Ricci Solitons on Submanifolds of Some Indefinite Almost Contact Manifolds.

Payel Karmakar[a], [1] and Arindam Bhattacharyya[b]

Department of Mathematics,
Jadavpur University, Kolkata-700032, India.

[a]*payelkarmakar632@gmail.com*
[b]*bhattachar1968@yahoo.co.in* .

Abstract

In this paper we study Ricci solitons on invariant and anti-invariant submanifolds of indefinite Sasakian manifolds, indefinite trans-Sasakian manifolds, indefinite Kenmotsu manifolds with respect to Riemannian connection and quarter symmetric metric connection.

Subject Classification: [2010] 53C15, 53C20, 53C25, 53C40, 53C50.

Keywords: Ricci soliton, invariant submanifold, anti-invariant submanifold, quarter symmetric metric connection, indefinite Sasakian manifold, indefinite trans-Sasakian manifold, indefinite Kenmotsu manifold.

1 Introduction

R.S. Hamilton[8] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. G. Perelman[17] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a riemannian manifold is a generalization of an Einstein metric such that[9]

$$\pounds_V g + 2S + 2\lambda g = 0,$$

where S is the Ricci tensor, $\pounds$ is the Lie derivative operator along with the vector field V on M and λ is a real number.

Ricci solitons have become more important in differential geometry after Perelman applied them to solve Poincare conjecture posed in 1904. R. Sharma[18] studied the Ricci solitons in contact geometry. Thereafter, Ricci solitons have been studied by various mathematicians such as C.L. Bejan and M. Crasmareanu[1], S.K. Hui et. al.([3],[11],[12],[13],[15],[16]), S. Deshmukh et. al.([4],[6]), C. He and M. Zhu[10], M.M. Tripathi [19] and many

[1] The first author is the corresponding author and has been sponsored by University Grants Commission(UGC) Junior Research Fellowship,India. *UGC-Ref. No.*: 1139/(CSIR-UGC NET JUNE 2018).

others. Motivated from their work in this paper we have established some new results regarding Ricci solitons on submanifolds of some indefinite contact and paracontact manifolds like indefinite Sasakian manifold, indefinite trans-Sasakian manifold, indefinite Kenmotsu manifold.

S. Golab[7] defined and studied quarter symmetric linear connection on a differentiable manifold. A linear connection $\bar{\nabla}$ in an n-dimensional Riemannian manifold is called a quarter symmetric connection[7] if torsion tensor T is of the form

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = A(Y)K(X) - A(X)K(Y) \qquad (1.1)$$

where A is a 1-form and K is a tensor of type (1,1). If a quarter symmetric linear connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0$$

for all $X, Y, Z \in \chi(M)$,where $\chi(M)$ is a Lie algebra of vector fields on the manifold M, then $\bar{\nabla}$ is called a quarter symmetric metric connection. For a contact metric manifold admitting quarter symmetric connection, we can take $A = \eta$ and $K = \phi$ and hence (1.1) becomes

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \qquad (1.2)$$

The relation between Levi-Civita connection ∇ and quarter symmetric metric connection $\bar{\nabla}$ of a contact metric manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \qquad (1.3)$$

Recently S.K. Hui et. al. studied invariant submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric metric connection.

A (2n+1)-dimensional semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called an indefinite almost contact manifold if it admits an indefinite almost contact structure (ϕ, ξ, η), where ϕ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form satisfying for all vector fields X, Y on $\tilde{M}$ [2],

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \eta \circ \phi = 0, \phi\xi = 0, \eta(\xi) = 1, \\ &\tilde{g}(\phi X, \phi Y) = \tilde{g}(X,Y) - \epsilon\eta(X)\eta(Y), \\ &\tilde{g}(X,\xi) = \epsilon\eta(X), \tilde{g}(\phi X, Y) = -\tilde{g}(X, \phi Y). \end{aligned}$$

Here $\epsilon = \tilde{g}(\xi,\xi) = \pm 1$ and $\tilde{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\tilde{g}$.

