Irrationality of Bilateral Mock Theta Functionsof Order Five at Infinite number of Points.

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Abstract

Ramanujan wrote about seventeen functions in his last letter to G. H. Hardy, called them Mock Theta Functions and assigned them of order three, five and seven. In this paper Irrationality of Bilateral Mock Theta Functions of Order Five has been discussed at infinite number of points.

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1 Introduction

It was early in 1920, three months before his death, Ramanujan wrote his last letter to G. H. Hardy. In course of it he said " I discovered very interesting functions recently which I call ' Mock Theta Functions'. He listed seventeen such functions and assigned them of order three, five and seven.

Srivastava [1999] has obtained the following eight Bilateral Mock Theta functions of order 'five' by using the transformation of Bailey ${}_2\Psi_2$ series and also given some alternative forms of these functions by using the the general transformation of Gasper and Rahman [1990, page - 129 (5.4.3)] for $r = 2$:

(1.1) $$f_0c_2(q) = \sum_{-\infty}^{\infty} \frac{q^{n^2}}{(-q)_n} = \sum_{-\infty}^{\infty} q^{\frac{n(n+1)}{2}} (-1)_n \ ,$$

(1.2) $$\Phi_0c_2(q) = \sum_{-\infty}^{\infty} q^{n^2} \left(-q;q^2\right)_n = \frac{1}{2}\sum_{-\infty}^{\infty} \frac{q^{2n^2}}{(-q;q^2)_n} \ ,$$

(1.3) $$\Psi_0c_2(q) = \sum_{-\infty}^{\infty} q^{\frac{(n+1)(n+2)}{2}} (-q)_n = \frac{1}{2}(1+q)\sum_{-\infty}^{\infty} \frac{q^{(n+1)^2}}{(-q^2)_n} \ ,$$

(1.4) $$F_0c_2(q) = \sum_{-\infty}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} = \sum_{-\infty}^{\infty} (-1)^n q^{n^2} \left(q;q^2\right)_n \ ,$$

(1.5) $$f_1c_2(q) = \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q)_n} = 2\sum_{-\infty}^{\infty} q^{\frac{n(n+1)}{2}} (-q)_n \ ,$$

(1.6) $$\Phi_1c_2(q) = \sum_{-\infty}^{\infty} q^{(n+1)^2} \left(-q;q^2\right)_n = \frac{1}{1+q}\sum_{-\infty}^{\infty} \frac{q^{2n(n+1)}}{(-q^3;q^2)_n} \ ,$$

(1.7) $$\Psi_1c_2(q) = \sum_{-\infty}^{\infty} q^{\frac{n(n+1)}{2}} (-q)_n = \frac{1}{2}\sum_{-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q)_n} \ ,$$

and

(1.8) $F_1c_2(q) = \sum_{-\infty}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} = \sum_{-\infty}^{\infty} (-1)^n q^{n(n+2)} \left(q;q^2\right)_n$.

In this paper, irrationality of Bilateral Mock Theta Functions of order "five" has been discussed. The discussion includes the Cantor series and the theorems given by Oppenheim [1954, 1955] regarding the Cantor series along with Bilateral Mock Theta Functions of order "five". In Section 3, Oppenheim's theorems regarding the Cantor series are given and in Section 4, the theorem on the irrationality of these functions has been stated and proved.

2 Notations

The following q - notations and some standard results have been used:

For $\left|q^k\right| < 1$, k a non-negative integer, then

$\left(a;q^k\right)_n = \prod_{j=0}^{n-1}\left(1-aq^{kj}\right),\ n \geq 1, \qquad \left(a;q^k\right)_0 = 1,$

$\left(a;q^k\right)_\infty = \prod_{j=0}^{\infty}\left(1-aq^{kj}\right),$

$(a)_n = (a;q)_n = (1-a)(1-aq)\ldots\left(1-aq^{n-1}\right),$

$\left(a_1,a_2,...,a_m;q^k\right)_n = \left(a_1;q^k\right)_n\left(a_2;q^k\right)_n\ldots\left(a_m;q^k\right)_n$,

