Some Weakly Symmetric Kähler Manifolds

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Abstract

In this paper we have studied weakly symmetric Kähler manifold which is pseudo-projectively flat and quasi-conformally flat.

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1 Introduction

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V) \\ &\quad + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X),\end{aligned} \tag{1.1}$$

where A,B,C,D,E are simultaneously non-vanishing 1-forms and X,Y,Z,U,V are vector fields and ∇ be the operator of covariant differentiation with respect to the Riemannian metric g. The 1- forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$.
In 1995 M. Prvanovic [1] proved that if the manifold M is a weakly symmetric manifold satisfying (1.1) then $B = C = D = E$. In this paper we consider $B = C = D = E = \omega$ and then (1.1) becomes

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + \omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V) \\ &\quad + \omega(U)R(Y,Z,X,V) + \omega(V)R(Y,Z,U,X),\end{aligned} \tag{1.2}$$

where $g(X, \rho) = \omega(X)$ and $g(X, \alpha) = A(X)$. where ρ and α are vector fields. In 2002 Prasad [2] defined and studied a tensor field $\overline{P}$ on a Riemannian manifold of dimension $n(n > 2)$ which includes the projective curvature tensor P. This tensor field $\overline{P}$ is known as pseudo-projective curvature tensor and is given by

$$\begin{aligned}\overline{P}(X,Y,Z) &= aR(X,Y,Z) + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y],\end{aligned}$$

(1.3)

where a and b are constants such that $a, b \neq 0, R$ is the curvature tensor, S is the Ricci tensor and r is the scalar curvature. A non-pseudo projectively flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be weakly pseudo projectively symmetric manifold if the pseudo-projective curvature tensor $\overline{P}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X\overline{P})(Y,Z,U,V) &= A(X)\overline{P}(Y,Z,U,V)+B(Y)\overline{P}(X,Z,U,V)+C(Z)\overline{P}(Y,X,U,V)\\ &\quad +D(U)\overline{P}(Y,Z,X,V)+E(V)\overline{P}(Y,Z,U,X),\end{aligned} \tag{1.4}$$

for all vectors fields X, Y, Z, U, V and A, B, C, D, E are non-vanishing 1-forms. Such an n-dimensional manifold is denoted by $(WPPS)_n$. Also, we give the definition of quasi-conformal curvature tensor given by Yano and Sawaki [8] as follows

$$\begin{aligned}\overline{C}(X,Y,Z) &= aR(X,Y,Z)+b[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]\\ &\quad -\frac{r}{n}\left[\frac{a}{n-1}+2b\right][g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{1.5}$$

where a and b are non zero constants. If $a = 1$ and $b = -\frac{1}{n-2}$, then quasi-conformal curvature tensor is reduced to the conformal curvature tensor. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be weakly quasi-conformally symmetric manifold if the quasi-conformally curvature tensor $\overline{C}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X\overline{C})(Y,Z,U,V) &= A(X)\overline{C}(Y,Z,U,V)+B(Y)\overline{C}(X,Z,U,V)+C(Z)\overline{C}(Y,X,U,V)\\ &\quad +D(U)\overline{C}(Y,Z,X,V)+E(V)\overline{C}(Y,Z,U,X),\end{aligned} \tag{1.6}$$

for all vectors fields X, Y, Z, U, V and A, B, C, D, E are non-vanishing 1-forms such an n-dimensional manifold is denoted by $(WQCS)_n$.

In this paper we have considered two types of Kähler manifold namely weakly pseudo projectively symmetric Kähler manifold and weakly quasi-conformally symmetric Kähler manifold.

2 Preliminaries

First of all, we define Kähler manifold in this section. A Kähler manifold is a Riemannian manifold M of even dimension n with complex structure F on the tangent space of M at each point satisfies the following relation

$$F^2(X) = -X, g(\overline{X}, \overline{Y}) = g(X, Y), (\nabla_X F)(Y) = 0$$

where F is a tensor field of type $(1, 1)$ such that $F(X) = \overline{X}$, g is a Riemannian metric and ∇ is the Levi-Civita Connection. Also in this section we derive some formulae which will be required to study of $(WPPS)_n$ and $(WQCS)_n$. Let $e_i, i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then from (1.3), we have the following:-

(a) $\sum_{i=1}^{n} \overline{P}(e_i, Y, Z, e_i) = [a + (n-1)b][S(Y, Z) - \frac{r}{n} g(Y, Z)]$