Based on the structure equations manifolds are as follows [2]:

• An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Sasakian structure if for all vector fields Z, W on $\tilde{M}$,

$$\begin{aligned} (\bar{\nabla}_Z \phi)W &= \epsilon\eta(W)Z - \tilde{g}(Z,W)\xi, \\ \bar{\nabla}_Z \xi &= -\epsilon\phi Z. \end{aligned} \qquad (1.4)$$

• An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite trans-Sasakian structure of type (α, β) if

$$\begin{aligned} (\tilde{\nabla}_Z \phi)W &= \alpha[\tilde{g}(Z,W)\xi - \epsilon\eta(W)Z] + \beta[\tilde{g}(\phi Z, W)\xi - \epsilon\eta(W)\phi Z], \\ \tilde{\nabla}_Z \xi &= -\epsilon\alpha\phi Z + \epsilon\beta[Z - \eta(Z)\xi]. \end{aligned} \qquad (1.5)$$

for smooth functions α, β on $\tilde{M}$ and for all vector fields Z, W on $\tilde{M}$.

• In [5], U.C. De and Avijit Sarkar introduced and studied the notion of ϵ-Kenmotsu manifolds with indefinite metric by giving an example of $\alpha = 0, \beta = 1$, then indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Kenmotsu structure. The structure equations thus become

$$(\tilde{\nabla}_Z \phi)W = \tilde{g}(\phi Z, W)\xi - \epsilon\eta(W)\phi Z,$$
$$\tilde{\nabla}_Z \xi = \epsilon Z - \epsilon\eta(Z)\xi. \tag{1.6}$$

Let M be a submanifold of dimension m of a manifold $\tilde{M}$ $(m < n)$ with induced metric g. Also let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.8}$$
$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{1.9}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\tilde{M}$. The second fundamental form h and the shape operator A_V are related by[20]

$$g(h(X, Y), V) = g(A_V X, Y), \tag{1.10}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H on M is given by $H = \frac{1}{m}\sum_{i=1}^{m} g(e_i, e_i)$, where $\{e_i\}_{i=1}^{m}$ is a local orthonormal frame of vector fields on M.

A submanifold M of a manifold $\tilde{M}$ is called totally umbilical if

$$h(X, Y) = g(X, Y)H \tag{1.11}$$

for $X, Y \in TM$. Moreover if $h(X, Y) = 0$ $\forall X, Y \in TM$, then M is called totally geodesic and if $H = 0$, then M is minimal in $\tilde{M}$.

A submanifold M of a manifold $\tilde{M}$ is called invariant (anti-invariant) if ϕX is tangent (normal) to M for every vector field X tangent to M, i.e. $\phi(TM) \subset TM$ $(\phi(TM) \subset T^{\perp}M)$ at every point of M.

Let $\bar{\tilde{\nabla}}$ be a linear connection and $\tilde{\nabla}$ be the Levi-Civita connection of a manifold $\tilde{M}$ such that

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + U(X, Y), \tag{1.12}$$

where U is a (1,1) type tensor and $X, Y \in \Gamma(T\tilde{M})$.

For $\bar{\tilde{\nabla}}$ to be a quarter symmetric metric connection on $\tilde{M}$, we have

$$U(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \tag{1.13}$$

where $$g(T'(X, Y), Z) = g(T(Z, X), Y). \tag{1.14}$$

From (1.2) and (1.14) we get

$$T'(X, Y) = \eta(X)\phi Y - g(Y, \phi X)\xi. \tag{1.15}$$

So, $$U(X, Y) = \eta(Y)\phi X - g(Y, \phi X)\xi. \tag{1.16}$$

Thus, a quarter symmetric metric connection $\bar{\tilde{\nabla}}$ in a manifold $\tilde{M}$ is given by

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{1.17}$$

2 Ricci solitons on submanifolds of indefinite Sasakian manifolds

In this section we discuss about Ricci solitons on invariant and anti-invariant submanifolds of indefinite Sasakian manifolds with respect to Riemannian connection and quarter symmetric metric connection.

2.1 Ricci solitons on submanifolds of indefinite Sasakian manifolds with respect to Riemannian connection

Let us take (g, ξ, λ) be a Ricci soliton on a submanifold M of an indefinite Sasakian manifold $\tilde{M}$.