A generalised basic hypergeometric series with base q is defined as

$${}_r\Phi_{r-1}\left[\begin{matrix} a_1,a_2,...,a_r \\ b_1,b_2,...,b_{r-1}\end{matrix};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q,z)_n}{(q;q)_n(b_1,b_2,...,b_{r-1};q,z)_n}z^n,\ |z|<1\ ,$$

A bilateral basic hypergeometric series with base q is defined as

$${}_r\Psi_r\left[\begin{matrix} a_1,a_2,...,a_r \\ b_1,b_2,...,b_r\end{matrix};q,z\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n}{(b_1,b_2,...,b_r;q)_n}z^n \text{ for } \left|\frac{b_1b_2...b_r}{a_1a_2...a_r}\right| < |z| < 1\ .$$

3 The Cantor Series and the Theorems of Oppenheim

Cantor [1869] presented a necessary and sufficient condition for series of the form

(3.1) $S = \sum_{n=1}^{\infty}\frac{b_n}{a_1a_2...a_n}$,

where the a_i, b_i are integers to have irrational sums. The basic conditons being of the form $a_i \geq 2$, $a_i - 1 \geq b_i \geq 0$ and for every integer $k \geq 1$ there is an n such that $k \mid a_1a_2...a_n$, Cantor showed that S is irrational if and only if the $b_i > 0$ infinitely often and $a_i - 1 > b_i$ infinitely ofen.

Oppenheim [1954, 1955] dropped the divisibility condition on the product of the first n, a's and an extension of the thoerem to the case where the b_i can have both signs. The following two theorems given by Oppenheim regarding the Cantor series are used:

Theorem 3.1 *(Oppenheim [1954], Theorem 4)*: Let (a_n), (b_n) be two sequences of integers with $a_n \geq 2$, $0 \leq b_n \leq a_n - 1$. If $b_n > 0$ infinitely often and if there is a subsequence i_n such that $a_{i_n} \to \infty$ and $b_{i_n}/a_{i_n} \to 0$ as $n \to \infty$, then S as defined in (3.1) is irrational.

Theorem 3.2 *(Oppenheim [1954], Theorem 8)*: Let (a_n), (b_n) be two sequences of integers with $a_n \geq 2, |b_n| \leq a_n - 1$. Furthermore, let $b_mb_n < 0$ for some $m > i, n > i$ for any assigned integer i. If there is a subsequence i_n such that $a_{i_n} \to \infty$ and $b_{i_n}/a_{i_n} \to 0$ as $n \to \infty$, then S given by (3.1) is irrational.

4 Irrationality of Bilateral Mock Theta Functions of order "Five"

The following theorem on the irrationality of Bilateral Mock Theta Functions of order "five" has been proved in this section:

Theorem 4.1. Bilateral Mock Theta Functions of order "five" (1.1) to (1.8) $f_0c_2(q)$, $\Phi_0c_2(q)$, $\Psi_0c_2(q)$, $F_0c_2\,(q)$, $f_1c_2(q)$, $\Phi_1c_2(q)$, $\Psi_1c_2(q)$, $F_1c_2\,(q)$ take on irrational values at $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$.

Proof of Theorem (4.1) : The above theorem has been proved for all the Bilateral Mock Theta Functions of order "five", (1.1) to (1.8) separately:

I. *Bilateral Mock Theta function* $F_0c_2\,(q)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$

The function $F_0c_2\,(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}$ can be written as

$$F_0c_2\,(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} + \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(q;q^2)_{-n}}\ ,$$

or

$$F_0c_2\,(q) = 1 + F_0S_2\,(q) + \sum_{n=1}^{\infty} (-1)^n\, q^{n^2} \left(q;q^2\right)_n\ . \tag{4.1}$$

We note that $F_0c_2\,(q)$ is irrational if and only if $F_0S_2\,(q)$ is irrational, and

$$F_0S_2\,(q) = \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(1-q)(1-q^3)\ldots(1-q^{2n-1})}\ . \tag{4.2}$$

