A Further Extension of Generalized Hurwitz - Lerch Zeta Function of Two Variables-II

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Abstract

In this paper we introduce an extension of generalized Hurwitz-Lerch Zeta Function defined by Pathan and Daman and represented its integral representation in terms of Third Appell Hypergeometric Function F_3 and then systematically investigate its several properties and various integral representations which provide certain known as well new extensions of earlier stated results.

Subject Classification: 11M35, 33B15, 33C05, 33C65, 33C70, 33C90.

Keywords: Generalized Hurwitz-Lerch Zeta function, Extended beta function, Extended hypergeometric function, Extended Hurwitz-Lerch Zeta function, Third Appell Hypergeometric Function.

1 Introduction

Hurwitz Zeta function $\varsigma(s,a)$, defined by [4, p. 249] and [1, p. 89]

$$\varsigma(s,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}, \quad Re(s) > 1,\ a \neq \{0,-1,-2\ldots\}. \tag{1.1}$$

A generalization of (1.1) is given by Hurwitz Lerch Zeta Function originally defined by Erdelyi [7] as:

$$\phi(z,s,a) = \sum_{m=0}^{\infty} \frac{z^m}{(a+m)^s}, \quad (|z|<1, a \neq \{0,-1,-2,\ldots\}). \tag{1.2}$$

A generalization of (1.2) is given by Goyal and Laddha [8] as:

$$\phi_{\mu}^{*}(z,s,a) = \sum_{m=0}^{\infty} \frac{(\mu)_m z^m}{(a+m)^s m!}, \tag{1.3}$$

where $a \neq \{-1,-2,\ldots\}$, $\mu \geq 1$ and either $|z|<1$, $Re(s)>0$ or $z=1$ and $Re(s)>\mu$.

Pathan and Daman [9] defined a generalization of (1.1) to (1.3) for two variables in the form

$$\phi^*_{\mu,\lambda}(z,t,s,a) = \sum_{m,k=0}^{\infty} \frac{(\mu)_m(\lambda)_k z^m t^k}{(a+m+k)^s m!k!}$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} \phi_\mu{}^*(z,s,a+k)\,, \tag{1.4}$$

where $a \neq \{-1,-2,\ldots\}$, $\mu,\ \lambda \geq 1$ and either $|z|,|t| < 1$, $Re(s) > 0$ or $t,z = 1$ and $Re(s) > \mu, \lambda$.

In my previous work [2], we defined a further extension of generalization stated in (1.4) as:

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} \frac{z^m t^n}{(a+m+n)^s}\,, \tag{1.5}$$

where $\alpha, \beta, \beta' \in \mathbb{C}$, $\gamma, \gamma', a \neq \{0,-1,-2,\ldots\}$, $s,z,t \in \mathbb{C}$ and $Re(s) > 0$ when $|z|,|t| < 1$ or $Re(\gamma+\gamma'+s-\alpha-\beta-\beta') > 0$ when $|z|,|t| = 1$.

Motivated by the generalizations provided by Daman and Pathan i.e. the extended Hurwitz-Lerch Zeta Function and continuing my previous work, we further investigate and introduce here a further extension of Generalized Hurwitz-Lerch Zeta Function.

2 Extension of Generalized Hurwitz-Lerch Zeta Function of two variables

Definition 2.1. *We define the extension of generalized Hurwitz-Lerch Zeta function of two variables (defined by Pathan and Daman [9]) in the form:*

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} \frac{z^m t^n}{(a+m+n)^s}\,, \tag{2.1}$$

where $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$, $\gamma, a \neq \{0,-1,-2,\ldots\}$, $s,z,t \in \mathbb{C}$ *and* $Re(s) > 0$ *when* $|z|,|t| < 1$ *or* $Re(\gamma+s-\alpha-\alpha'-\beta-\beta') > 0$ *when* $|z|,|t| = 1$.

Remark 1. *Amongst various other things, we consider the following special cases and limiting cases of extended Hurwitz-Lerch Zeta Function* $\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)$ *in* (2.1) *in our present investigation.*

Case 1: Consider the following limiting case of (2.1) which reduces it to the generalized Hurwitz-Lerch Zeta Function of Pathan and Daman [9]:

$$\phi^*_{\beta,\beta'}(z,t,s,a)$$

$$= \lim_{\alpha,\alpha',\gamma\to\infty} \left\{ \phi_{\alpha,\alpha',\beta,\beta';\gamma}\left(\frac{\gamma z}{\alpha}, \frac{\gamma t}{\alpha'}, s, a\right) \right\}$$

$$= \lim_{\alpha,\alpha',\gamma\to\infty} \left\{ \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} \frac{\gamma^m z^m \gamma^n t^n}{\alpha^m (\alpha')^n (a+m+n)^s} \right\}. \tag{2.2}$$

After certain simplifications we get,

$$\phi^*_{\beta,\beta'}(z,t,s,a)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{m!n!}\frac{z^m t^n}{(a+m+n)^s}\,, \tag{2.3}$$

where $\beta,\beta'\in\mathbb{C}$, $a\neq\{0,-1,-2,\ldots\}$; $s,z,t\in\mathbb{C}$ and $Re(s)>0$ when $|z|,|t|<1$ or $Re(s-\beta-\beta')>0$ when $|z|,|t|=1$.