(b) $\sum_{i=1}^{n} \overline{P}(X, Y, e_i, e_i) = 0$

(c) $\sum_{i=1}^{n} \overline{C}(e_i, Y, Z, e_i) = [a + (n-2)b][S(Y, Z) - \frac{r}{n} g(Y, Z)]$

(d) $\sum_{i=1}^{n} \overline{C}(X, Y, e_i, e_i) = 0$

Now we have proved the following proposition:

Proposition (2.1.) In a Riemannian manifold $(M^n, g)(n > 2)$ the pseudo-projective curvature tensor and quasi-conformally curvature tensor satisfies the following relation:

$$(I) \quad \overline{P}(X, Y, Z, U) + \overline{P}(Y, Z, X, U) + \overline{P}(Z, X, Y, U) = 0$$

$$(II) \quad \overline{P}(X, Y, U, Z) + \overline{P}(Y, Z, U, X) + \overline{P}(Z, X, U, Y) = 0$$

$$(III) \quad \overline{C}(X, Y, Z, U) + \overline{C}(Y, Z, X, U) + \overline{C}(Z, X, Y, U) = 0$$

$$(IV) \quad \overline{C}(X, Y, U, Z) + \overline{C}(Y, Z, U, X) + \overline{C}(Z, X, U, Y) = 0$$

3 Weakly Pseudo Projectively Symmetric K$\ddot{a}$hler Manifold

If the manifold M is a weakly pseudo projectively symmetric Kähler manifold, then we have proved

$$\overline{P}(\overline{Y}, \overline{Z}, U, V) = \overline{P}(Y, Z, U, V). \tag{3.1}$$

Taking the covariant derivative, we get

$$(\nabla_X \overline{P})(\overline{Y}, \overline{Z}, U, V) = (\nabla_X \overline{P})(Y, Z, U, V). \tag{3.2}$$

Using(1.2) and (1.4) in (3.2), we obtain

$$\omega(Y)\overline{P}(X, Z, U, V) + \omega(Z)\overline{P}(Y, X, U, V) = \omega(\overline{Y})\overline{P}(X, \overline{Z}, U, V) + \omega(\overline{Z})\overline{P}(\overline{Y}, X, U, V).$$

(3.3)

By, putting $Z = U = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\begin{aligned}
&(a-b)\omega(Y)S(X, V) - \frac{(a-1)br}{n}\omega(Y)g(X, V) - aR(Y, X, V, \rho) + bg(Y, V)S(X, \rho) \\
&-2bg(X, V)S(Y, \rho) + \frac{2r}{n}\left[\frac{a}{n-1} + b\right] g(X, V)g(Y, \rho) - \frac{r}{n}\left[\frac{a}{n-1} + b\right] g(Y, V)g(X, \rho) \\
&= (a+b)\omega(\overline{Y})S(X, \overline{V}) - \frac{r}{n}\left[\frac{a}{n-1} + b\right] g(X, \overline{V})\omega(\overline{Y}) + aR(\overline{Y}, X, V, \overline{\rho}) \\
&+\frac{r}{n}\left[\frac{a}{n-1} + \frac{(r-n)b}{r}\right] g(\overline{Y}, V)S(X, \overline{\rho}).
\end{aligned} \tag{3.4}$$

Again, putting $X = V = e_i$, $1 \le i \le n$ and summing over i, we get

$$rg(Y,\rho)\left[a(1-b)+(2-\frac{1}{n})(\frac{a}{n-1}+b)\right] = S(Y,\rho)\left[2a+2b(n-1)+\frac{ar}{n(n-1)}+\frac{br}{n}\right].$$

(3.5)

We get,

$$S(Y,\rho) = fg(Y,\rho).$$

(3.6)

This is an Einstein manifold for every vector field ρ.

