Coupled Fixed Point On Modular Space

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Abstract

This paper is based on existance of coupled fixed point on Modular space or weak normed linear spaces. In this space we have generalized Branciari integral contraction to get coupled fixed point.

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1 Introduction

In 1950 Nakno[8] introduced modular space by generalizing normed linear space. In 1959 Musielak[7] and Orlic redefined it. Latter some authors extended clasical Banach contraction principle in this space to get fixed point results. The present paper we have generalized Branciari integral contraction in this space to get coupled fixed point.

Theorem 1.1. *(Branciari integral contraction) Let (X, d) be a complete metric space, f be a map on X satisfying,*

$$\int_0^{d(f(x),f(y))} \phi(t)dt \leq k \int_0^{d(x,y)} \phi(t)dt, \tag{1.1}$$

for all $x, y \in X$, for $k \in (0, 1)$, then f has a fixed point in X.

The function $\phi(t)$ used by Branciari integral contraction has the following properties:
(i) $\phi(t)$ is Lebesgue integrable and summable on each compact subset of $\mathbb{R}$,
(ii) $\int_0^r \phi(t)dt > 0$, for each $r > 0$,
(iii) If (r_n) be a +ve sequence with $lim\ r_n = a$, then $lim \int_0^{r_n} \phi(t)dt = \int_0^a \phi(t)dt$,
(iv) If (r_n) be a +ve sequence with $lim\ r_n = 0$, then $lim \int_0^{r_n} \phi(t)dt = 0$.

Definition 1 (8). *Let X be a arbitrary vector space, R be a map on X to $[0, \infty]$ satisfying:*
(i) $R(x) = 0$ iff $x = 0$,

(ii) $R(ax) = R(x)$, *for all* $x \in X$, *any scalar* a *with* $|a| = 1$,
(iii) $R(ax + by) \leq R(x) + R(y)$, *for all* $x, y \in X$, *for all scalar* a, b *with* $a + b = 1$ *and* $a, b \geq 0$.
Then R *is called modular and* X_R *is called modular space.*

Theorem 1.2 (8). *Let* X_R *be a modular space and* (x_n) *be a sequence in* X_R, *then we have the following:*
(i) (x_n) *converges to* x, *if* $R(x_n - x)$ *converges to* 0 *as* n *tends to* ∞.
(ii) (x_n) *is Cauchy, if* $R(x_n - x_m)$ *converges to* 0 *as* n *tends to* ∞,
(iii) X_R *is complete, if every Cauchy sequence is convergent in* X_R.
(iv) B *a subset of* X_R, *is called closed, if for any sequence* (x_n) *converging to* x, $x \in B$.
(v) B *a subset of* X_R, *is called bounded, if for all* $x, y \in B$ *sup* $R(x - y) < \infty$.
(vi) R *is said to have Fatou property, if* $R(x - y) \leq lim \inf R(x_n - y_n)$.
(vii) R *is called to satisfy* $\triangle_2$ *condition, if* $R(2x_n)$ *converges to* 0 *whenever* $R(x_n)$ *converges to* 0.

2 Main result

Theorem 2.1. *Let* X_R *be a modular space satisfying* $\triangle_2$ *condition and* c, l, d *be positive numbers with* $\frac{l}{c} + \frac{1}{d} = 1$ *and* $\phi(t)$ *be integrand function in Branciari integral contraction. Suppose* f *be a map from* $X_R \times X_R$ *to* X_R *satisfying the following properties:*

$$\int_0^{R(c(f(x,y)-(f(u,v))} \phi(t)dt \leq a \int_0^{R(l(x-u))} \phi(t)dt$$

and

$$\int_0^{R(c(f(y,x)-(f(v,u))} \phi(t)dt \leq a \int_0^{R(l(y-v))} \phi(t)dt, \text{ for all } x, y, u, v \in X \text{ and some } a \in (0,1),$$

then f *has a coupled fixed point in* X_R.