Case 2: Consider the following limiting case:

$$\begin{aligned}\phi^*_{\beta,\beta';\gamma}(z,t,s,a)&=\lim_{\alpha,\alpha'\to\infty}\left\{\phi_{\alpha,\alpha',\beta,\beta';\gamma}\left(\frac{z}{\alpha},\frac{t}{\alpha'},s,a\right)\right\}\\&=\lim_{\alpha,\alpha'\to\infty}\left\{\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}\frac{z^m t^n}{\alpha^m(\alpha')^n(a+m+n)^s}\right\}\\&=\sum_{m,n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}\frac{z^m t^n}{(a+m+n)^s}\,, \qquad (2.4)\end{aligned}$$

where γ, $a\neq\{0,-1,-2,\ldots\}$ and $\beta,\beta'\in\mathbb{C}$; $s,z,t\in\mathbb{C}$ and $Re(s)>0$ when $|z|,|t|<1$ or $Re(\gamma+s-\beta-\beta')>0$ when $|z|,|t|=1$.

Case 3: The following limiting case will be used:

$$\begin{aligned}&\phi^*_{\alpha,\alpha',\beta;\gamma}(z,t,s,a)\\&=\lim_{\beta'\to\infty}\left\{\phi_{\alpha,\alpha',\beta,\beta';\gamma}\left(z,\frac{t}{\beta'},s,a\right)\right\}\\&=\lim_{\beta'\to\infty}\left\{\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}\frac{z^m t^n}{\beta'^n(a+m+n)^s}\right\}\\&=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}m!n!}\frac{z^m t^n}{(a+m+n)^s}\,, \qquad (2.5)\end{aligned}$$

where $\gamma,a\neq\{0,-1,-2,\ldots\}$ and $\alpha,\alpha',\beta\in\mathbb{C}$; $s,z,t\in\mathbb{C}$ and $Re(s)>0$ when $|z|,|t|<1$ or $Re(\gamma+s-\alpha-\alpha'-\beta)>0$ when $|z|,|t|=1$.

Case 4: The following limiting case of extended generalized Hurwitz Zeta Function of two variables will be used:

$$\begin{aligned}&\phi^*_{\beta;\gamma}(z,t,s,a)\\&=\lim_{\alpha,\alpha',\beta'\to\infty}\left\{\phi_{\alpha,\alpha',\beta,\beta';\gamma}\left(\frac{z}{\alpha},\frac{t}{\alpha'\beta'},s,a\right)\right\}\\&=\lim_{\alpha,\alpha',\beta'\to\infty}\left\{\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}\;m!n!}\frac{z^m t^n}{\alpha^m\beta^m\alpha'^n(a+m+n)^s}\right\}\\&=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}m!n!}\frac{z^m t^n}{(a+m+n)^s}\,, \qquad (2.6)\end{aligned}$$

where $\gamma, a \neq \{0, -1, -2, \ldots\}$ and $\beta \in \mathbb{C}$; $s, z, t \in \mathbb{C}$ and $Re(s) > 0$ when $|z|, |t| < 1$ or $Re(\gamma + s - \beta) > 0$ when $|z|, |t| = 1$.

Case 5: Consider the following limiting case:

$$\begin{aligned}
&\phi^*_{\alpha,\beta;\gamma}(z,t,s,a)\\
&\quad = \lim_{\alpha',\beta' \to \infty} \left\{ \phi_{\alpha,\alpha',\beta,\beta';\gamma}\left(z, \frac{t}{\alpha'\beta'}, s, a\right) \right\}\\
&\quad = \lim_{\alpha',\beta' \to \infty} \left\{ \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}\ m!n!} \frac{z^m t^n}{(\alpha')^n (\beta')^n (a+m+n)^s} \right\}\\
&\quad = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_{m+n} m!n!} \frac{z^m t^n}{(a+m+n)^s}, \qquad (2.7)
\end{aligned}$$

where $\gamma, a \neq \{0, -1, -2, \ldots\}$ and $\alpha, \beta \in \mathbb{C}$; $s, z, t \in \mathbb{C}$ and $Re(s) > 0$ when $|z|, |t| < 1$ or $Re(\gamma + s - \alpha - \beta) > 0$ when $|z|, |t| = 1$.