Thus we state the following theorem:

Theorem 3.1. *A weakly pseudo projectively symmetric Kähler manifold is an Einstein manifold with respect to vector field ρ defined by $g(X,\rho) = \omega(X)$.*

From theorem (3.1.), we have the following corollary

Corollary 3.1. *For a weakly pseudo projectively symmetric Kähler manifold if the vector field ρ is a unit vector field and Y=ρ, then the expression for scalar curvature is, $r = \frac{2nh[a+(n-1)b]}{2na+(n-h-1)(a+b)}$ provided $2na + (n-h-1)(a+b) \neq 0$ where $h = S(\rho,\rho)$. In addition if $a + (n-1)b = 0$, then the scalar curvature vanishes.*

4 Pseudo-Projectively Flat Weakly Symmetric K$\ddot{a}$hler manifold

For pseudo-projectively flat curvature tensor, $\overline{P}(Y,Z,U,V) = 0$, then

$$\begin{aligned} aR(Y,Z,U,V)+bS(Z,U)g(Y,V) &- bS(Y,U)g(Z,V)-\frac{r}{n}\left[\frac{a}{n-1}+b\right]g(Z,U)g(Y,V) \\ &+ \frac{r}{n}\left[\frac{a}{n-1}+b\right]g(Y,U)g(Z,V)=0. \end{aligned} \tag{4.1}$$

Then

$$\begin{aligned} R(Y,Z,U,V) &= -\frac{b}{a}[S(Z,U)g(Y,V)-S(Y,U)g(Z,V)] \\ &\quad +\frac{r}{an}\left[\frac{a}{n-1}+b\right][g(Z,U)g(Y,V)-g(Y,U)g(Z,V)]. \end{aligned} \tag{4.2}$$

Taking covariant differentiation w.r.t. X,we get

$$(\nabla_X R)(Y,Z,U,V) = -\frac{b}{a}[g(Y,V)(\nabla_X S)(Z,U)-g(Z,V)(\nabla_X S)(Y,U)], \tag{4.3}$$

then (4.3) reduces to

$$\begin{aligned}&A(X)R(Y,Z,U,V)+\omega(Y)R(X,Z,U,V)+\omega(Z)R(Y,X,U,V)+\omega(U)R(Y,Z,X,V)\\&+\omega(V)R(Y,Z,U,X)=-\frac{b}{a}[g(Y,V)A(X)S(Z,U)+\omega(Z)S(X,U)+\omega(U)S(Z,X)\\&-g(Z,V)A(X)S(Y,U)+\omega(Y)S(X,U)+\omega(U)S(Y,X)].\end{aligned} \tag{4.4}$$

By, putting $Y=V=e_i$, $1\leq i\leq n$ and summing over i, we get

$$\left[1+\frac{b}{a}(n-1)\right][A(X)S(Z,U)+\omega(Z)S(X,U)+\omega(U)S(Z,X)]=0. \tag{4.5}$$

Again,taking $X=U=e_i$, $1\leq i\leq n$ and summing over i, we get

$$\left[1+\frac{b}{a}(n-1)\right][S(Z,\alpha)+r\omega(Z)+S(Z,\rho)]=0, \tag{4.6}$$

for any vector field ρ defined by $g(X,\rho)=\omega(X)$ and $g(X,\alpha)=A(X)$,then we have

$$S(Z,\alpha)+S(Z,\rho)=-r\omega(Z). \tag{4.7}$$

Then we get the theorem,

Theorem 4.1. *In a pseudo projectively flat weakly symmetric Kähler manifold, the Ricci tensor satisfies the relation $S(Z,\alpha)+S(Z,\rho)=-r\omega(Z)$.*

5 Weakly Quasi-Conformally Symmetric K$\ddot{a}$hler manifold

If the manifold M is a weakly quasi-conformally symmetric Kähler manifold, then we have proved

$$\overline{C}(\overline{Y},\overline{Z},U,V)=\overline{C}(Y,Z,U,V). \tag{5.1}$$

Taking the covariant derivative, we get

$$(\nabla_X\overline{C})(\overline{Y},\overline{Z},U,V)=(\nabla_X\overline{C})(Y,Z,U,V). \tag{5.2}$$

Using(1.5) in (5.2), we obtain

$$\omega(Y)\overline{C}(X,Z,U,V)+\omega(Z)\overline{C}(Y,X,U,V)=\omega(\overline{Y})\overline{C}(X,\overline{Z},U,V)+\omega(\overline{Z})\overline{C}(\overline{Y},X,U,V).$$

(5.3)

