Integral Operator Defined by Polylogarithm Function for Ceratin Subclass Of Analytic Functions.

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Abstract

For analytic function f in the open unit disc E, a linear operator by the polylogarithm functions introduced. The object of the paper is to study some properties for $\Im_c^\delta f(z)$ belonging to some classes by applying Jack's lemma.

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1 Introduction

Let A denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of A, which consists of functions of the form (1.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in E. In addition, Silverman(1975) introduced the class T of analytic functions with negative coefficients consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (z \in E) \tag{1.2}$$

The Hadamard product (or convolution) of two power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n \text{ is given by}$$

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.3}$$

Let $\phi_\delta(c;z)$denote the well-known generalization of the Riemann zeta and polylogarithm functions, or simply the δth order polylogarithm function, given by

$$\phi_\delta(c;z) = \sum_{n=2}^{\infty} \frac{z^n}{(n+c)^\delta}$$

where any term with $n + c = 0$ is excluded [see Lerch(1887)]. Using the definition of the Gamma function [for details see Bateman(1953)] a simple transformation produces the integral formula

$$\phi_\delta(c;z) = \frac{1}{\Gamma(\delta)} \int_0^1 z \left(log\frac{1}{t} \right)^{\delta-1} \frac{t^c}{1-tz} dt$$

where $Re(c) > -1$ and $Re(\delta) > 1$. For more details about polylogarithm function, see Ponnusamy(1998) and Ponnusamy & Sabapathy(1996).

For $f \in A$ of the form (1.1), Al-Shasqi(2014) defined the following integral operator

$$\Im_c^\delta f(z) = (1+c)^\delta \phi_\delta(c;z) * f(z) = -\frac{(1+c)^\delta}{\Gamma(\delta)} \int_0^1 t^{c-1} \left(log\frac{1}{t} \right)^{\delta-1} f(tz) dt \tag{1.4}$$

where $c > 0, \delta > 1$ and $z \in E$ Al-Shaqsi (2014) noted that the operator defined by (1.3) can be expressed by the series expansion below.

$$\Im_c^\delta f(z) = z + \sum_{n=1}^{\infty} \left(\frac{1+c}{n+c} \right)^\delta a_n z^n. \tag{1.5}$$

In the following definition, we introduce a new class of analytic functions containing an integral operator defined by a polylogarithm function of equation (1.4).

Definition 1. *Let a function $f \in A$, then $f \in \Im_c^\delta f(z)$ if and only if*

$$Re\left\{ \frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)} \right\} > \beta, \quad z \in E, 0 \le \beta < 1 \tag{1.6}$$

Let f and g be analytic in E. Then f is said to be subordinate to g if there exists an analytic function w satisfying $w(0) = 0$ and $|w(z) < 1$, such that $f(z) = g(w(z)), z \in E$, We denote this subordination as $f(z) \prec g(z)$ or $(f \prec z), z \in E$

Lemma 1. *Let $w(z)$ be analytic in E with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in E$, then we have $z_0 w'(z_0)$, where $k \ge 1$ is a real number.*

2 MAIN RESULTS

In the present paper, we follow similar works done by Shireishi and Owa(2009) and Ochiai et al.(2005), we derive the following result:

Theorem 2.1. *If $f \in A$ satisfies*

$$Re\left\{\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}\right\} < \frac{\beta-3}{2(\beta-1)}, \quad z \in E, for\ some\ \beta(-1<\beta\leq 0), then$$

$$\frac{\Im_c^\delta f(z)}{z} \prec \frac{1+\beta z}{1-z}, \quad z \in E$$

This implies that

$$Re\left\{\frac{\Im_c^\delta f(z)}{z}\right\} > \frac{1-\beta}{2}$$

Proof. Lets us define the function $w(z)$ by

$$\frac{\Im_c^\delta f(z)}{z} = \frac{1-\beta w(z)}{1-w(z)}, \quad (w(z)\neq 1)$$