Proof: Let (x_0, y_0) be a arbitrary point of $X_R \times X_R$.
Define $x_{n+1} = f(x_n, y_n)$ and $y_{n+1} = f(y_n, x_n)$.
Now

$$\int_0^{R(l(x_{n+1}-x_n))} \phi(t)dt = \int_0^{R(\frac{l}{c}c(x_{n+1}-x_n))} \phi(t)dt$$
$$\leq \int_0^{R(c(f(x_n,y_n)-f(x_{n-1},y_{n-1})))} \phi(t)dt$$
$$\leq a \int_0^{R(l(f(x_{n-1},y_{n-1})-f(x_{n-2},y_{n-2})))} \phi(t)dt$$
$$= a \int_0^{R(\frac{cl}{c}(f(x_{n-1},y_{n-1})-f(x_{n-2},y_{n-2}))}) \phi(t)dt$$
$$\leq \int_0^{R(c(f(x_{n-1},y_{n-1})-f(x_{n-2},y_{n-2})))} \phi(t)dt \leq ....a^n A,$$

where

$$A = \int_0^{R(l((x_1,y_1)-(x_0,y_0)))} \phi(t)dt.$$

Taking limt as n tends to ∞ we have

$$\begin{aligned} &lim\ a^n A = 0 \\ &\Rightarrow lim \int_0^{R(l(x_{n+1}-x_n))} \phi(t)dt = 0 \\ &\Rightarrow lim R(l(x_{n+1} - x_n)) = 0. \end{aligned}$$

Now we have to show that (x_{n+1}) is a Cauchy sequence with respect to R.
Suppose (x_{n+1}) is not a Cauchy sequence with respect to modular metric, then there exists $\epsilon > 0$ such that we have two sequences of integer $m(s)$ and $n(s)$ such that $R(l(x_{m(s)} - x_{n(s)})) \geq \epsilon$ with $n(s) \geq m(s) \geq s$ and $n(s)$ should be least positive integer for which the above inequality happens otherwise

$$R(c(x_{m(s)} - x_{n(s)-1})) \leq \epsilon.$$

Now

$$\begin{aligned} &\int_0^{R(l(x_{m(s)+1}-x_{n(s)+1}))} \phi(t)dt \\ &= \int_0^{R(l(f(x_{m(s)},y_{m(s)})-f(x_{n(s)},x_{n(s)}))} \phi(t)dt \\ &\leq a \int_0^{R(l(x_{m(s)-1}-x_{n(s)-1}))} \phi(t)dt. \end{aligned}$$

Now

$$\begin{aligned} &R(l(x_{m(s)-1} - x_{n(s)-1})) \\ &= R(l(x_{m(s)-1} - x_{m(s)} + x_{m(s)} - x_{n(s)-1})) \\ &= R(\frac{dl}{d}(x_{m(s)-1} - x_{m(s)}) + \frac{cl}{c}(x_{m(s)} - x_{n(s)-1})) \\ &\leq R(dl(x_{m(s)-1} - x_{m(s)})) + R(c(x_{m(s)} - x_{n(s)-1})). \end{aligned}$$

By using $\triangle_2$ condition and limit as s tends to ∞, we get

$$\begin{aligned} &\lim R(dl(x_{m(s)-1} - x_{m(s)})) = 0. \\ &\Rightarrow lim R(l(x_{m(s)-1} - x_{n(s)-1})) \leq \epsilon. \end{aligned}$$

Now

$$\begin{aligned}
&\lim \int_0^{R(l(x_{m(s)-1}-x_{n(s)-1}))} \phi(t)dt \\
&\leq \int_0^{R(\epsilon)} \phi(t)dt \\
&\leq \int_0^{R(l(x_{m(s)}-x_{n(s)}))} \phi(t)dt \\
&\leq a \int_0^{R(l(x_{m(s)-1}-x_{n(s)-1}))} \phi(t)dt \\
&\Rightarrow \int_0^{R(\epsilon)} \phi(t)dt \leq a \int_0^{R(\epsilon)} \phi(t)dt \\
&\Rightarrow 1 \leq a,
\end{aligned}$$

which is a contradiction. Hence (x_n) is a Cauchy sequence. Since X is complete we have a point $x \in X$ such that (x_n) converges to x.
Now

$$\begin{aligned}
&lim \int_0^{R(l(f(x_{n+1},y_{n+1})-f(x,y))} \phi(t)dt \\
&= lim \int_0^{R(cl/c(f(x_{n+1},y_{n+1})-f(x,y))} \phi(t)dt \\
&\leq lim\ a \int_0^{R(c(x_n-x))} \phi(t)dt) \\
&= lim \int_0^{R(c(x_n-x))} \phi(t)dt = 0,
\end{aligned}$$

$\Rightarrow (x_n)$ converges to $f(x,y)$.
Since the limit is unique $f(x,y) = x$. In a similar way we can prove that $f(y,x) = y$.
Hence (x,y) is a coupled fixed point of f.
Our next result is following.