Let $p, q \in C$, $q \neq 0$, then

$$F_0S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2}}{q^{n^2}(q-p)(q^3-p^3)\ldots(q^{2n-1}-p^{2n-1})}\ . \tag{4.3}$$

Inserting $p = 1$ in preceding expression, we get

$$F_0S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2}(q-1)(q^3-1)\ldots(q^{2n-1}-1)}\ ,$$

or

$$F_0S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q(q-1)q^3(q^3-1)\ldots q^{2n-1}(q^{2n-1}-1)}\ . \tag{4.4}$$

Since for every $q \geq 2$, $q \in Z$, the above expression is a Cantor series as given by (3.1) with the identifications $a_n = q^{2n-1}\left(q^{2n-1}-1\right)$ and $b_n = 1$ for all n. For every $q \geq 2$, and an integer $n \geq 1, a_n \geq 2, a_n - 1 > b_n > 0, a_n \to \infty$, $b_n/a_n \to 0$ as $n \to \infty$, so by Theorem 3.1, $F_0S_2\left(\frac{1}{q}\right)$is irrational for every integer $q \geq 2$.

Again inserting $p = -1$ in (4.3), we obtain

$$F_0S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2}(q+1)(q^3+1)\ldots(q^{2n-1}+1)}\ ,$$

This implies

$$F_0S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q(q+1)q^3(q^3+1)\ldots q^{2n-1}(q^{2n-1}+1)}\ . \tag{4.5}$$

Hence $F_0S_2\left(-\frac{1}{q}\right)$ is Cantor series as given by (3.1) with the identifications $a_n = q^{2n-1}\left(q^{2n-1}+1\right), b_n = 1$ for all n. For every $q \geq 2$, an integer and any $n \geq 1, b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0, a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$. Therefore by Theorem 3.1, $F_0S_2\left(-\frac{1}{q}\right)$is irrational for every integer $q \geq 2$.

Since $F_0S_2\left(\frac{1}{q}\right)$ and $F_0S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, so $F_0S_2\left(q\right)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$.

II. *Bilateral Mock Theta function $F_1c_2\left(q\right)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$*

The function $F_1c_2\left(q\right) = \sum_{-\infty}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}$ can be written as

$$F_1c_2\left(q\right) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} + \sum_{n=1}^{\infty} \frac{q^{2n(n-1)}}{(q;q^2)_{-n+1}} ,$$

or

$$F_1c_2\left(q\right) = \frac{1}{1-q} + F_1S_2\left(q\right) + \sum_{n=1}^{\infty} \left(-1\right)^{n-1} q^{n^2+1}\left(q;q^2\right)_{n-1} . \tag{4.6}$$

It is noted that $F_1c_2\left(q\right)$ is irrational if and only if $F_1S_2\left(q\right)$ is irrational, and

$$F_1S_2\left(q\right) = \sum_{n=1}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} = \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})} . \tag{4.7}$$

As before, for $p, q \in C$, $q \neq 0$, we have

$$F_1S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2+2n}}{q^{n^2-1}(q-p)(q^3-p^3)\ldots(q^{2n+1}-p^{2n+1})} . \tag{4.8}$$

Substituting $p = 1$ in (4.8), we obtain

$$F_1S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q}{q^{n^2}(q-1)(q^3-1)\ldots(q^{2n+1}-1)} ,$$

or

$$F_1S_2\left(\frac{1}{q}\right) = \frac{1}{q-1}\sum_{n=1}^{\infty} \frac{q}{q(q^3-1)q^3(q^5-1)\ldots q^{2n-1}(q^{2n+1}-1)} . \tag{4.9}$$

It is observed that the sum of the right of (4.9) is Cantor series as given by (3.1) with the identifications $a_n = q^{2n-1}\left(q^{2n+1}-1\right)$ and $b_n = q$ for every n. Also, for every $q \geq 2$, an integer and any $n \geq 1, b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0$, $a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$, therefore by Theorem 3.1, $F_1S_2\left(\frac{1}{q}\right)$is irrational for every integer $q \geq 2$.