Then we have $(\pounds_\xi g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0.$ (2.1.1)

From (1.4) and (1.8) we get,

$$-\epsilon\phi X = \tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{2.1.2}$$

If M is invariant in $\tilde{M}$, then $\phi X \in TM$, hence equating tangential and normal components of (2.1.2) we get,

$$\nabla_X \xi = -\epsilon\phi X, \ h(X, \xi) = 0. \tag{2.1.3}$$

Using (2.1.3) we have,

$$\begin{aligned}(\pounds_\xi g)(Y,Z) &= g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) \\ &= -\epsilon[g(\phi Y, Z) + g(Y, \phi Z)] \\ &= 0.\end{aligned} \tag{2.1.4}$$

In view of (2.1.4), (2.1.1) yields

$$S(Y,Z) = -\lambda g(Y,Z),$$

which implies that M is Einstein. Also from (1.11) and (2.1.3) we get $\eta(X)H = 0$, i.e., $H = 0$, since $\eta(X) \neq 0$. Consequently, M is minimal in $\tilde{M}$. Thus we can state the following:

Theorem 2.1.1: *If (g, ξ, λ) is a Ricci soliton on an invariant submanifold M of an indefinite Sasakian manifold $\tilde{M}$, then M is Einstein and also M is minimal in $\tilde{M}$.*

Again, if M is anti-invariant in $\tilde{M}$, then for any $X \in TM, \ \phi X \in T^\perp M$ and hence from (2.1.2) we get,

$$\nabla_X \xi = 0, \ h(X, \xi) = -\epsilon\phi X. \tag{2.1.5}$$

Using (2.1.4) we have,

$$(\pounds_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = 0,$$

which means that ξ is a Killing vector field and consequently (2.1.1) yields

$$S(Y,Z) = -\lambda g(Y,Z),$$

which implies that M is Einstein. Thus we can state the following:

Theorem 2.1.2: *If (g, ξ, λ) is a Ricci soliton on an anti-invariant submanifold M of an indefinite Sasakian manifold $\tilde{M}$, then M is Einstein and ξ is a Killing vector field.*

Also $\nabla_X \xi = 0 \Rightarrow R(X,Y)\xi = 0 \Rightarrow S(Y, \xi) = 0 = -\lambda\eta(Y) \Rightarrow \lambda = 0$ and hence:

Theorem 2.1.3: *A Ricci soliton (g, ξ, λ) on an anti-invariant submanifold M of an indefinite Sasakian manifold $\tilde{M}$ is always steady.*

2.2 Ricci solitons on submanifolds of indefinite Sasakian manifolds with respect to quarter symmetric metric connection

Let us consider that (g, ξ, λ) is a Ricci soliton on a submanifold M of an indefinite Sasakian manifold $\tilde{M}$ with respect to quarter symmetric metric connection, where $\bar{\nabla}$ is the induced connection on M from the connection $\bar{\tilde{\nabla}}$. Then we have

$$(\bar{\mathcal{L}}_\xi g)(Y, Z) + 2\bar{S}(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{2.2.1}$$

Let $\bar{h}$ be the second fundamental form of $\bar{M}$ with respect to induced connection $\bar{\nabla}$. Then we have

$$\bar{\tilde{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y), \tag{2.2.2}$$

and hence by virtue of (1.8), (1.17) we have,

$$\bar{\nabla}_X Y + \bar{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.2.3}$$

If M is an invariant submanifold of $\tilde{M}$, then $\phi X \in TM$ for any $X \in TM$ and hence equating tangential parts from (2.2.1) we get,

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{2.2.4}$$

which means that M admits quarter symmetric metric connection.

Also from (2.2.4) we get $\bar{\nabla}_X \xi = (-\epsilon + 1)\phi X$ and hence

$$\begin{aligned}(\bar{\mathcal{L}}_\xi g)(Y, Z) &= g(\bar{\nabla}_Y \xi, Z) + g(Y, \bar{\nabla}_Z \xi)\\ &= (-\epsilon + 1)[g(\phi Y, Z) + g(Y, \phi Z)]\\ &= 0.\end{aligned}$$

Hence from (2.2.1) we get $\bar{S}(Y, Z) = -\lambda g(Y, Z)$. Thus we can state:

Theorem 2.2.1: *Let (g, ξ, λ) be a Ricci soliton on an invariant submanifold M of an indefinite Sasakian manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\bar{\tilde{\nabla}}$. Then M is Einstein with respect to induced Riemannian connection.*

Again if M is an anti-invariant submanifold of $\tilde{M}$ with respect to quarter symmetric metric connection, then from (2.2.4) we have, $\bar{\nabla}_X \xi = 0$.

