $h(X, Y)$ and $h^*(X, Y)$ are symmetric bilinear forms on each tangent space $T_p(\mathbf{M})$ for p in $\mathbf{M}$. It is easily verified that S and S^* are tensor fields of type $(1, 1)$ and τ, τ^* are 1-forms. We call S (S^*) the shape operator and τ (τ^*) the transversal connection form for f and the induced connections ∇ (∇^*). Thus on the manifold $\mathbf{M}$ we have induced connections ∇ and ∇^* and affine fundamental forms

$$h(h^*) : T_p\mathbf{M} \times T_p\mathbf{M} \longrightarrow \mathbb{R}$$

Definition 3.2. *Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a statistical immersion of co-dimension one. Then f is called non-degenerate if h is non-degenerate.*

A statistical immersion of co-dimension one is called equiaffine if $\tau = 0$.
Note that from above proposition statistical immersion is equiaffine if and only if $\tau^* = 0$.

Suppose $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ has constant curvature $\tilde{k}$, then we have the following fundamental equations

$$\begin{aligned} R^{\nabla}(X, Y)Z &= \tilde{k}\{g(Y, Z)X - g(X, Z)Y\} + h(Y, Z)SX \\ &\quad -h(X, Z)SY (Gauss) \\ (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) &= (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z) (Codazzi) \\ (\nabla_X S)(Y) - \tau(X)SY &= (\nabla_Y S)(X) - \tau(Y)SX (Codazzi) \\ h(X, SY) - h(SX, Y) &= d\tau(X, Y).(Ricci) \end{aligned}$$

where $R^{\nabla}(X, Y)Z$ denotes the curvature tensor with respect to ∇ in $\mathbf{M}$. Also for dual connections the equations are

$$\begin{aligned} R^{\nabla^*}(X, Y)Z &= \tilde{k}\{g(Y, Z)X - g(X, Z)Y\} + h^*(Y, Z)S^*X \\ &\quad -h^*(X, Z)S^*Y \\ (\nabla^*_X h^*)(Y, Z) + \tau^*(X)h^*(Y, Z) &= (\nabla^*_Y h^*)(X, Z) + \tau^*(Y)h^*(X, Z) \\ (\nabla^*_X S^*)(Y) - \tau^*(X)S^*Y &= (\nabla^*_Y S^*)(X) - \tau^*(Y)S^*X \\ h^*(X, S^*Y) - h^*(S^*X, Y) &= d\tau^*(X, Y). \end{aligned}$$

Proposition 3.1. *Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a non-degenerate equiaffine statistical immersion of co-dimension one. Then $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \nabla^*, h^*)$ are statistical manifolds*

Proof. Since f is equiaffine $\tau = 0$, then

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) \;=\; (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z)$$

reduces to

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z).$$

This implies $(\mathbf{M}, \nabla, h)$ is a statistical manifold. Similarly $(\mathbf{M}, \nabla^*, h^*)$ also a statistical manifold. □

Definition 3.3. *Two statistical manifolds $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \nabla^*, h^*)$ are said to be dual to each other if $h = h^*$ and ∇, ∇^* are dual with respect to h.*

Theorem 3.1. *Let $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ be a non-degenerate equiaffine statistical immersion of co-dimension one. If $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \nabla^*, h^*)$ are dual to each other, then $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ is a statistical immersion of co-dimension one with the Gauss equation*

$$\tilde{\nabla}^{(\alpha)}_X f_* Y = f_*(\nabla^{(\alpha)}_X Y) + h(X,Y)\xi; \quad \forall \alpha \in \mathbb{R} \tag{3.4}$$

Proof. Consider

$$\begin{aligned}
f_*(\nabla^{(\alpha)}_X Y) &= f_*\left(\frac{1+\alpha}{2}(\nabla_X Y) + \frac{1-\alpha}{2}(\nabla^*_X Y)\right)\\
&= \left(\frac{1+\alpha}{2}\right) f_*(\nabla_X Y) + \left(\frac{1-\alpha}{2}\right) f_*(\nabla^*_X Y)\\
&= \left(\frac{1+\alpha}{2}\right)(\tilde{\nabla}_X f_* Y - h(X,Y)\xi) + \left(\frac{1-\alpha}{2}\right)(\tilde{\nabla}^*_X f_* Y - h^*(X,Y)\xi)\\
&= \left(\frac{1+\alpha}{2}\right)\tilde{\nabla}_X f_* Y + \left(\frac{1-\alpha}{2}\right)\tilde{\nabla}^*_X f_* Y - \left(\frac{1+\alpha}{2}\right)h(X,Y)\xi - \left(\frac{1-\alpha}{2}\right)h^*(X,Y)\xi
\end{aligned}$$

Since $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \nabla^*, h^*)$ are dual to each other we have,

$$f_*(\nabla^{(\alpha)}_X Y) = \tilde{\nabla}^{(\alpha)}_X f_* Y - h(X,Y)\xi$$

Hence,

$$\tilde{\nabla}^{(\alpha)}_X f_* Y = f_*(\nabla^{(\alpha)}_X Y) + h(X,Y)\xi; \quad \forall \alpha \in \mathbb{R}$$

□

4 Minimal Statistical Immersions

In [3] H Furuhatha has given a necessary and sufficient condition for a statistical structure to be realized as a minimal affine hyper-surface. Also he proved a necessary and sufficient condition for a statistical structure to be realized as a minimal centro-affine immersion of co-dimension two. In this section we give a necessary and sufficient condition for the existence of a minimal statistical immersion $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ of co-dimension one, where $\tilde{\mathbf{M}}$ is dually flat. Also obtained a necessary condition for the existence of a minimal statistical immersion from $(\mathbf{M}, \nabla^{(\alpha)}, g)$ into $(\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$.