Clearly, $w(z)$ is analytic in E and $w(0)=0$. We want to prove that $|w(z)|\neq 1)$ in E. Since

$$\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)} = \frac{-\beta z w'(z)}{1-\alpha w(z)} + \frac{zw'(z)}{1-w(z)} + 1,$$

we see that

$$Re\left\{\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}\right\} = Re\left\{\frac{-\beta z w'(z)}{1-\alpha w(z)} + \frac{zw'(z)}{1-w(z)} + 1\right\} < \frac{\beta-3}{2(\beta-1)}, \quad (z\in E)$$

For $-1<\beta\leq 0$, if there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$$

Then Lemma 1.1 gives us that $w(z_0)=e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0),\ k\geq 1.$ Thus we have

$$\frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z_0)} = \frac{-\beta z_0 w'(z_0)}{1-\alpha w(z_0)} + \frac{z_0 w'(z_0)}{1-w(z_0)} + 1 = 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\alpha e^{i\theta}}$$

It follows that

$$Re\left\{\frac{1}{1-w(z_0)}\right\} = Re\left\{\frac{1}{1-e^{i\theta}}\right\} = \frac{1}{2}$$

and

$$Re\left\{\frac{1}{1-\beta w(z_0)}\right\} = Re\left\{\frac{1}{1-\beta e^{i\theta}}\right\} = \frac{1}{2} - \frac{1-\beta^2}{2(1+\beta^2-2\beta cos\theta)}.$$

Therefore, we have

$$Re\left\{\frac{z_0\Im_c^\delta f(z_0)}{\Im_c^\delta f(z)}\right\}=1-\frac{k(\beta^2-1)}{2(1+\beta^2-2\beta cos\theta)}.$$

This implies that, $-1<\beta\leq 0$

$$Re\left\{\frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z)}\right\}\geq 1-\frac{1-\beta^2}{2(\beta-1)^2}=\frac{\beta-3}{2(\beta-1)}.$$

This contradicts the condition in the theorem. Then, there exists no $z_0\in E$ such that $|w(z_0)|=1$ for all $z\in E$, that is

$$\frac{\Im_c^\delta f(z}{z}\prec\frac{1+\beta z}{1-z},\quad z\in E$$

Furthermore, since

$$w(z)=\frac{\frac{\Im_c^\delta f(z}{z}-1}{\frac{\Im_c^\delta f(z}{z}-\beta}\quad z\in E$$

and $|w(z)|<1$, $(z\in E)$, we conclude that

$$Re\left\{\frac{\Im_c^\delta f(z)}{z}\right\}>\frac{1-\beta}{2}$$

□

Taking $\beta=0$ in the theorem, we have the following corollary:

Corollary 1. *If $f\in A$ satisfies*

$$Re\left\{\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}\right\}>\frac{3}{2},\quad z\in E$$

then

$$\frac{\Im_c^\delta f(z)}{z}\prec\frac{1}{1-z},\quad z\in E$$

and

$$Re\left\{\frac{\Im_c^\delta f(z)}{z}\right\}>\frac{1}{2},\quad z\in E$$

Theorem 2.2. *If $f\in A$ satisfies*

$$Re\left\{\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}\right\}>\frac{3\beta-1}{2(\beta-1)},\quad z\in E, for\ some\ \beta(-1<\beta\leq 0), then$$

$$\frac{z}{\Im_c^\delta f(z)}\prec\frac{1+z}{1-z},\quad z\in E$$

and

$$\left|\frac{\Im_c^\delta f(z)}{z}-\frac{1}{1-\beta}\right|<\frac{1}{1-\beta},\quad z\in E$$

This implies that

$$Re\left\{\frac{\Im_c^\delta f(z)}{z}\right\}>0,\quad z\in E$$

Proof. Let us define the function $w(z)$ by

$$\frac{z}{\Im_c^\delta f(z)}=\frac{1-\beta w(z)}{1-w(z)},\quad w(z)\neq 1 \tag{2.1}$$