Hence $(\bar{\mathcal{L}}_\xi g)(Y, Z) = 0.$ (2.2.5)

Using (2.2.5) in (2.2.1) we get, $\bar{S}(Y, Z) = -\lambda g(Y, Z)$. Thus we can state:

Theorem 2.2.2: *Let (g, ξ, λ) be a Ricci soliton on an anti-invariant submanifold of an indefinite Sasakian manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\bar{\tilde{\nabla}}$. Then M is Einstein with respect to induced Riemannian connection.*

3 Ricci solitons on submanifolds of indefinite trans-Sasakian manifolds

In this section we discuss about Ricci solitons on invariant and anti-invariant submanifolds of indefinite trans-Sasakian manifolds with respect to Riemannian connection and quarter symmetric metric connection.

3.1 Ricci solitons on submanifolds of indefinite trans-Sasakian manifolds with respect to Riemannian connection

Let us take (g, ξ, λ) be a Ricci soliton on a submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$.

Then we have $(\pounds_\xi g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0.$ (3.1.1)

From (1.5) and (1.8) we get,

$$-\epsilon\alpha\phi X + \epsilon\beta[X - \eta(X)\xi] = \tilde{\nabla}_X \xi = \nabla_X \xi + h(X,\xi). \tag{3.1.2}$$

If M is invariant in $\tilde{M}$, then $\phi X \in TM$, hence equating tangential and normal components of (3.1.2) we get,

$$\nabla_X \xi = -\epsilon\alpha\phi X + \epsilon\beta[X - \eta(X)\xi],\ h(X,\xi) = 0. \tag{3.1.3}$$

Using (3.1.3) we have,

$$\begin{aligned}(\pounds_\xi g)(Y,Z) &= g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)\\ &= 2\epsilon\beta[g(Y,Z) - \eta(Y)\eta(Z)].\end{aligned} \tag{3.1.4}$$

In view of (3.1.4), (3.1.1) yields

$$S(Y,Z) = -(\epsilon\beta + \lambda)g(Y,Z) + \epsilon\beta\eta(Y)\eta(Z),$$

which implies that M is η-Einstein. Also from (1.11) and (3.1.3) we get $\eta(X)H = 0$, i.e., $H = 0$, since $\eta(X) \neq 0$. Consequently, M is minimal in $\tilde{M}$. Thus we can state the following:

Theorem 3.1.1: *If (g, ξ, λ) is a Ricci soliton on an invariant submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$, then M is η-Einstein and also M is minimal in $\tilde{M}$.*

Again if M is anti-invariant in $\tilde{M}$, then for any $X \in TM$, $\phi X \in T^\perp M$ and hence from (3.1.2) we have, $\nabla_X \xi = \epsilon[\beta X - \eta(X)\xi]$, $h(X,\xi) = -\epsilon\alpha\phi X$.

Hence, $(\pounds_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = 2\epsilon\beta[g(Y,Z) - \eta(Y)\eta(Z)]$.

Hence from (3.1.1) we obtain,

$$S(Y,Z) = -(\epsilon\beta + \lambda)g(Y,Z) + \epsilon\beta\eta(Y)\eta(Z).$$

Thus we have the following:

Theorem 3.1.2: *If (g, ξ, λ) is a Ricci soliton on an anti-invariant submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$, then M is η-Einstein.*

3.2 Ricci solitons on submanifolds of indefinite trans-Sasakian manifolds with respect to quarter symmetric metric connection

Let us consider that (g, ξ, λ) is a Ricci soliton on a submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$ with respect to quarter symmetric metric connection, where $\bar{\nabla}$ is the induced connection on M from the connection $\bar{\tilde{\nabla}}$. Also let $\bar{h}$ be the second fundamental form of M with respect to induced connection $\bar{\nabla}$. Then we can consider the equations (2.2.1), (2.2.2), (2.2.3).

If M is an invariant submanifold of $\tilde{M}$, then we have the equation (2.2.4) which means that M admits quarter symmetric metric connection.

