Stability of Cubic Mappings in Fuzzy Normed Spaces: A Fixed Point Approach

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Abstract

Using the fixed point method, we prove the Ulam-Hyers stability of the cubic functional equation $f(3x+y)+f(3x-y)-3f(x)-3f(x-y) = 12f(x)$ in non-Archimedean fuzzy normed spaces. In this paper we use the concept of stability of cubic mappings in non-Archimedean fuzzy normed space from Renu Chugh et al.

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1. Introduction

In order to construct a fuzzy structure on a linear space, in 1984, Kataras [5] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. At the same year Wu and Fang [17] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. In [11], Biswas defined and studied fuzzy inner product spaces in a linear space. In 1994, Cheng and Mordeso introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [7]. In 2003, Bag and Samanta [15] modified the definition of Cheng and Mordeson [12] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [13] posed the first stability problem. In the next year, Hyers [6] gave an affirmative answer to the equation of Ulam. Hyers's theorem was generalized by Aoki [14] for additive mappings and by Rassias [16] for linear mappings by considering an unbounded Cauchy differences. The concept of the generalized Hyers-Ulam stability was originated form Rassias's paper [16] for the stability of functional equations.

During the last three decades, the theory of non-Archimedean spaces has gained the interest of Physicists for their research, in particular the problems that emerge in quantum physics, p-adic strings and superstrings. Although many results in the classical normal space theory have a non-Archimedean counterpart, their proofs are different and require a rather new kind of intuitions. It may be noted that $|n| \leq 1$ in each valuation field every triangle is isosceles and there may be no unit vector in a non-Archimedean framework is of the special interest. In 1994, P.Gauruta [10] provided a further generalization of Th.M. Rassias's theorem in which he replaced the bound $\varepsilon(\| x \|^p + \| y \|^p)$ by a general control function $\emptyset(x, y)$ for the existence of a unique linear mapping. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to [1, 2, 3, 4, 8, 9].

In this paper, we obtain some results regarding stability of cubic mappings in non-Archimedean fuzzy normed spaces.

2. Preliminaries

Definition 2.1. Let K be a field. A non-Archimedean absolute value on K is a function $|\cdot| : K \to R$ such that for any $a, b \in K$ we have
(i) $|a| \geq 0$ and equality holds if and only if $a = 0$;
(ii) $|ab| = |a||b|$;
(iii) $|a + b| \leq \max\{|a|, |b|\}$
The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $|\cdot|$ is non-trivial, i.e. that there an $a_0 \in K$ such that $|a_0| \notin \{0, 1\}$.

Definition 2.2. Let X be a linear space over a non-Archimedean field K. A function $N : X \times R \to [0, 1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $x, t \in K$
(i) $N(x, c) = 0$ for $c \leq 0$;
(ii) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(iii) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
(iv) $N\big(x + y, \max\{s, t\}\big) \geq \min\{N(x, s), N(y, t)\}$; (v) $\lim_{n\to\infty} N(x, t) = 1$.
The pair (X, N) is called a non-Archimedean fuzzy normed space. Clearly, if (iv) holds then so is
(vi) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$.
A classical vector space over a complex or real field satisfying (i)-(v) is called fuzzy normed space. It is easy to see that (iv) is equivalent to the following condition
(iv) $N(x + y, t) \geq \min\{N(x, t), N(y, t)\}(x, y \in X; t \in R)$.
Example 2.3. Let $(X, \|\cdot\|)$ be a non-Archimedean normed linear space. Then

$$N(x, t) = \left\{ \begin{array}{ll} \frac{t}{t+\|x\|} & t > 0,\ x \in X \\ 0 & t \leq 0,\ x \in X \end{array} \right\}.$$

is a non-Archimedean fuzzy norm on X.
Example 2.4. Let $(X, \|\cdot\|)$ be a non-Archimedean linear space. Then

$$N(x, t) = \left\{ \begin{array}{ll} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{array} \right\}.$$

is a non-Archimedean fuzzy norm on X.
Definition 2.5. Let (X, N) be a non-Archimedean fuzzy normed linear space. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1 \text{ for all} t > 0$$