Then, we have $w(z)$ is analytic in E and $w(0)=0$. We want to prove that $|w(z)|<1$ in E. Differentiating equation (2.1), we obtain

$$\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}=-\frac{zw'(z)}{1-w(z)}+\frac{\alpha zw'(z)}{1-\alpha w(z)}+1,$$

and hence

$$\begin{aligned}Re\left\{\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}\right\}&=Re\left\{-\frac{zw'(z)}{1-w(z)}+\frac{\beta zw'(z)}{1-\beta w(z)}+1\right\}\\&>\frac{3\beta-1}{2(\beta-1)},\quad z\in E\end{aligned}$$

For $(-1<\beta\leq 0)$. If there exists a point $z_0 inE$ such that Lemma (1.1) gives us that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$, $k\geq 1$. Thus we have

$$\begin{aligned}\frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z_0)}&=-\frac{z_0w'(z_0)}{1-w(z_0)}+\frac{\beta z_0w'(z_0)}{1-\beta w(z_0)}+1\\&=1-\frac{k}{1-e^{i\theta}}+\frac{k}{1-\beta e^{i\theta}},\quad z\in E\end{aligned}$$

Therefore, we have

$$Re\left\{\frac{z_0(\Im_c^\delta f(z))'}{\Im_c^\delta f(z_0)}\right\}=1+\frac{k(\beta^2-1)}{2(1+\beta^2-2\beta cos\theta)} \tag{2.2}$$

This implies that, for $-1<\alpha\leq 0$,

$$\begin{aligned}Re\left\{\frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z_0)}\right\}&=1-\frac{k(1-\alpha^2)}{2(1+\alpha^2-2\alpha cos\theta)}\\&\leq\frac{3\alpha-1}{2(\alpha-1)}\end{aligned}$$

This contradicts the condition in the theorem. Hence, there no $z_0 \in E$ such that $|w(z_0)| = 1$ for all $z \in E$, that is

$$\frac{z}{\Im_c^\delta f(z)} \prec \frac{1+z}{1-z}, \quad z \in E$$

Furthermore, since

$$w(z) = \frac{1 - \frac{\Im_c^\delta f(z)}{z}}{\beta - \frac{\Im_c^\delta f(z)}{z}} \quad z \in E$$

and $|w(z)| = 1$, $(z \in E)$, we conclude that

$$\left| \frac{\Im_c^\delta f(z)}{z} - \frac{1}{1-\beta} \right| < \frac{1}{1-\beta}, \quad z \in E$$

which implies that

$$Re\left\{ \frac{\Im_c^\delta f(z)}{z} \right\} > 0, \quad z \in E$$

We complete the proof of the theorem. By setting $\beta = 0$ in Theorem 2.2, we readily obtain the following: □

Corollary 2. *If $f \in A$ satisfies*

$$Re\left\{ \frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)} \right\} > \frac{1}{2}, \quad z \in E,$$

then

$$\frac{z}{\Im_c^\delta f(z)} \prec \frac{1+z}{1-z}, \quad z \in E$$

and

$$\left| \frac{\Im_c^\delta f(z)}{z} - 1 \right| < 1, \quad z \in E$$

Theorem 2.3. *If $f \in A$ satisfies*

$$Re\left\{ \frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)} \right\} < \frac{\beta(2-\gamma) - (2+\gamma)}{2(\beta - 1)}, \quad z \in E,$$

for some $\beta(-1 < \beta \le 0)$ and $0 < \gamma \le 1$, then

$$\left(\frac{\Im_c^\delta f(z)}{z} \right)^{\frac{1}{\gamma}} \prec \frac{1+\beta z}{1-z}, \quad z \in E$$

This implies that

$$Re\left\{ \frac{\Im_c^\delta f(z)}{z} \right\}^{\frac{1}{\gamma}} > \frac{1-\beta}{2}, \quad z \in E$$

Proof. Let us define the function $w(z)$ by

$$\frac{\Im_c^\delta f(z)}{z} > \left(\frac{1-\beta w(z)}{1-w(z)}\right)^\gamma, \quad w(z) \neq 1$$

Clearly, $w(z)$ is analytic in E and $w(0)=0$. We want to prove that $|w(z)|<1$ in E. Since

$$\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)} = \gamma\left(\frac{zw'(z)}{1-w(z)} - \frac{\beta z w'(z)}{1-\beta w(z)}\right) + 1.$$