Also from (2.2.4) we get $\bar{\nabla}_X \xi = (-\epsilon\alpha + 1)\phi X + \epsilon\beta[X - \eta(X)\xi]$ and hence

$$(\bar{\pounds}_\xi g)(Y,Z) = g(\bar{\nabla}_Y \xi, Z) + g(Y, \bar{\nabla}_Z \xi)$$
$$= 2\epsilon\beta[g(Y,Z) - \eta(Y)\eta(Z)].$$

Hence from (2.2.1) we get $\bar{S}(Y,Z) = -(\lambda+\epsilon\beta)g(Y,Z)+\epsilon\beta\eta(Y)\eta(Z)$. Thus we can state:

Theorem 3.2.1: *Let (g,ξ,λ) be a Ricci soliton on an invariant submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\bar{\tilde{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

Again if M is an anti-invariant submanifold of $\tilde{M}$ with respect to quarter symmetric metric connection, then from (2.2.4) we have, $\bar{\nabla}_X \xi = \epsilon[\beta X - \eta(X)\xi]$.

Hence $(\bar{\pounds}_\xi g)(Y,Z) = 2\epsilon\beta[g(Y,Z) - \eta(Y)\eta(Z)]$. (3.2.1)

Using (3.2.1) in (2.2.1) we get, $\bar{S}(Y,Z) = -(\lambda+\epsilon\beta)g(Y,Z)+\epsilon\beta\eta(Y)\eta(Z)$. Thus we can state:

Theorem 3.2.2: *Let (g,ξ,λ) be a Ricci soliton on an anti-invariant submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\bar{\tilde{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

4 Ricci solitons on submanifolds of indefinite Kenmotsu manifolds

In this section we discuss about Ricci solitons on invariant and anti-invariant submanifolds of indefinite Kenmotsu manifolds with respect to Riemannian connection and quarter symmetric metric connection.

4.1 Ricci solitons on submanifolds of indefinite Kenmotsu manifolds with respect to Riemannian connection

Let us take (g,ξ,λ) be a Ricci soliton on a submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$.

Then we have $(\pounds_\xi g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0.$ (4.1.1)

From (1.6), (1.8) we obtain,

$$\epsilon[X - \eta(X)\xi] = \tilde{\nabla}_X \xi = \nabla_X \xi + h(X,\xi). \tag{4.1.2}$$

Equating tangential and normal components of (4.1.2) we get,

$$\nabla_X \xi = \epsilon[X - \eta(X)\xi] \text{ and } h(X,\xi) = 0. \tag{4.1.3}$$

Using (4.1.3) we have,

$$(\pounds_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)$$

$$= 2\epsilon[g(Y,Z) - \eta(Y)\eta(Z)]. \tag{4.1.4}$$

Using (4.1.4) in (4.1.1) we get, $S(Y,Z) = -(\lambda+\epsilon)g(Y,Z) + \epsilon\eta(Y)\eta(Z)$, and also from (1.11) and (4.1.3) we obtain, $h(X,\xi) = \eta(X)H = 0 \Rightarrow H = 0$ as $\eta(X) \neq 0$. Hence we can state the following:

Theorem 4.1.1: *If (g,ξ,λ) is a Ricci soliton on a submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$, then M is η-Einstein and also M is minimal in $\tilde{M}$.*

4.2 Ricci solitons on submanifolds of indefinite Kenmotsu manifolds with respect to quarter symmetric metric connection

Let us consider that (g,ξ,λ) is a Ricci soliton on a submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$ with respect to quarter symmetric metric connection, where $\bar{\nabla}$ is the induced connection on M from the connection $\bar{\bar{\nabla}}$. Also let $\bar{h}$ be the second fundamental form of M with respect to induced connection $\bar{\nabla}$. Then we can consider the equations (2.2.1), (2.2.2), (2.2.3).

If M is an invariant submanifold of $\tilde{M}$, then we have the equation (2.2.4) which means that M admits quarter symmetric metric connection.