Again substituting $p = -1$ in (4.8), we find that

$$F_1S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2-1}(q+1)(q^3+1)\ldots(q^{2n+1}+1)} ,$$

or

$$F_1S_2\left(-\frac{1}{q}\right) = \frac{1}{q+1}\sum_{n=1}^{\infty} \frac{q}{q(q^3+1)q^3(q^5+1)\ldots q^{2n-1}(q^{2n+1}+1)} . \tag{4.10}$$

We observe that the sum of the right of (4.10) is Cantor series as given by (3.1) with the identifications $a_n = q^{2n-1}\left(q^{2n+1}+1\right)$ and $b_n = q$ for all n. For every $q \geq 2$, an integer and any $n \geq 1, b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0,\ a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$, therefore by Theorem 3.1, $F_1S_2\left(-\frac{1}{q}\right)$is irrational for every integer $q \geq 2$.

Since $F_1S_2\left(\frac{1}{q}\right)$ and $F_1S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $F_1S_2\left(q\right)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$.

III. *Bilateral Mock Theta function* $\Phi_0 c_2\left(q\right)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$

In this case, we write the function $\Phi_0 c_2\left(q\right) = \frac{1}{2}\sum_{-\infty}^{\infty} \frac{q^{2n^2}}{(-q;q^2)_n}$ as

$$\Phi_0 c_2\left(q\right) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q^2)_n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{q^{2n^2}}{(-q;q^2)_{-n}}\ ,$$

or

$$\Phi_0 c_2\left(q\right) = \frac{1}{2} + \frac{1}{2}\Phi_0 S_2\left(q\right) + \frac{1}{2}\sum_{n=1}^{\infty} q^{n^2}\left(-q;q^2\right)_n\ . \tag{4.11}$$

It follows that $\Phi_0 c_2\left(q\right)$ is irrational if and only if $\Phi_0 S_2\left(q\right)$ is irrational, and

$$\Phi_0 S_2\left(q\right) = \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(-q;q^2)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(1+q)(1+q^3)\ldots(1+q^{2n-1})}\ . \tag{4.12}$$

If $p, q \in C,\ q \neq 0$, we see that

$$\Phi_0 S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2}}{q^{n^2}(q+p)(q^3+p^3)\ldots(q^{2n-1}+p^{2n-1})}\ . \tag{4.13}$$

Now set $p = 1$ in (4.13), we obtain

$$\Phi_0 S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2}(q+1)(q^3+1)\ldots(q^{2n-1}+1)}\ ,$$

or

$$\Phi_0 S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q(q+1)q^3(q^3+1)\ldots q^{2n-1}(q^{2n-1}+1)}\ . \tag{4.14}$$

The sum $\Phi_0 S_2\left(\frac{1}{q}\right)$ is same as the sum (4.5) and we have verified that the sum (4.5) is irrational for every integer $q \geq 2$. Hence the sum $\Phi_0 S_2\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again set $p = -1$ in (4.13), we get

$$\Phi_0 S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2}(q-1)(q^3-1)\ldots(q^{2n-1}-1)}\ ,$$

or

$$\Phi_0 S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q(q-1)q^3(q^3-1)\ldots q^{2n-1}(q^{2n-1}-1)}\ . \tag{4.15}$$

The above sum $\Phi_0 S_2\left(-\frac{1}{q}\right)$ is same as the sum (4.4) and it has been verified that the sum (4.4) is irrational for every integer $q \geq 2$. Hence the sum $\Phi_0 S_2\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

As it have seen that $\Phi_0 S_2\left(\frac{1}{q}\right)$ and $\Phi_0 S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $\Phi_0 S_2(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$.

