Coefficient inequality for a subclass of Starlike function generated by symmetric points.

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Abstract

In this investigation, we obtained the fekete szegö functional and the sharp upper bounds for the function belonging to certain subclass of starlike function generated by symmetric points by using subordination.

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1 Introduction

$G(\mathcal{U})$ denote the class of functions which are analytic in open unit disk $\mathcal{U} = \{z; |z| < 1\}$. Let $\mathbf{A}$ be the class of all those functions from $G(\mathcal{U})$ which are normalized by the conditions $f(0) = 0, f'(0) = 1$ having the taylor expansion $f(z) = z + a_2z^2 + a_3z^3 + ...$ which are analytic in $\mathcal{U} = \{z; |z| < 1\}$. Class $\mathbf{S}$ consists of all the functions from class $\mathbf{A}$ which are univalent in $\mathcal{U}$. Let $\mathbf{P}$ denote the class of function of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ analytic in the $\mathcal{U}$ and satisfying the condition $Re(p(z)) > 0$ above stated function is known as Caratheodory function or function with positive real part. For two analytic function $f, g \in \mathbf{A}$, subordination for $f(z)$ and $g(z)$ written as $f \prec g$, if there exists an analytic function $w(z)$ such that $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$ i.e $f(z) = g(w(z))$. Also if g is univalent in $\mathcal{U}$ then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. This concept was introduced by Lindelof [3]. Miller et.al.[5] proved a very important following lemma for $w(z)$

Lemma 1. *If* $w(z) = \sum_{k=1}^{\infty} c_k z^k$, *then* $|c_1| \leq 1$ *and* $|c_2| \leq 1 - |c_1|^2$

For the taylor expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ from class $\mathbf{A}$ relationship between the coefficients of f has been investigated and still continuing for different classes of analytic function. Relation between a_2 and a_3 under the perimeter μ i.e. $|a_3 - \mu a_2^2|$ was obtained by Fekete and Szegö [2] with the help of Löwner's perametric method[4]. The well known result due to them states that if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is analytic and univalent in $\mathcal{U}$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & if \quad \mu \leq 0, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) & if \quad 0 \leq \mu \leq 1, \\ 4\mu - 3 & if \quad \mu \geq 1. \end{cases}$$

the result is sharp in the sense that for each μ there is a function in the class under consideration for which equality holds. The coefficient functional $a_3 - \mu a_2^2 = \Lambda_\mu(f)$ plays an important role in the function theory. The problem of maximizing the absolute value of the functional $\Lambda_\mu(f)$ is called the Fekete Szegö problem.

Lemma 2. ***Caratheodory Lemma*** *[1] If $p(z) \in \mathbf{P}$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ then*

$$|p_n| \leq 2, \qquad n \in \mathbb{N} \tag{1.1}$$

equality holds for the analytic and univalent function $p(z)$ defined as $p(z) = \frac{1+z}{1-z}$ **class** $S^*(ff')$ consists of all the functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathbf{A}$ and satisfying the condition

$$Re\left(\frac{2f(z)}{z(f'(z) + f'(-z))}\right) > 0, z \in \mathcal{U}$$

$$i.e. Re\left(\frac{2f(z)}{z(f'(z) + f'(-z))}\right) \prec \frac{1+z}{1-z} \tag{1.2}$$

and subclasses of above class are as

$$S^*(ff', \delta) = \left(f \in \mathbf{A}, \frac{2f(z)}{z(f'(z) + f'(-z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \delta > 0\right) \tag{1.3}$$

$$S^*(ff', A, B) = \left(f \in \mathbf{A}, \frac{2f(z)}{z(f'(z) + f'(-z))} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1\right) \tag{1.4}$$

(1.5)
$$S^*(ff', A, B, \delta) = \left(f \in \mathbf{A}, \frac{2f(z)}{z(f'(z) + f'(-z))} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}, \delta > 0, -1 \leq B < A \leq 1\right)$$

2 Fekete-Szegö problem

Theorem 2.1. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathcal{U}$ belongs to $S^*(ff')$ then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -1 - 4\mu & if \quad (\mu \leq \frac{-1}{2}) \\ 1 & if \quad (\frac{-1}{2} \leq \mu \leq 0) \\ 1 + 4\mu & if \quad (\mu \geq 0) \end{cases} \tag{2.1}$$

the result is sharp.