By, putting $Z = U = e_i$, $1 \le i \le n$ and summing over i, we get

$$\begin{aligned}
&[a+(n-4)b]\omega(Y)S(X,V) - \frac{r}{n}[a+(n-2)b]\omega(Y)g(X,V) - aR(Y,X,V,\rho)\\
&+bg(Y,V)S(X,\rho) - 2bg(X,V)S(Y,\rho) + bg(X,\rho)S(Y,V)\\
&= (a+2b)\omega(\overline{Y})S(X,\overline{V}) - \frac{r}{n}\left[\frac{a}{n-1}+2b\right]\omega(\overline{Y})g(X,\overline{V}) + aR(\overline{Y},X,V,\overline{\rho})\\
&-bg(Y,\overline{V})S(X,\overline{\rho}) + \frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(\overline{Y},V)g(X,\overline{\rho}) - bg(X,\overline{\rho})S(\overline{Y},V).
\end{aligned} \tag{5.4}$$

Again, putting $X = V = e_i$, $1 \le i \le n$ and summing over i, we get

$$-rg(Y,\rho)\left[2b + \frac{a}{n(n-1)} + \frac{2b}{n}\right] = S(Y,\rho)[2a - 3b + 2bn]. \tag{5.5}$$

We get

$$S(Y,\rho) = fg(Y,\rho). \tag{5.6}$$

This is again an Einstein manifold for every vector field ρ.

Theorem 5.1. *A weakly quasi-conformally symmetric Kähler manifold is an Einstein manifold with respect to vector field ρ defined by $g(X,\rho) = \omega(X)$.*

From theorem (5.1.), we have the following corollary

Corollary 5.1. *For a weakly quasi-conformally symmetric Kähler manifold if the vector field ρ is a unit vector field and Y=ρ, then the expression for scalar curvature is, $r = -\frac{n(n-1)h[2a+(2n-3)b]}{a+(n^2-1)2b}$ provided $a + (n^2-1)2b \neq 0$ where $h = S(\rho,\rho)$. In addition if $2a + (2n-3)b = 0$, then the scalar curvature vanishes.*

6 Quasi-Conformally Flat Weakly Symmetric Kähler manifold

For quasi-conformally flat curvature tensor, $\overline{C}(Y,Z,U,V) = 0$, then

$$\begin{aligned}
aR(Y,Z,U,V) + bS(Z,U)g(Y,V) &- bS(Y,U)g(Z,V) + g(Z,U)g(QY,V) - bg(Y,U)g(QZ,V)\\
&- \frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(Z,U)g(Y,V)\\
&+ \frac{r}{n}\left[\frac{a}{n-1}+2b\right]g(Y,U)g(Z,V) = 0.
\end{aligned} \tag{6.1}$$

Then

$$\begin{aligned}
R(Y,Z,U,V) = &- \frac{b}{a}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) + g(Z,U)g(QY,V) - g(Y,U)g(QZ,V)]\\
&+ \frac{r}{an}\left[\frac{a}{n-1}+2b\right][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)].
\end{aligned} \tag{6.2}$$

Taking covariant differentiation w.r.t. X,we get

$$(\nabla_X R)(Y,Z,U,V) = -\frac{b}{a}[g(Y,V)(\nabla_X S)(Z,U) - g(Z,V)(\nabla_X S)(Y,U)], \tag{6.3}$$

then

$$\begin{aligned} &A(X)R(Y,Z,U,V) + \omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V) + \omega(U)R(Y,Z,X,V)\\ &+\omega(V)R(Y,Z,U,X) = -\frac{b}{a}[g(Y,V)A(X)S(Z,U) + \omega(Z)S(X,U) + \omega(U)S(Z,X)\\ &-g(Z,V)A(X)S(Y,U) + \omega(Y)S(X,U) + \omega(U)S(Y,X)]. \end{aligned} \tag{6.4}$$

By, putting $Y = V = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\left[1 + \frac{b}{a}(n-1)\right][A(X)S(Z,U) + \omega(Z)S(X,U) + \omega(U)S(Z,X)] = 0. \tag{6.5}$$

Again,taking $X = U = e_i$, $1 \leq i \leq n$ and summing over i, we get

$$\left[1 + \frac{b}{a}(n-1)\right][S(Z,\alpha) + r\omega(Z) + S(Z,\rho)] = 0, \tag{6.6}$$

for any vector field ρ defined by $g(X,\rho) = \omega(X)$ and $g(X,\alpha) = A(X)$,then we have

$$S(Z,\alpha) + S(Z,\rho) = -r\omega(Z). \tag{6.7}$$

Thus we state the following theorem:

Theorem 6.1. *In a quasi-conformally flat weakly symmetric Kähler manifold, the Ricci tensor satisfies the relation $S(Z,\alpha) + S(Z,\rho) = -r\omega(Z)$.*

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