Definition 4.1. *Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a statistical immersion of co-dimension one. Then f is called minimal immersion if the mean curvature $H = \frac{1}{n} trS$ vanishes identically.*

Theorem 4.1. *Let $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ be a statistical immersion of co-dimension one and $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{\nabla}^*, \tilde{g})$ be dually flat. Then f is a minimal immersion if and only if the scalar curvature $Scal^{(\nabla,h)}$ vanishes identically.*

Proof. Since $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{\nabla}^*, \tilde{g})$ is dually flat, we have $\tilde{k} = 0$. Then the equation

$$R^{\nabla}(X,Y)Z = \tilde{k}\{g(Y,Z)X - g(X,Z)Y\} + h(Y,Z)SX - h(X,Z)SY$$

reduces to

$$R^{\nabla}(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$$

Then

$$Ric^{\nabla}(Y,Z) = h(Y,Z)trS - h(SY,Z) \tag{4.1}$$

Then from the definition of scalar curvature

$$Scal^{(\nabla,h)} = 2H$$

where $H = \frac{1}{n} trS$. Then f is a minimal immersion if and only if the scalar curvature $Scal^{(\nabla,h)}$ vanishes identically □

Proposition 4.1. *The scalar curvature $Scal^{(\nabla^\alpha,h)}$ for ∇^α is related to $Scal^{(\nabla,h)}$ as*

$$Scal^{(\nabla^\alpha,h)} = Scal^{(\nabla,h)} + \frac{1-\alpha^2}{4}K \tag{4.2}$$

where

$$K = \sum_{m,k,i,j} h^{ij}(K^m_{ik}K^k_{jm} - K^m_{ij}K^k_{km}). \tag{4.3}$$

Proof. Consider $h(R^{\nabla}(X,Y)Z,W) = -h(Z, R^{\nabla*}(X,Y)W)$. Writing this equation in component form gives

$$\sum_k h_{km}R^k_{\ell ij} + \sum_k h_{\ell k}R^{*k}_{mij} = 0$$

multiplying $h^{tm}h^{s\ell}$ and sum over ℓ, m indices we get

$$\sum_\ell h_{\ell s}R^t_{\ell ij} + \sum_m h_{mt}R^{*s}_{mij} = 0. \tag{4.4}$$

Since ∇ is torsion free

$$R^{\nabla}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{\nabla_X Y}Z + \nabla_{\nabla_Y X}Z$$

in local co-ordinate we have

$$R^t_{\ell ij} = -R^t_{\ell ji}. \tag{4.5}$$

From (4.4) and (4.5) we get

$$Scal^{(\nabla,h)} = Scal^{(\nabla^*,h)}.$$

Now consider

$$R^{(\alpha)}_{ij} = \frac{1+\alpha}{2}R_{ij} + \frac{1-\alpha}{2}R^*_{ij} + \frac{1-\alpha^2}{4}\Big(\sum_{m,k} K^m_{ik}K^k_{jm} - \sum_{m,k} K^m_{ij}K^k_{km}\Big)$$

multiply h^{ij} and sum over i, j then we get

$$Scal^{(\nabla^{(\alpha)},h)} = \frac{1+\alpha}{2}Scal^{(\nabla,h)} + \frac{1-\alpha}{2}Scal^{(\nabla^*,h)} + \frac{1-\alpha^2}{4}K$$

where

$$K = \sum_{m,k,i,j} h^{ij}(K^m_{ik}K^k_{jm} - K^m_{ij}K^k_{km}).$$

Since

$$Scal^{(\nabla,h)} = Scal^{(\nabla^*,h)}$$

we have

$$Scal^{(\nabla^\alpha,h)} = Scal^{(\nabla,h)} + \frac{1-\alpha^2}{4}K.$$

□

Theorem 4.2. *Let $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ be a non-degenerate statistical immersion of co-dimension one. Then f is a minimal immersion if $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ is a minimal immersion and $\nabla_X Y = \nabla^*_X Y$*

Proof. Since $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ is a minimal immersion, we have

$$Scal^{(\nabla,h)} = 0$$

Then from the above proposition

$$Scal^{(\nabla^\alpha,h)} = \frac{1-\alpha^2}{2}K$$

Since $\nabla_X Y = \nabla^*_X Y$, we get $Scal^{(\nabla^\alpha,h)} = 0$. Then from the theorem(4.1) we get $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ is a minimal immersion for every $\alpha \in \mathbb{R}$. □

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