Theorem 2.2. *Let X_R be a modular space satisfying $\triangle_2$ condition and c, l, d be positive numbers such that $\frac{l}{c}+\frac{1}{d} = 1$ and $\phi(t)$ be integrand function of Branciari integral contraction. Suppose f be a map from $X_R \times X_R$ to X_R satisfying the following properties:*

$$\int_0^{R(c(T(x,y)-(T(u,v))} \phi(t)dt \leq a \int_0^{R(l(x-u))} \phi(t)dt + b \int_0^{R(l(y-v))} \phi(t)dt$$

with

$$\lim[\binom{n}{0}a^n k + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}b^n k] = 0,$$

where

$$k = max\{\int_0^{R(l(x_1-x_0)}\phi(t)dt, \int_0^{R(l(y_1-y_0)}\phi(t)dt\}$$

defined in the proof of Theorem 2.1, then f has a coupled fixed point.

Proof: Let (x_n) and (y_n) be two sequences defined in Theorem 2.1
Now we claim that

$$\int_0^{R(l(x_{n+1}-x_n)}\phi(t)dt \leq \binom{n}{0}a^nk + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}b^n$$

and

$$\int_0^{R(l(y_{n+1}-y_n)}\phi(t)dt \leq \binom{n}{0}a^nk + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}b^n.$$

We have to show it by method of induction. Suppose p_n is a statement which stands for

$$\int_0^{R(c(x_{n+1}-x_n)}\phi(t)dt \leq \binom{n}{0}a^nk + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}b^n$$

and

$$\int_0^{R(l(y_{n+1}-y_n)}\phi(t)dt \leq \binom{n}{0}a^nk + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}b^n.$$

Suppose $n = 1$ then

$$\begin{aligned}&\int_0^{R(c(x_2-x_1)}\phi(t)dt\\ &\leq a\int_0^{R(l(x_1-x_0)}\phi(t)dt + b\int_0^{R(l(y_1-y_0)}\phi(t)dt\\ &\leq ak + bk.\end{aligned}$$

So it is true for $n = 1$.
Suppose p_n is true for some n, we wish to show that it is true for $n + 1$.
Now

$$\begin{aligned}&\int_0^{R(c(x_{n+2}-x_{n+1})}\phi(t)dt\\ &\leq a\int_0^{R(l(x_{n+1}-x_n)}\phi(t)dt + b\int_0^{R(l(y_{n+1}-y_n)}\phi(t)dt\end{aligned}$$

$$= a\int_0^{R(cl/c(x_{n+1}-x_n)}\phi(t)dt + b\int_0^{R(cl/c(y_{n+1}-y_n)}\phi(t)dt$$

$$\leq a\int_0^{R(c(x_n-x_{n-1})}\phi(t)dt + b\int_0^{R(c(y_n-y_{n-1})}\phi(t)dt$$

$$\leq a(\binom{n}{0}a^nk + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}kb^n) + b(\binom{n}{0}a^nk + \binom{n}{1}a^{n-1}bk + \cdots + \binom{n}{n}kb^n)$$

$$= \binom{n+1}{0}a^{n+1}k + \binom{n+1}{1}a^nbk + \cdots + \binom{n+1}{n+1}b^{n+1}k.$$

Similarly we can show

$$\int_0^{R(l(y_{n+2}-y_{n+1})}\phi(t)dt$$
$$\leq \int_0^{R(c\frac{l}{c}(y_{n+2}-y_{n+1})}\phi(t)dt$$
$$\leq \binom{n+1}{0}a^{n+1}k + \binom{n+1}{1}a^nbk + \cdots + \binom{n+1}{n+1}kb^{n+1}.$$