IV. *Bilateral Mock Theta function* $\Phi_1 c_2(q)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$

The function $\Phi_1 c_2(q) = \frac{1}{1+q}\sum_{-\infty}^{\infty} \frac{q^{2n(n+1)}}{(-q^3;q^2)_n}$ can be expressed as

$$\Phi_1 c_2(q) = \frac{1}{1+q}\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(-q^3;q^2)_n} + \frac{1}{1+q}\sum_{n=1}^{\infty} \frac{q^{2n(n-1)}}{(-q^3;q^2)_{-n}},$$

or

$$\Phi_1 c_2(q) = \frac{1}{1+q} + \frac{1}{1+q}\Phi_1 S_2(q) + \frac{1}{1+q}\sum_{n=1}^{\infty} q^{n^2}\left(-1/q;q^2\right)_n. \tag{4.16}$$

It is clear that $\Phi_1 c_2(q)$ is irrational if and only if $\Phi_1 S_2(q)$ is irrational, and

$$\Phi_1 S_2(q) = \sum_{n=1}^{\infty} \frac{q^{2n(n+1)}}{(-q^3;q^2)_n} = \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1+q^3)(1+q^5)\ldots(1+q^{2n+1})}. \tag{4.17}$$

Whenever $p, q \in C$, $q \neq 0$, then

$$\Phi_1 S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{2n^2+2n}}{q^{n^2}(q^3+p^3)(q^5+p^5)\ldots(q^{2n+1}+p^{2n+1})}. \tag{4.18}$$

Setting $p = 1$ in (4.18), we obtain

$$\Phi_1 S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2}(q^3+1)(q^5+1)\ldots(q^{2n+1}+1)},$$

or

$$\Phi_1 S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q(q^3+1)q^3(q^5+1)\ldots q^{2n-1}(q^{2n+1}+1)}. \tag{4.19}$$

The series (4.19) is a Cantor series with $a_n = q^{2n-1}\left(q^{2n+1}+1\right)$ and $b_n = 1$ for all n. For every integer $q \geq 2$ and $n \geq 1, b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0, a_n \to \infty, \frac{b_n}{a_n} \to 0$ as $n \to \infty$. Hence by Theorem 3.1, $\Phi_1 S_2\left(\frac{1}{q}\right)$ is irrational for $q \geq 2$.

Again, setting $p = -1$ in (4.18) gives

$$\Phi_1 S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q^{n^2}(q^3-1)(q^5-1)\ldots(q^{2n+1}-1)},$$

or

$$\Phi_1 S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{q(q^3-1)q^3(q^5-1)\ldots q^{2n-1}(q^{2n+1}-1)}. \tag{4.20}$$

Hence $\Phi_1 S_2\left(-\frac{1}{q}\right)$ is a Cantor series as given by (3.1) with the identifications $a_n = q^{2n-1}\left(q^{2n+1}-1\right)$ and $b_n = 1$ for all n. For every $q \geq 2$, an integer and any $n \geq 1, b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0$, $a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$, therefore by Theorem 3.1, $\Phi_1 S_2\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Since $\Phi_1 S_2\left(\frac{1}{q}\right)$ and $\Phi_1 S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $\Phi_1 S_2(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$.

V. *Bilateral Mock Theta function* $f_0 c_2(q)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$

As in the preceding cases, we write the function $f_0c_2\left(q\right) = \sum_{-\infty}^{\infty} \frac{q^{n^2}}{(-q)_n}$ as

$f_0c_2\left(q\right) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n} + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_{-n}}$,

or

$$f_0c_2\left(q\right) = 1 + f_0S_2\left(q\right) + \sum_{n=1}^{\infty} q^{\frac{n^2+n}{2}} \left(-1\right)_n \ . \tag{4.21}$$

It follows that $f_0c_2\left(q\right)$ is irrational if and only if $f_0S_2\left(q\right)$ is irrational, and

$$f_0S_2\left(q\right) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_n} = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)...(1+q^n)} \ . \tag{4.22}$$

For $p, q \in C$, $q \neq 0$, we have

$$f_0S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{n^2} q^{-\left(\frac{n^2+n}{2}\right)}}{(q+p)(q^2+p^2)...(q^n+p^n)} \ . \tag{4.23}$$

Setting $p = 1$ in (4.23), we get

$$f_0S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{-\left(\frac{n^2-n}{2}\right)}}{(q+1)(q^2+1)...(q^n+1)} \ . \tag{4.24}$$

Comparing $f_0S_2\left(\frac{1}{q}\right)$ with the Cantor series given by (3.1), we get

$b_n = q^{-\left(\frac{n^2-n}{2}\right)}, a_n = q^n + 1$. For every $q \geq 2$, an integer and any $n \geq 1, b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0, a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$, therefore by Theorem 3.1, $f_0S_2\left(\frac{1}{q}\right)$is irrational for every integer $q \geq 2$.