We see that

$$\begin{aligned} Re\left\{\frac{z(\Im_c^\delta f(z))'}{\Im_c^\delta f(z)}\right\} &= Re\left\{\gamma\left(\frac{zw'(z)}{1-w(z)} - \frac{\beta z w'(z)}{1-\beta w(z)}\right) + 1\right\} \\ &< \frac{\beta(2-\gamma)-(2+\gamma)}{2(\beta-1)}, \quad z \in E \end{aligned}$$

for $(-1<\beta\leq 0)$ and $0<\gamma\leq 1$. If there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$$

then by Lemma 1.1, gives us that $w(z_0)=e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. Thus we have

$$\begin{aligned} \frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z_0)} &= \gamma\left(\frac{z_0 w'(z_0)}{1-w(z_0)} - \frac{\beta z_0 w'(z_0)}{1-\beta w(z_0)}\right) + 1 \\ &= 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\beta e^{i\theta}} \end{aligned}$$

Therefore, we have

$$Re\left\{\frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z_0)}\right\} = 1 + \frac{\gamma k(1-\beta^2)}{2(1+\beta^2-2\beta cos\theta)}$$

This implies that, for $-1<\beta\leq 0$ and $0<\gamma\leq 1$

$$Re\left\{\frac{z_0(\Im_c^\delta f(z_0))'}{\Im_c^\delta f(z_0)}\right\} \geq \frac{\beta(2-\gamma)-(2+\gamma)}{2(\beta-1)}$$

This contradicts the condition in the theorem. Hence, there is $z_0 \in E$ such that $|w(z_0|=1$ for all $z \in E$, that is

$$\left(\frac{\Im_c^\delta f(z)}{z}\right)^{\frac{1}{\gamma}} \prec \frac{1+\beta z}{1-z}, \quad z \in E$$

Furthermore, since

$$w(z) = \frac{\frac{\Im_c^\delta f(z}{z} - 1}{\frac{\Im_c^\delta f(z}{z} - \beta} \quad z \in E$$

and $|w(z)| < 1$ $(z \in E)$, we conclude that

$$Re\left\{\frac{\Im_c^\delta f(z)}{z}\right\}^{\frac{1}{\gamma}} > \frac{1-\beta}{2}, \quad z \in E$$

The proof of the theorem is completed. □

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References

[1] AL-Shaqsi K.,Strong differential subordinations obtained with new integral operator defined by polylogarithm function, *Int. J. Math. Math. Sci.*, **2014**, (2014), 260198.

[2] Bateman, H. Higher, Transcendental Functions, *McGraw-Hill: New York, NY, USA,*, **1**, 1953.

[3] Jack, I.S., Functions starlike and convex of order α., *J. London Math. Soc.*, **1**, (1971), 469-474.

[4] Lerch M., Note sur la fonction $k(w, x, s) = \sum_{k=0}^{\infty} e^{2k\pi x}$, *Acta Math.*,**11**, (1887), 19–24.

[5] Miller, S. S.; Mocanu, P. T., Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65**, (1978), 289-305.

[6] Ochiai, K.; Owa, S.; Acu, M., Applications of Jack's lemma for certain subclasses of analytic functions., *General Math.*, **13**,(2005), 73-82.

[7] Ponnusamy, S.; Sabapathy, S., Polylogarithms in the theory of univalent functions, *Results Math.*, **30**, (1996), 136–150.

[8] Ponnusamy, S. , Inclusion theorems for convolution product of second order polylogarithms and Functions with the derivative in a half plane., *Rocky Mt. J. Math.*, **28**, (1998), 695–733.

[9] Shiraishi, H.; Owa, S., Starlike and convexity for analytic functions concerned with Jack's lemma., *J. open problems Compt. Math*, **2**, (2009), 37-47.

[10] Silverman, H., Univalent functions with negative coefficients, *Proc. Am. Math. Soc.*, **51**, (1975), 09–116.