Hence it is true for $n+1$. Now take limit as n tends to ∞ in above inequality, we get

$$lim\int_0^{R(l(x_{n+1}-x_n)}\phi(t)dt = 0$$
$$\Rightarrow limR(l(x_{n+1}-x_n) = 0.$$

Now, we claim that (x_n) is a Cauchy sequence. Suppose it is not cauchy, then there exists $\epsilon > 0$ such that we have two sequences $m(s)$ and $n(s)$ such that

$$R(l(x_{m(s)} - x_{n(s)})) \geq \epsilon$$

with
$n(s) \geq m(s) \geq s$ and $n(s)$ be the least +ve integer that the above inequality happens otherwise

$$R(c(x_{m(s)} - x_{n(s)-1})) \leq \epsilon.$$

Now

$$R(l(x_{m(s)-1} - x_{n(s)-1}))$$
$$= R(l(x_{m(s)-1} - x_{m(s)} + x_{m(s)} - x_{n(s)-1}))$$
$$= R(\frac{dl}{d}(x_{m(s)-1} - x_{m(s)}) + \frac{cl}{c}(x_{m(s)} - x_{n(s)-1}))$$
$$\leq R(dl(x_{m(s)-1} - x_{m(s)})) + R(c(x_{m(s)} - x_{n(s)-1})).$$

Applying $\triangle_2$ condition and limit as s tends to ∞, we get

$$limR(dl(x_{m(s)-1} - x_{m(s)})) = 0.$$

Then we have

$$limR(l(x_{m(s)-1} - x_{n(s)-1})) \leq \epsilon.$$

Similarly we can show

$$R(l(y_{m(s)-1} - y_{n(s)-1})) \leq \epsilon.$$

Now

$$\begin{aligned}
&\int_0^{R(l(x_{m(s)-1}-x_{n(s)-1}))} \phi(t)dt \\
&\leq \int_0^{R(\epsilon)} \phi(t)dt \\
&\leq \int_0^{R(l(x_{m(s)}-x_{n(s)}))} \phi(t)dt \\
&= \int_0^{R(\frac{lc}{c}(x_{m(s)}-x_{n(s)}))} \phi(t)dt \\
&= \int_0^{R(c(\frac{l}{c})(x_{m(s)}-x_{n(s)}))} \phi(t)dt \\
&\leq \int_0^{R(c(x_{m(s)}-x_{n(s)}))} \phi(t)dt \\
&\leq a\int_0^{R(l(x_{m(s)-1}-x_{n(s)-1}))} \phi(t)dt + b\int_0^{R(l(y_{m(s)-1}-y_{n(s)-1}))} \phi(t)dt \\
&\Rightarrow \int_0^{R(\epsilon)} \phi(t)dt \leq a\int_0^{R(\epsilon)} \phi(t)dt \leq a\int_0^{R(\epsilon)} \phi(t) + b\int_0^{R(\epsilon)} \phi(t) \\
&\Rightarrow 1 \leq a + b,
\end{aligned}$$

by contradiction, Hence (x_n) is a Cauchy sequence,i.e., (x_n) converges to some $x \in X$. Similarly we can show that (y_n) converges to some $y \in X$.

Next we have to show that (x, y) is a coupled fixed point of f.

Now

$$
\begin{aligned}
& lim \int_0^{R(l(f(x_{n+1},y_{n+1})-f(x,y))} \phi(t)dt \\
& = lim \int_0^{R(\frac{lc}{c}(f(x_{n+1},y_{n+1})-f(x,y))} \phi(t)dt \\
& \leq lim\ a \int_0^{R(c(x_n-x))} \phi(t)dt + b\ lim \int_0^{R(c(y_n-y))} \phi(t)dt \\
& = lim\ a \int_0^{R(c(x_n-x))} \phi(t)dt) + b\ lim \int_0^{R(c(y_n-y))} \phi(t)dt) = 0.
\end{aligned}
$$

Using $\triangle_2$ condition, we obtain (x_n) converges to $f(x,y)$. Since the limit is unique therefore, $f(x,y) = x$. In a similar way, we can prove $f(y,x) = y$. Hence (x,y) is a coupled fixed point of f.

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