Again setting $p = -1$ in (4.23), we obtain

$f_0S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n^2} q^{-\frac{\left(n^2-n\right)}{2}}}{(q-1)(q^2+1)(q^3-1)...(q^n+(-1)^n)}$,

or $\quad f_0S_2\left(-\frac{1}{q}\right) = -\frac{1}{(q-1)} + \sum_{n=2}^{\infty} \frac{(-1)^n q^{-\frac{n(n-1)}{2}}}{(q-1)(q^2+1)(q^3-1)...(q^n+(-1)^n)}$,

or

$$f_0S_2\left(-\frac{1}{q}\right) = -\frac{1}{(q-1)} + \frac{1}{q-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{-\frac{n(n+1)}{2}}}{(q^2+1)(q^3-1)...\left(q^{n+1}+(-1)^{n+1}\right)} \ . \tag{4.25}$$

The sum on the right of (4.25) is the form of Cantor series given by (3.1) with identifications $b_n = (-1)^{n+1} q^{-\frac{n(n+1)}{2}}, a_n = q^{n+1} + (-1)^{n+1}$. It is observed that for any $n \geq 1, |b_n| > 0, a_n \geq 2, a_n - 1 > |b_n| > 0$. Furthermore $a_n \to \infty, \frac{b_n}{a_n} \to 0$ as $n \to \infty$. Hence all the conditions of Theorem 3.2 are satisfied, therefore $f_0S_2\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

As we have shown that $f_0S_2\left(\frac{1}{q}\right)$ and $f_0S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, then $f_0S_2\left(q\right)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, ...$.

VI. *Bilateral Mock Theta function* $f_1c_2(q)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, ...$

The function $f_1c_2(q) = \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q)_n}$ can be written as

$$f_1c_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q)_n} + \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(-q)_{-n}} ,$$

or

$$f_1c_2(q) = 1 + f_1S_2(q) + \sum_{n=1}^{\infty} q^{\frac{n^2-n}{2}} (-1)_n , \tag{4.26}$$

where

$$f_1S_2(q) = \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(-q)_n} = \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^2)...(1+q^n)} . \tag{4.27}$$

Whenever it is defined, $f_1c_2(q) \notin Q$,iff $f_1S_2(q) \notin Q$. Let $p, q \in C$, $q \neq 0$, then

$$f_1S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{n(n+1)}q^{-\frac{n(n+1)}{2}}}{(q+p)(q^2+p^2)...(q^n+p^n)} . \tag{4.28}$$

Setting $p = 1$ in (4.28), we get

$$f_1S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{-\frac{n(n+1)}{2}}}{(q+1)(q^2+1)...(q^n+1)} . \tag{4.29}$$

The sum $f_1S_2\left(\frac{1}{q}\right)$ is a Cantor series given by (3.1) with the identifications $b_n = q^{-\frac{n(n+1)}{2}}$, $a_n = q^n + 1$ for all n. For every integer $q \geq 2$ and any $n \geq 1$, $b_n > 0$, $a_n \geq 2$, $a_n - 1 > b_n$, $a_n \to \infty$ and $\frac{b_n}{a_n} \to 0$ as $n \to \infty$. Hence by Theorem 3.1, $f_1S_2\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again setting $p = -1$ in (4.28), we get

$$f_1S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{\frac{-n(n+1)}{2}}}{(q-1)(q^2+1)...(q^n+(-1)^n)} . \tag{4.30}$$

Now compare the above sum with Cantor series (3.1) and identify

$b_n = q^{\frac{-n(n+1)}{2}}, a_n = q^n + (-1)^n$ for all n. For $q \geq 2$ an integer and $n \geq 1$, we observe that $b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0, a_n \to \infty$, $\frac{b_n}{a_n} \to 0$ as $n \to \infty$. Hence by Theorem 3.1, $f_1S_2\left(-\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Since $f_1S_2\left(\frac{1}{q}\right)$ and $f_1S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $f_1S_2(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, ...$.