Proof For $f(z) \in S^*(ff')$, by definition (1.2) by using principle of subordination then there exists $w(z) = c_1 z + c_2 z^2 + ...$ bounded analytic function such that

$$Re\left(\frac{2f(z)}{z(f'(z) + f'(-z))}\right) = \frac{1+w(z)}{1-w(z)} \tag{2.2}$$

expanding equation (2.2), we get

$$1 + a_2z + a_3z^2 + ... = 1 + 2c_1z + (3a_3 + 2(c_2 + c_1^2))z^2 + ...$$

we obtain $a_2 = 2c_1$ and $a_3 = -(c_2 + c_1^2)$ therefore we have

$$|a_3 - \mu a_2^2| \leq |c_2| + |-1 - 4\mu||c_1|^2$$

$$\leq 1 + (|-1 - 4\mu| - 1)|c_1|^2 \tag{2.3}$$

Case 1 when $\mu \leq \frac{-1}{4}$ in above (2.3) equation,

$$|a_3 - \mu a_2^2| \leq 1 + (-2 - 4\mu)|c_1|^2 \tag{2.4}$$

Subcase 1(a) when $\mu \leq \frac{-1}{2}$ in above (2.4) equation,

$$|a_3 - \mu a_2^2| \leq -1 - 4\mu \tag{2.5}$$

Subcase 1(b) when $\mu \geq \frac{-1}{2}$ in above (2.4) equation,

$$|a_3 - \mu a_2^2| \leq 1 \tag{2.6}$$

Case 2 when $\mu \geq \frac{-1}{4}$, in above (2.3)equation,

$$|a_3 - \mu a_2^2| \leq 1 + 4\mu|c_1|^2 \tag{2.7}$$

Subcase 2(a) when $\mu \leq 0$ in above (2.7) equation,

$$|a_3 - \mu a_2^2| \leq 1 \tag{2.8}$$

Subcase 2(b) when $\mu \geq 0$ in above (2.7) equation,

$$|a_3 - \mu a_2^2| \leq 1 + 4\mu \tag{2.9}$$

By combining (2.5), (2.6), (2.8), (2.9), we get our desired result. The extremal function for 1st and 3rd inequality is $z(1+3z)^{\frac{2}{3}}$ and extremal function for 2nd inequality is $z(1+z^2)$

Theorem 2.2. *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathcal{U}$ *belongs to* $S^*(ff', \delta)$ *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\delta(1 + 4\delta\mu) & if \quad (\mu \leq \frac{1}{2\delta}) \\ \delta & if \quad (\frac{1}{2\delta} \leq \mu \leq 0) \\ -\delta(1 + 4\delta\mu) & if \quad (\mu \geq 0) \end{cases} \tag{2.10}$$

the result is sharp.

Proof For $f(z) \in S^*(ff',\delta)$, by definition (1.3) by using principle of subordination then there exists $w(z) = c_1 z + c_2 z^2 + ...$ bounded analytic function such that

$$Re\left(\frac{2f(z)}{z(f'(z)+f'(-z))}\right) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta} \tag{2.11}$$

expanding equation (2.11), we get

$$1 + a_2 z + a_3 z^2 + ... = 1 + 2\delta c_1 z + (3a_3 + 2\delta(c_2 + c_1^2))z^2 + ...$$