3 Integral Representation of $\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)$

We begin by recalling the third Appell hypergeometric function of two variables F_3.
For $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$, $\gamma \in \mathbb{C}/(\mathbb{Z}^- \cup \{0\})$,

$$F_3[\alpha, \alpha', \beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \qquad (3.1)$$

Also, the following are the confluent forms of Appell hypergeometric function F_3

$$\phi_2[\beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \qquad (3.2)$$

$$\phi_3[\beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \qquad (3.3)$$

$$E_1[\alpha, \alpha', \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| < 1. \qquad (3.4)$$

$$E_2[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| < 1. \qquad (3.5)$$

Theorem 3.1. *The following integral representation of* $\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)$ *of* (2.1) *holds true:*

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} F_3(\alpha, \alpha', \beta, \beta'; \gamma; ze^{-x}, te^{-x}) dx, \qquad (3.6)$$

where $\min\{Re(s), Re(a)\} > 0$ *when* $|z| < 1$, $|t| < 1$ *and* $Re(s) > 1$ *when* $|z|, |t| = 1$.

Proof. Consider the following Eulerian Integral

$$\frac{1}{(a+m+n)^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(a+m+n)t} dt\,, \tag{3.7}$$

where $\min\{Re(s), Re(a)\} > 0$; $m, n \in \mathbb{N}_0$.

Using the above equation in the definition of $\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)$ stated in (2.1) and simplifying we get,

$$\begin{aligned}
&\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)\\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} z^m t^n \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-(a+m+n)x} dx\\
&= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}\ m!n!} (ze^{-x})^m (te^{-x})^n x^{s-1} e^{-ax} dx\\
&= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} F_3(\alpha,\alpha',\beta,\beta';\gamma; ze^{-x}, te^{-x}) dx.
\end{aligned}$$

□

Similarly, on using Eulerian Integral mentioned in (3.7), in (2.4), (2.5), (2.6) and (2.7) and using (3.2), (3.3), (3.4) and (3.5) respectively, we get the integral representation asserted by the corollaries stated below:

Corollary 3.2. *The following integral representation holds true:*

$$\phi^*_{\beta,\beta';\gamma}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \phi_2[\beta,\beta';\gamma; ze^{-x}, te^{-x}] dx. \tag{3.8}$$

$$\phi^*_{\beta;\gamma}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \phi_3[\beta;\gamma; ze^{-x}, te^{-x}) dx. \tag{3.9}$$

$$\phi^*_{\alpha,\alpha',\beta;\gamma}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} E_1[\alpha,\alpha',\beta;\gamma; ze^{-x}, te^{-x}) dx,\ \ |ze^{-x}| < 1. \tag{3.10}$$

$$\phi^*_{\alpha,\beta;\gamma}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} E_2[\alpha,\beta;\gamma; ze^{-x}, te^{-x}) dx,\ \ |ze^{-x}| < 1 \tag{3.11}$$

where $\min\{Re(s), Re(a)\} > 0$ *when* $|z|, |t| < 1$ *and* $Re(s) > 1$ *when* $z = 1,\ t = 1$.

Corollary 3.3. *In this section, we represent the integral of* $\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)$ *stated in* (2.1) *in terms of Appell Hypergeometric function* F_1 *by recalling a relation:*

$$F_3\left(\alpha, \gamma-\alpha, \beta, \beta'; \gamma; x, \frac{y}{y-1}\right) = (1-y)^{\beta'} F_1(\alpha,\beta,\beta';\gamma;x,y). \tag{3.12}$$

Let $\alpha' = \gamma - \alpha$ *in integral representation* (3.6) *and replacing* t *by* $\frac{t}{t-1}$ *we get,*

$$\begin{aligned}
&\phi_{\alpha,\gamma-\alpha,\beta,\beta';\gamma}\left(z, \frac{t}{t-e^x}, s, a\right) \\
&= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} (1-te^{-x})^{\beta'} F_3\left(\alpha, \gamma-\alpha, \beta, \beta'; \gamma'; ze^{-x}, \left(\frac{t}{t-e^x}\right) e^{-x}\right) dx \\
&= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} (1-te^{-x})^{\beta'} F_3\left(\alpha, \gamma-\alpha, \beta, \beta'; \gamma'; ze^{-x}, \frac{te^{-x}}{te^{-x}-1}\right) dx \\
&= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} (1-te^{-x})^{\beta'} F_1(\alpha, \beta, \beta'; \gamma'; ze^{-x}, te^{-x}) dx. \qquad (3.13)
\end{aligned}$$

4 A Connection with Generalized Hypergeometric Function

In this section, we establish a relationship between extension of Generalized Hurwitz-Lerch Zeta Function $\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ and generalized hypergeometric functions ${}_pF_q$ for $p = 2$ and $q = 1$.