In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim x_n = x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$. There exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. Due to

$$N(x_{n+p} - x_n, t) \geq \min\{N(x_{n+p} - x_{n+p-1}, t)..., N(x_{n+1} - x_n, t)\}$$

the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exists n_0 such that for all $n \geq n_0$ we have

$$N(x_{n+1} - x_n, t) > 1 - \varepsilon$$

It is easy to see that every convergent sequence in a non-Archimedean fuzzy normed space Cauchy. If each Cauchy sequence is convergent, then non-Archimedean fuzzy normed space is called a non-Archimedean fuzzy Banach space.

3. Main Result

Let X be a linear space, $(Z, N^{'})$ be a non-Archimedean fuzzy normed space. Suppose that an even function $\varphi : X \times X \to Z$ be a function such that for some $0 < \alpha < 9$

$$N^{'}\big(\varphi(3x, 0), t\big) \geq N^{'}\big(\alpha\varphi(x, 0), t\big) \forall x \in X,\ t > 0, f(0) = 0 \tag{3.1}$$

satisfies $\lim_{n\to\infty} N^{'}\big(\varphi(3^n x, 3^n y), 9^n t\big) = 1$ for all $x, y \in X$ and all $t > 0$. Let (Y, N) be a complete non-Archimedean fuzzy space. If a map $f : X \to Y$ define as

$$N\big(f(3x+y) + f(3x-y) - 3f(x) - 3f(x-y) - 12f(x), t\big) \geq N^{'}\big(\varphi(x, y), t\big) \tag{3.2}$$

then there exist a unique cubic mapping $C : X \to Y$ which satisfies (3.1) such that

$$N\big(f(x) - C(x), t\big) \geq N^{'}\Big(\varphi(x, 0), 3(9-\alpha)t\Big). \tag{3.3}$$

Proof. From (3.2) it follows that

$$\begin{aligned}&\min\Big\{t > 0; N\Big(f(3x+y) + f(3x-y) - 3f(x) - 3f(x-y) - 12f(x), t\Big) > k\Big\} \leq \\ &\qquad \min\Big\{t > 0; N^{'}\Big(\varphi(x, y), t\big) > 1 - \lambda\Big\}, \forall x, y \in X, \lambda \in (0, 1)\end{aligned} \tag{3.4}$$

Putting $y = 0$ in (3.4), we get

$$\min\left\{t > 0;\ N\Big(\frac{f(3x)}{9} - f(x), t\Big) > k\right\} \leq \frac{1}{18}\min\Big\{t > 0;\ B^{'}\big(\varphi(x, 0), t\big)\Big\} \forall x \in X \tag{3.5}$$

Replacing x by $3^n x$ in (3.5) and using (3.1), we obtain

$$\min\left\{t>0;\ N\left(\frac{f(3^{n+1}x)}{9^{n+1}}-\frac{f(3^n x)}{9^n}f(x),9^n t\right)>k\right\}\leq\frac{1}{18}\min\left\{t>0;\ N'\left(\varphi(3^n x,0)t\right)\right\}$$

$$\min\left\{t>0;\ N\left(\frac{f(3^{n+1}x)}{9^{n+1}}-\frac{f(3^n x)}{9^n}f(x),t\right)>k\right\}\leq\frac{1}{18\times 9^n}\min\left\{t>0;\ N'\left(\alpha^n\varphi(x,0),t\right)\right\}$$

$$\min\left\{t>0;\ N\left(\frac{f(3^{n+1}x)}{9^{n+1}}-\frac{f(3^n x)}{9^n}f(x),t\right)>k\right\}\leq\frac{\alpha^n}{18\times 9^n}\min\left\{t>0;\ N'\left(\varphi(x,0),t\right)\right\}$$