we obtain $a_2 = 2\delta c_1$ and $a_3 = -\delta(c_2 + c_1^2)$ therefore we have

$$|a_3 - \mu a_2^2| \leq \delta|c_2| + \delta|-1-4\delta\mu||c_1|^2$$

$$\leq \delta + \delta(|-1-4\delta\mu|-1)|c_1|^2 \tag{2.12}$$

Case 1 when $\mu \leq \frac{-1}{4\delta}$ in above (2.12) equation,

$$|a_3 - \mu a_2^2| \leq \delta + (-2-4\delta\mu)|c_1|^2 \tag{2.13}$$

Subcase 1(a) when $\mu \leq \frac{-1}{2\delta}$ in above (2.13) equation,

$$|a_3 - \mu a_2^2| \leq -\delta(1+4\delta\mu) \tag{2.14}$$

Subcase 1(b) when $\mu \geq \frac{-1}{2\delta}$ in above (2.13) equation,

$$|a_3 - \mu a_2^2| \leq \delta \tag{2.15}$$

Case 2 when $\mu \geq \frac{-1}{4\delta}$, in above (2.12)equation,

$$|a_3 - \mu a_2^2| \leq \delta + 4\delta^2\mu|c_1|^2 \tag{2.16}$$

Subcase 2(a) when $\mu \leq 0$ in above (2.16) equation,

$$|a_3 - \mu a_2^2| \leq \delta \tag{2.17}$$

Subcase 2(b) when $\mu \geq 0$ in above (2.17) equation,

$$|a_3 - \mu a_2^2| \leq \delta(1+4\delta\mu) \tag{2.18}$$

By combining (2.14), (2.15), (2.17), (2.18), we get our desired result. The extremal function for 1st and 3rd inequality is $z(1+(1+2\delta)^{\frac{2\delta}{1+2\delta}})$ and extremal function for 2nd inequality is $z(1+\delta z^2)$

Theorem 2.3. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathcal{U}$ belongs to $S^*(ff', A, B)$ then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} (A-B)\left(\frac{B}{2} - (A-B)\mu\right) & if \quad (\mu \leq \frac{B-1}{2(A-B)} \\ \frac{(A-B}{2} & if \quad (\frac{B-1}{2(A-B)} \leq \mu \leq \frac{-(B+1)}{2(A-B)}) \\ -(A-B)\left(\frac{B}{2} - (A-B)\mu\right) & if \quad (\mu \geq \frac{-(B+1)}{2(A-B)}) \end{cases} \tag{2.19}$$

the result is sharp.

Proof For $f(z) \in S^*(ff', A, B)$, by definition (1.4)

$$Re\left(\frac{2f(z)}{z(f'(z) + f'(-z))}\right) = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2.20}$$

expanding equation (2.20), we get

$$1 + a_2 z + a_3 z^2 + ... = 1 + (A-B)c_1 z + (3a_3 + (A-B)c_2 + (B^2 - AB)c_1^2)z^2 + ...$$

we obtain $a_2 = (A-B)c_1$ and $a_3 = -\frac{1}{2}((A-B)c_2 + (B^2 - AB)c_1^2)$
therefore we have

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{2}|c_2| + (A-B)\left|-\frac{B}{2} - (A-B)\mu\right| |c_1|^2$$

$$\leq \frac{(A-B)}{2} + (A-B)\left(\left|\frac{B}{2} - (A-B)\mu\right| - \frac{1}{2}\right)|c_1|^2 \tag{2.21}$$

Case 1 when $\mu \leq \frac{B}{2(A-B)}$ in above (2.21) equation,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{2} + (A-B)\left(\frac{B-1}{2} - (A-B)\mu\right)|c_1|^2 \tag{2.22}$$

Subcase 1(a) when $\mu \leq \frac{B-1}{2(A-B)}$ in above (2.22) equation,

$$|a_3 - \mu a_2^2| \leq (A-B)\left(\frac{B}{2} - (A-B)\mu\right) \tag{2.23}$$

Subcase 1(b) when $\mu \geq \frac{B-1}{2(A-B)}$ in above (2.22) equation,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B}{2} \tag{2.24}$$

Case 2 when $\mu \geq \frac{B}{2(A-B)}$, in above (2.21)equation,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{2} + (A-B)\left(-\frac{B-1}{2} + (A-B)\mu\right)|c_1|^2 \tag{2.25}$$

Subcase 2(a) when $\mu \leq \frac{-(B+1)}{2(A-B)}$ in above (2.25) equation,

$$|a_3 - \mu a_2^2| \leq \frac{(A-B}{2} \tag{2.26}$$

Subcase 2(b) when $\mu \geq \frac{-(B+1)}{2(A-B)}$ in above (2.25) equation,

$$|a_3 - \mu a_2^2| \leq -(A-B)\left(\frac{B}{2} - (A-B)\mu\right) \tag{2.27}$$

By combining (2.23), (2.24), (2.26), (2.27), we get our required result. The extremal function for 1st and 3rd inequality is $z(1+(A-2B)z)^{\frac{(A-B)}{(A-2B)}}$ and extremal function for 2nd inequality is $z(1+\frac{A-B}{2}z^2)$