Theorem 4.1. *The following integral representation of* $\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ *holds true:*

$$\begin{aligned}
\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) &= \frac{\Gamma(\gamma)}{\Gamma(s)\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^\infty \int_0^1 x^{s-1} e^{-ax} (1-\mu)^{\beta-1} \\
&\cdot \frac{\mu^{\gamma-\beta-1}}{(1-(1-\mu)ze^{-x})^{\alpha}} \, {}_2F_1(\alpha', \ \beta'; \gamma-\beta; \mu te^{-x}) d\mu dx, \qquad (4.1)
\end{aligned}$$

where $Re(\beta) > 0$*,* $Re(\beta') > 0$*,* $Re(\gamma) > 0$*,* $Re(\gamma - \beta) > 0$*,* $Re(\gamma - \beta - \beta') > 0$*.*

Proof. Consider the classical double integration of F_3 [3]:

$$\begin{aligned}
&F_3(a, a', b, b'; c; x, y) \\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \int_0^1 \int_0^{1-u} \frac{u^{b-1} v^{b'-1} (1-u-v)^{c-b-b'-1}}{(1-ux)^a (1-vy)^{a'}} dv du, \qquad (4.2)
\end{aligned}$$

where $R = \{(u,v)|u \geq 0, v \geq 0, u+v \leq 1\}$, $Re(c)$, $Re(b)$, $Re(b')$, $Re(c-b-b') > 0$.
Using this relation in (3.6) we get,

$$\begin{aligned}
&\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a) \\
&= \frac{\Gamma(\gamma)}{\Gamma(s)\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \int_0^\infty \int_0^1 \int_0^{1-\mu} x^{s-1} \\
&\times e^{-ax} \frac{\mu^{\beta-1} \nu^{\beta'-1} (1-\mu-\nu)^{\gamma-\beta-\beta'-1}}{(1-\mu ze^{-x})^{\alpha} (1-\nu te^{-x})^{\alpha'}} d\nu d\mu dx, \qquad (4.3)
\end{aligned}$$

where $R = \{(\mu,\nu)|\mu \geq 0, \nu \geq 0, \mu+\nu \leq 1\}$.

The integral representation of gauss hypergeometric function is given as:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-z\tau)^{-a} d\tau, \tag{4.4} $$

where $Re(c) > Re(b) > 0$, $|arg(1-z)| < \pi - \epsilon$; $0 < \epsilon < \pi$.

Using the equation (4.3) in (4.2) and further simplifying we get,

$$ \begin{aligned} \phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a) = & \frac{\Gamma(\gamma)}{\Gamma(s)\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^\infty \int_0^1 x^{s-1} e^{-ax} (1-\mu)^{\beta-1} \\ & \times \frac{\mu^{\gamma-\beta-1}}{(1-(1-\mu)ze^{-x})^\alpha} {}_2F_1(\alpha', \beta'; \gamma-\beta; \mu t e^{-x}) \, d\mu dx, \end{aligned} $$

where $Re(\beta) > 0$, $Re(\beta') > 0$, $Re(\gamma) > 0$, $Re(\gamma-\beta) > 0$, $Re(\gamma-\beta-\beta') > 0$. □

Theorem 4.2. *The following summation formula for* $\phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a)$ *in* (2.1) *holds true:*

$$ \phi_{\alpha,\alpha',\beta,\beta';\gamma}(z,t,s,a-x) = \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \phi_{\alpha,\alpha',\beta,\beta';\,\gamma}(z,t,s+k,a) x^k, \tag{4.5} $$

where $|x| < |a|$; $s \neq 1$.

Proof. Consider

$$ \begin{aligned} & \phi_{\alpha,\alpha',\beta,\beta';\,\gamma}(z,t,s,a-x) \\ & = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m! n!} \frac{z^m t^n}{(a-x+m+n)^s} \\ & = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m! n!} \frac{z^m t^n}{(a+m+n)^s} (1 - \frac{x}{(a+m+n)})^{-s} \\ & = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m! n!} \frac{z^m t^n}{(a+m+n)^s} \left\{ \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \frac{x^k}{(a+m+n)^k} \right\} \\ & = \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \left\{ \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m! n!} \frac{z^m t^n}{(a+m+n)^{s+k}} \right\} x^k, \qquad (4.6) \end{aligned} $$

which on using (2.1) leads to the desired formula in (4.5). □

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