(3.6)

It follows from

$$f\left(\frac{3^n x}{9^n}\right)-f(x)=\sum_{k=0}^{n-1}\left[\frac{f(3^{k+1}x)}{9^{k-1}}-\frac{f(3^k x)}{9^k}\right]$$

and (3.6) using (iv) that

$$\begin{aligned}\min\left\{t>0;\ N\left(\frac{f(3^n x)}{9^n}-f(x),t\right)>k\right\}&=\min\left\{t>0;\ N\left(\sum_{k=0}^{n-1}\left(\frac{f(3^{k+1}x)}{9^{k-1}}-\frac{f(3^k x)}{9^k}\right),t\right)>k\right\}\\&\leq\sum_{k=0}^{n-1}\min\left\{t>0;\ N\left(\frac{f(3^{k+1}x)}{9^{k+1}}-\frac{f(3^k x)}{9^k},t\right)>k\right\}\\&\leq\sum_{k=0}^{n-1}\frac{1}{18\times 9^k}\min\left\{t>0;\ N'\left(\varphi(3^k x,0),t\right)>k\right\}\\&\leq\sum_{k=0}^{n-1}\frac{\alpha^k}{18\times 9^k}\min\left\{t>0;\ N'\left(\varphi(x,0),t\right)>k\right\}\end{aligned}\tag{3.7}$$

Replacing x with $3^m x$ in (3.7), we have

$$\begin{aligned}\min\left\{t>0;\ N\left(\frac{f(3^{n+m}x)}{9^{n+m}}-\frac{f(3^m x)}{9^m},t\right)>k\right\}&\leq\sum_{k=0}^{n-1}\frac{\alpha^k}{18\times 9^{k+m}}\min\left\{t>0;\ N'\left(\varphi(3^m x,0),t\right)>k\right\}\\&\leq\sum_{k=0}^{n-1}\frac{\alpha^{k+m}}{18\times 9^{k+m}}\min\left\{t>0;\ N'\left(\varphi(x,0),t\right)>k\right\}\\&\leq\sum_{k=m}^{m+n-1}\frac{\alpha^k}{18\times 9^k}\min\left\{t>0;\ N'\left(\varphi(x,0),t\right)>k\right\}\\&=\frac{1}{18}\min\left\{t>0;\ N'\left(\varphi(x,0),t\right)>k\right\}\sum_{k=m}^{m+n-1}\left(\frac{\alpha}{9}\right)^k\end{aligned}\tag{3.8}$$

Then $\left(\frac{f(3^n x)}{9^n}\right)$ is a Cauchy sequence in (Y, N). Since (Y, N) is a complete non-Archimedean fuzzy normed space, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.8) then we obtain

(3.9)
$$\min\left\{t > 0;\ N\left(\frac{f(3^n x)}{9^n} - f(x), t\right) > k\right\} \leq \frac{1}{18} \min\left\{t > 0;\ N'\big(\varphi(x,0), t\big) > k\right\} \sum_{k=0}^{n-1} \left(\frac{\alpha}{9}\right)^k$$

and so

$$\begin{aligned} \min\left\{t > 0;\ N\big(C(x) - f(x), t\big) > k\right\} &\leq \min\left\{t > 0;\ N\left(C(x) - \frac{f(3^n x)}{9^n}, t\right) > k\right\} \\ &\quad + \min\left\{t > 0;\ N\left(\frac{f(3^n x)}{9^n} - f(x), t\right) > k\right\} \\ &\leq \min\left\{t > 0;\ N\left(C(x) - \frac{f(3^n x)}{9^n}, t\right) > k\right\} \\ &\quad + \frac{1}{18} \min\left\{t > 0;\ N'\big(\varphi(x,0), t\big) > k\right\} \sum_{k=0}^{n-1} \left(\frac{\alpha}{9}\right)^k \end{aligned} \tag{3.10}$$