Theorem 2.4. *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathcal{U}$ *belongs to* $S^*(ff', \delta, A, B)$ *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta(A-B)}{2}\left(B - \frac{(\delta-1)(A-B)}{2} - 2\delta(A-B)\mu\right) & if \quad (\mu \leq \lambda_1) \\ \frac{\delta(A-B}{2} & if \quad (\lambda_1 \leq \mu \leq \lambda_2) \\ \frac{\delta(A-B)}{2}\left(-B + \frac{(\delta-1)(A-B)}{2} + 2\delta(A-B)\mu\right) & if \quad (\mu \geq \lambda_2) \end{cases} \tag{2.28}$$

where $\lambda_1 = \frac{(B-1)-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ *and* $\lambda_2 = \frac{(B-1)-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ *and the result is sharp.*

Proof For $f(z) \in S^*(ff', \delta, A, B)$, by definition (1.5)

$$Re\left(\frac{2f(z)}{z(f'(z)+f'(-z))}\right) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \tag{2.29}$$

expanding equation (2.29), we get $1 + a_2 z + a_3 z^2 + ...$

$$= 1 + \delta(A-B)c_1 z + (3a_3 + \delta(A-B)c_2 + \delta(B^2 - AB)c_1^2 + \frac{\delta(\delta-1)(A-B)^2}{2})z^2 + ...$$

we obtain $a_2 = \delta(A-B)c_1$ and

$$a_3 = -\frac{1}{2}(\delta(A-B)c_2 + \delta(B^2 - AB)c_1^2 + \frac{\delta(\delta-1)(A-B)^2}{2})$$

$$\therefore |a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{2}|c_2| +$$

$$\frac{\delta(A-B)}{2}\left|B - \frac{(\delta-1)(A-B)}{2} - 2\delta(A-B)\mu\right| |c_1|^2$$

(2.30)
$$\leq \frac{\delta(A-B)}{2} + \frac{\delta(A-B)}{2}\left(\left|B - \frac{(\delta-1)(A-B)}{2} - 2\delta(A-B)\mu\right| - 1\right)|c_1|^2$$

Case 1 when $\mu \leq \frac{B-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ in above (2.30) equation,

(2.31)
$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{2} + \frac{\delta(A-B)}{2}\left((B-1) - \frac{(\delta-1)(A-B)}{2} - 2\delta(A-B)\mu\right)|c_1|^2$$

Subcase 1(a) when $\mu \leq \frac{(B-1)-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ in above (2.31) equation,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{2}\left(B - \frac{(\delta-1)(A-B)}{2} - 2\delta(A-B)\mu\right) \tag{2.32}$$

Subcase 1(b) when $\mu \geq \frac{(B-1)-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ in above (2.31) equation,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B}{2} \tag{2.33}$$

Case 2 when $\mu \geq \frac{B-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$, in above (2.30)equation,

(2.34)
$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{2} + \frac{\delta(A-B)}{2}\left(-(B+1) + \frac{(\delta-1)(A-B)}{2} + 2\delta(A-B)\mu\right)|c_1|^2$$

Subcase 2(a) when $\mu \leq \frac{(B+1)-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ in above (2.34) equation,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B}{2} \tag{2.35}$$

Subcase 2(b) when $\mu \geq \frac{(B+1)-\frac{(\delta-1)(A-B)}{2}}{2\delta(A-B)}$ in above (2.34) equation,

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{2}\left(-B + \frac{(\delta-1)(A-B)}{2} + 2\delta(A-B)\mu\right) \tag{2.36}$$

hence by combining (2.23), (2.24), (2.26), (2.27), we get our desired result. The extremal function for 1st and 3rd inequality is

$$z(1 + \left(\frac{(A-B)(3\delta-1)}{2} - B\right)z)^{\frac{2\delta(A-B)}{(A-B)(3\delta-1)-2B}}$$

and extremal function for 2nd inequality is $z(1 + \frac{\delta(A-B)}{2}z^2)$

Corollary 1. *If $\delta = 1$ in (2.2) we get result of (2.1).*
If $A = 1, B = -1$ in (2.3) we get result of (2.1).
If $\delta = 1$ in (2.4) we get result of (2.3).
If $A = 1, B = -1$ in (2.4) we get result of (2.2).
and if $\delta = 1, A = 1, B = -1$ in (2.4) we get result of (2.1).

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