VII. *Bilateral Mock Theta function* $\Psi_0c_2(q)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, ...$

The function $\Psi_0c_2(q) = \frac{(1+q)}{2} \sum_{-\infty}^{\infty} \frac{q^{(n+1)^2}}{(-q^2)_n}$ can be written as

$$\Psi_0c_2(q) = \frac{(1+q)}{2} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(-q^2)_n} + \frac{(1+q)}{2} \sum_{n=1}^{\infty} \frac{q^{(-n+1)^2}}{(-q^2)_{-n}} ,$$

or

$$\Psi_0 c_2(q) = \tfrac{q(1+q)}{2} + \tfrac{(1+q)}{2}\Psi_0 S_2(q) + \tfrac{(1+q)}{2}\sum_{n=1}^{\infty} q^{\frac{n^2-n+2}{2}}\left(\tfrac{-1}{q};q\right)_n , \tag{4.31}$$

where

$$\Psi_0 S_2(q) = \sum_{n=1}^{\infty} \tfrac{q^{(n+1)^2}}{(-q^2)_n} = \sum_{n=1}^{\infty} \tfrac{q^{(n+1)^2}}{(1+q^2)(1+q^3)...(1+q^{n+1})} . \tag{4.32}$$

It is clear that $\Psi_0 c_2(q)$ is irrational if and only if $\Psi_0 S_2(q)$ is irrational.

For $p, q \in C, q \neq 0$, then

$$\Psi_0 S_2\left(\tfrac{p}{q}\right) = \sum_{n=1}^{\infty} \tfrac{p^{(n+1)^2} q^{-\frac{\left(n^2+n+2\right)}{2}}}{(q^2+p^2)(q^3+p^3)...(q^{n+1}+p^{n+1})} . \tag{4.33}$$

Substituting $p = 1$ in (4.33), we get

$$\Psi_0 S_2\left(\tfrac{1}{q}\right) = \sum_{n=1}^{\infty} \tfrac{q^{-\frac{\left(n^2+n+2\right)}{2}}}{(q^2+1)(q^3+1)...(q^{n+1}+1)} . \tag{4.34}$$

It is observed that the above series is a Cantor series given by (3.1) with the identifications $b_n = q^{-\frac{\left(n^2+n+2\right)}{2}}, a_n = q^{n+1} + 1$. For $q \geq 2$ an integer and $n \geq 1$, we observe that $b_n > 0, a_n \geq 2, a_n - 1 > b_n > 0, a_n \to \infty$, $\frac{b_n}{a_n} \to 0$ as $n \to \infty$. Hence by Theorem 3.1, $\Psi_0 S_2\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again substituting $p = -1$ in (4.33), we get

$$\Psi_0 S_2\left(-\tfrac{1}{q}\right) = \sum_{n=1}^{\infty} \tfrac{(-1)^{(n+1)^2} q^{-\frac{\left(n^2+n+2\right)}{2}}}{(q^2+1)(q^3-1)...\left(q^{n+1}+(-1)^{n+1}\right)} . \tag{4.35}$$

Comparing the above series with Cantor series (3.1), we get

$b_n = (-1)^{(n+1)^2} q^{-\frac{\left(n^2+n+2\right)}{2}}, a_n = q^{n+1} + (-1)^{n+1}$for all n. For every integer $q \geq 2$, and any $n \geq 1, a_n \geq 2, a_n - 1 > |b_n| > 0, a_n \to \infty$ & $b_n/a_n \to 0$ as $n \to \infty$. Hence by Theorem 3.2, $\Psi_0 S_2\left(-\frac{1}{q}\right)$is irrational for every integer $q \geq 2$.