From (2.2.4) we have, $\bar{\nabla}_X\xi = \epsilon[X - \eta(X)\xi] + \phi X$ and hence

$$\begin{aligned}(\bar{\mathcal{L}}_\xi g)(Y,Z) &= g(\bar{\nabla}_Y\xi, Z) + g(Y, \bar{\nabla}_Z\xi)\\ &= 2\epsilon[g(Y,Z) - \eta(Y)\eta(Z)].\end{aligned} \tag{4.2.1}$$

Using (4.2.1) in (2.2.1) we get, $\bar{S}(Y,Z) = -(\epsilon+\lambda)g(Y,Z) + \epsilon\eta(Y)\eta(Z)$. Hence we have:

Theorem 4.2.1: *Let (g,ξ,λ) be a Ricci soliton on an invariant submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\bar{\bar{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

Again if M is an anti-invariant submanifold of $\tilde{M}$ with respect to quarter symmetric metric connection, then from (2.2.4) we have, $\bar{\nabla}_X\xi = \epsilon[X - \eta(X)\xi]$ and hence

$$(\bar{\mathcal{L}}_\xi g)(Y,Z) = 2\epsilon[g(Y,Z) - \eta(Y)\eta(Z)]. \tag{4.2.2}$$

Hence using (4.2.2) in (2.2.1) we obtain, $\bar{S}(Y,Z) = -(\epsilon+\lambda)g(Y,Z) + \epsilon\eta(Y)\eta(Z)$ and hence we can state:

Theorem 4.2.2: *Let (g,ξ,λ) be a Ricci soliton on an anti-invariant submanifold of an indefinite Kenmotsu manifold $\tilde{M}$ with respect to quarter symmetric metric connection $\bar{\bar{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.*

References

[1] C.L. Bejan and M. Crasmareanu, Ricci solitons in manifolds with quasi-constant curvature, Publ. Math. Debrecen, 78(1), 235-243(2011).

[2] D.E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math., Vol.509, Springer Verlag, Berlin(1976).

[3] S. Chandra, S.K. Hui and A.A. Shaikh, Second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds, Commun. Korean Math. Soc., 30(2), 123-130(2015).

[4] B.Y. Chen and S. Deshmukh, Geometry of compact shrinking Ricci solitons, Balkan J. Geom. Appl., 19, 13-21(2014).

[5] U.C. De and A. Sarkar, On ϵ-Kenmotsu manifolds, Hadronic J., 32, 231-242(2009).

[6] S. Deshmukh, H. Al-Sodais and H. Alodan, Anote on Ricci solitons, Balkan J. Geom. Appl., 16, 48-55(2011).

[7] S. Golab, On semi-symmetric and quarter symmetric linear connections, Tensor(N.S.), 29, 249-254(1975).

[8] R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom., 17, 255-306(1982).

[9] R.S. Hamilton, The Ricci flow on surfaces, Mathematics and general relativity, Contemp. Math., 71, American Math. Soc., 237-262(1988).

[10] C. He and M. Zhu, Ricci solitons on Sasakian manifolds, arXiv:1109.4407 V2 [math.DG](2011).

[11] S.K. Hui and D. Chakraborty, Generalized Sasakian-space-forms and Ricci almost solitons with a conformally Killing vector field, New Trends in Math. Sciences, 4, 263-269(2016).

[12] S.K. Hui and D. Chakraborty, η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds, Acta Univ. Palack. Olamuc. Fac. Rerum Natur. Math., 55(2), 101-109(2016).

[13] S.K. Hui and D. Chakraborty, Ricci almost solitons on Concircular Ricci pseudosymmetric β-Kenmotsu manifolds, Hacet. J. Math. Stat., 47(3), 579-587(2018).

[14] S.K. Hui, L.I. Piscoran and T. Pal, Invariant submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric metric connection, arXiv:1706.09159[math.DG](2017).

[15] S.K. Hui, R. Prasad and T. Pal, Ricci solitons on submanifolds of $(LCS)_n$-manifolds, Ganita, 68(1),359-372(2018).

[16] S.K. Hui, S. Uddin and D. Chakraborty, Infinitesimal CL-transformations on $(LCS)_n$-manifolds, Palestine J. Math., 6(special issue: II), 190-195(2017).

[17] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:0211159[math.DG](2002).

[18] R. Sharma, Certain results on k-contact and (k, μ)-contact manifolds, J. Geom., 89, 138-147(2008).

[19] M.M. Tripathi, Ricci solitons in contact metric manifolds, arXiv:0801.4222(2008).

[20] K. Yano and M. Kon, Structures on manifolds, World Scientific Publishing(1984).