Taking the limit as $n \to \infty$ and using (3.10) we get

$$\min\left\{t > 0;\ N\big(C(x) - f(x), t\big) > k\right\} \leq \frac{1}{18 - 3\alpha} \min\left\{t > 0;\ N'\big(\varphi(x,0), t\big) > k\right\} \tag{3.11}$$

i.e.

$$\min\left\{t > 0;\ N\big(C(x) - f(x), t\big) > k\right\} \leq \min\left\{t > 0;\ N'\big(\varphi(x,0), 3t(9-\alpha)\big) > k\right\} \tag{3.12}$$

$$N\big(C(x) - f(x), t\big) \geq N'\big(\varphi(x,0), 3t(9-\alpha)\big) \tag{3.13}$$

then replacing x and y by $3^n x$ and $3^n y$ in (3.2), we get

$$\begin{aligned} &N\left(\frac{f\big(3^n(3x+y)\big)}{9^n} + \frac{f\big(3^n(3x-y)\big)}{9^n} - \frac{3^{n+1} f(x)}{9^n} - \frac{3^{n+1} f(x-y)}{9^n} - \frac{12 f(3^n x)}{9^n}, t\right) \geq \\ &\qquad N'\big(\varphi(3^n x, 3^n y), 9^n t\big),\ \forall\, x, y \in X,\ t > 0 \end{aligned} \tag{3.14}$$

Since $\lim_{n\to\infty} N'\big(f(3^n x, 3^n y), 9^n t\big) = 1$
Thus C satisfies

$$f(3x+y) + f(3x-y) - 3f(x) - 3f(x-y) = 12 f(x).$$

To prove the uniqueness of cubic mapping C assume that there exist a cubic mapping $D : X \to Y$ which satisfies (3.3). Fix $x \in X$, clearly

$$C(3^n x) = 9^n\big(C(x)\big) \text{ and } D(3^n x) = 9^n D(x), \ \forall\, n \in N$$

From (3.3), using (iv)

$$N\big(C(x) - D(x), t\big) = \lim_{n\to\infty} N\left(\frac{C(3^n x)}{9^n} - \frac{D(3^n x)}{9^n}, t\right)$$

$$N\left(\frac{C(3^n x)}{9^n} - \frac{D(3^n x)}{9^n}, t\right) \geq \min N'\left(\varphi(3^n x, 0), 9^n 3(9-\alpha)t\right)$$

$$\geq N\left(\varphi(x,0), \frac{9^n 3(9-\alpha)t}{\alpha^n}\right) \tag{3.15}$$

Since $\lim_{n\to\infty}\left(\frac{9^n 3(9-\alpha)t}{\alpha^n}\right) = \infty$,
we get

$$\lim_{n\to\infty} N'\left(\varphi(x,0), \frac{9^n 3(9-\alpha)t}{\alpha^n}\right) = 1.$$

Therefore it implies

$$N\big(C(x) - D(x), t\big) = 1 \ \text{ for all } \ t > 0$$

and so $C(x) = D(x)$.

Corollary 3.1. Let X be a linear space (Z, N') be a non-Archimedean fuzzy normed linear space, (Y, N) be complete non-Archimedean fuzzy normed space. Let p, q be non-negative real numbers and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that

$$N\Big(f(3x+y) + f(3x-y) - 3f(x) - 3f(x-y) - 12f(x), t\Big)$$
$$\leq \varepsilon\, N'\Big(\| x \|^{2p} + \| y \|^{2p} + \| x \|^{p} \| y \|^{q}, t\Big)\ \forall\, x, y \in X,\ t > 0, f(0) = 0 \text{ and } p, q < 3,$$

then there exist a unique cubic mapping $C : X \to Y$ such that

$$N\big(f(x) - C(x), t\big) \geq N'\Big(\| x \|^{p} z_0, 3(9 - 3^p), t\Big)\ \forall\, x \in X,\ t > 0.$$

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