As we have shown that $\Psi_0 S_2\left(\frac{1}{q}\right)$ and $\Psi_0 S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, then it follows that $\Psi_0 S_2(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, ...$

VIII. *Bilateral Mock Theta function* $\Psi_1 c_2(q)$ *is irrational for* $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, ...$

In this case, we write $\Psi_1 c_2(q) = \frac{1}{2}\sum_{-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q)_n}$ in the form

$$\Psi_1 c_2(q) = \tfrac{1}{2}\sum_{n=0}^{\infty} \tfrac{q^{n(n+1)}}{(-q)_n} + \tfrac{1}{2}\sum_{n=1}^{\infty} \tfrac{q^{n(n-1)}}{(-q)_{-n}} ,$$

or

$$\Psi_1 c_2(q) = \tfrac{1}{2} + \tfrac{1}{2}\Psi_1 S_2(q) + \tfrac{1}{2}\sum_{n=1}^{\infty} q^{\frac{n^2-n}{2}}(-1)_n , \tag{4.36}$$

where

(4.37) $$\Psi_1 S_2(q) = \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(-q)_n} = \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^2)\dots(1+q^n)} .$$

Whenever it is defined, $\Psi_1 c_2(q) \notin Q$,iff $\Psi_1 S_2(q) \notin Q$. For $p, q \in C$ and $q \neq 0$, we have

(4.38) $$\Psi_1 S_2\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{p^{n(n+1)} q^{-\frac{n(n+1)}{2}}}{(q+p)(q^2+p^2)\dots(q^n+p^n)} .$$

Setting $p = 1$ in (4.38), we get

(4.39) $$\Psi_1 S_2\left(\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{-\frac{n(n+1)}{2}}}{(q+1)(q^2+1)\dots(q^n+1)} .$$

It is observed that the sum $\Psi_1 S_2\left(\frac{1}{q}\right)$ is a Cantor series as given by (3.1) with the identifications $b_n = q^{-\frac{n(n+1)}{2}}, a_n = q^n + 1$ for all n. For every integer $q \geq 2$ and any $n \geq 1$, all the conditions of Theorem 3.1 are satisfied, therefore $\Psi_1 S_2\left(\frac{1}{q}\right)$ is irrational for every integer $q \geq 2$.

Again setting $p = -1$ in (4.38), we obtain

$$\Psi_1 S_2\left(-\frac{1}{q}\right) = \sum_{n=1}^{\infty} \frac{q^{\frac{-n(n+1)}{2}}}{(q-1)(q^2+1)\dots(q^n+(-1)^n)} ,$$

or $$\Psi_1 S_2\left(-\frac{1}{q}\right) = \frac{1}{q-1} + \frac{1}{q-1} \sum_{n=2}^{\infty} \frac{q^{\frac{-n(n+1)}{2}}}{(q^2+1)(q^3-1)\dots(q^n+(-1)^n)} ,$$

or

(4.40) $$\Psi_1 S_2\left(-\frac{1}{q}\right) = \frac{1}{q-1} + \frac{1}{q-1} \sum_{n=1}^{\infty} \frac{q^{\frac{-(n+1)(n+2)}{2}}}{(q^2+1)(q^3-1)\dots\dots\left(q^{n+1}+(-1)^{n+1}\right)} .$$

Viewing the series on the right of above expression as a Cantor series given by $b_n = q^{\frac{-(n+1)(n+2)}{2}}, a_n = q^{n+1} + (-1)^{n+1}$ for all $n \geq 1$. It can be easily shown that the conditions of Theorem 3.1 are satisfied for every integer $q \geq 2$ and any $n \geq 1$. So the series on the right of (4.40) is irrational for all integers $q \geq 2$.

Since $\Psi_1 S_2\left(\frac{1}{q}\right)$ and $\Psi_1 S_2\left(-\frac{1}{q}\right)$ are irrational for every integer $q \geq 2$, therefore $\Psi_1 S_2(q)$ is irrational for $q = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots$.

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