Applications of Differential Subordination to a Class of Analytic Functions

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Abstract

Let $\mathbb{E}$ denotes open unit disc in the complex plane and $\mathcal{A}$ the usual class of normalized analytic function defined in $\mathbb{E}$. In the present paper best dominant for a differential subordination of the type

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)+\alpha(1+\mu)\left(1-\frac{zf'(z)}{f(z)}\right)+\alpha\frac{zf''(z)}{f'(z)} \prec q(z)+\alpha\frac{zq'(z)}{q(z)}; z\in\mathbb{E}$$

is obtained. Here $\prec$ stands for subordination between two analytic functions. As applications of this differential subordination, sufficient conditions for starlikeness, parabolic starlikeness, univalence and uniformly close-to-convexity are obtained for a function $f \in \mathcal{A}$.

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1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open disk $\mathbb{E} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ (the complex plane) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}$[a,n] be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$$

Let $\mathcal{A}$ denote the class of all analytic functions f which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore the functions of the class $\mathcal{A}$ are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function $f \in \mathcal{A}$ is starlike of order α if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in \mathbb{E}.$$

The class of starlike functions of order α is denoted by $\mathcal{S}^*(\alpha)$. We write $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of univalent starlike functions in $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be convex univalent in $\mathbb{E}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{E}.$$

The class of all convex univalent functions in $\mathbb{E}$ is denoted by $\mathcal{K}$.
A special subclass of $\mathcal{K}$ is the class of convex functions of order α, with $0 \leq \alpha < 1$, which is analytically defined as

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{E} \right\}.$$

Clearly $\mathcal{K}(0) = \mathcal{K}$.

An analytic function g said to be subordinate to an analytic function f in $|z| < 1$, written as $g \prec f$, if $g(z) = f(w(z))$, where $|w(z)| < 1$ in $|z| < 1$.
Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let p be an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of the differential subordination (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.
Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), h(0) = \Psi(p(0), 0; 0). \tag{1.2}$$

An analytic function q is called a subordinant of the differential superordination (1.2), if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant of (1.2). The best subordinant

is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha,\ 0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{E},$$

or, equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha},\ z \in \mathbb{E}.$$

Let $\overline{\mathcal{S}}(\alpha)$ denote the class of all such functions. Note that $\overline{\mathcal{S}}(1) \equiv \mathcal{S}^*$. The class $\overline{\mathcal{S}}(\alpha)$ was introduced and studied independently by Brannan and Kirwan [1] and Stankiewicz [5].
A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{E}. \tag{1.3}$$

The class of parabolic starlike functions is denoted by $\mathcal{S}_p$. A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, z \in \mathbb{E}, \tag{1.4}$$

for some $g \in \mathcal{S}_p$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_p$. Therefore, for $g(z) = z$, condition (1.4) becomes

$$\Re(f'(z)) > |f'(z) - 1|, z \in \mathbb{E}. \tag{1.5}$$

Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}.$$

Note that the conditions (1.3) and (1.5) are equivalent to the condition that $\dfrac{zf'(z)}{f(z)}$ and $f'(z)$ take values in the parabolic domain Ω.
Ronning [7] and Ma and Minda [6] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}} \tag{1.6}$$

maps the open unit disk $\mathbb{E}$ onto the parabolic domain Ω. Therefore, condition (1.3) is equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), z \in \mathbb{E}$$

and condition (1.5) is same as

$$f'(z) \prec q(z), z \in \mathbb{E},$$

where q is given by (1.6).

A number of sufficient conditions for $f \in \mathcal{A}$ to be starlike univalent have been established in the past e.g. Miller, Mocanu and Reade [3] studied the class of α-convex functions and proved that if a function $f \in \mathcal{A}$ satisfies the differential inequality

$$\Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in \mathbb{E},$$

where α is any real number, then f is starlike univalent in $\mathbb{E}$.

In 1972, Ozaki and Nunokawa [8] proved certain conditions for univalence of function $f \in \mathcal{A}$ given by following result:

Theorem 1.1. *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *be regular in* $z \in \mathbb{E}$ *and* $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1$ *or* $\Re\left(\frac{f^2(z)}{z^2 f'(z)}\right) \geq \frac{1}{2}$ *in* $\mathbb{E}$. *Then* f *is univalent in* $\mathbb{E}$.

It is known [9, 10] that the functions in $U(\lambda)$ are univalent if $0 \leq \lambda \leq 1$, where

$$U(\lambda, \mu) = \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \neq 0 \ \ \textit{and} \ \ \left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < \lambda, z \in \mathbb{E} \right\}$$

and $U(\lambda) = U(\lambda, 1)$.

In 2007, Fournier and Ponnusamy [2] also studied the class involving the operator $f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}$.

The objective of this paper is to obtain certain sufficient conditions for starlikeness and convexity in terms of certain differential subordinations involving operator $f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}$, where $f \in \mathcal{A}$ and μ is a real number.

The class

$$U(\mu) = \left\{ f \in \mathcal{A} : \frac{z}{f(z)} \neq 0 \ \ \textit{and} \ \ f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} > 0, z \in \mathbb{E} \right\}$$

forms a subclass of the class of starlike functions at $\mu = 0$; for $\mu < 0$, we obtain the subclass of class of Bazilevic functions and $\mu > 0$ is also an interesting case.

2 Preliminary Notes

To prove the main results, we shall use the following definition and lemmas of Miller-Mocanu [4] :

Definition 2.1. *([4], Definition 2, p. 817) Denote by* $\mathbb{Q}$*, the set of all functions* $f(z)$ *that are analytic and injective on* $\overline{\mathbb{E}} \setminus \mathbb{E}(f)$*, where*

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ *for* $\zeta \in \partial\overline{\mathbb{E}} \setminus \mathbb{E}(f)$*.*

Lemma 2.2. *([4]) Let* q *be univalent in the unit disk* $\mathbb{E}$ *and let* θ *and* ϕ *be analytic in domain* $\mathbb{D}$ *containing* $q(\mathbb{E})$ *with* $\phi(w) \neq 0$ *when* $w \in q(\mathbb{E})$*. Set* $Q_1(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q_1(z)$ *and suppose that*
(i) either h *is convex or* Q_1 *is starlike in* $\mathbb{E}$*, and*
(ii) $\Re \dfrac{zh'(z)}{Q_1(z)} > 0, z \in \mathbb{E}$*.*
If p *is analytic in* $\mathbb{E}$ *with* $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\theta[(p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

3 Main Results

Theorem 3.1. *Let* $\alpha(\neq 0)$ *and* μ *are real numbers and* q *is a univalent function defined on* $\mathbb{E}$ *with* $q(0) = 1$ *such that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > \max\left\{0, -\Re\left(\frac{q(z)}{\alpha}\right)\right\}. \tag{3.1}$$

If $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) + \alpha(1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha\frac{zf''(z)}{f'(z)} \prec q(z) + \alpha\frac{zq'(z)}{q(z)} \tag{3.2}$$

then

$$\left[\frac{z}{f(z)}\right]^{1+\mu} .f'(z) \prec q(z); z \in \mathbb{E},$$

where $q(z)$ *is the best dominant.*

Proof. Define the function p by $p(z) = \left[\dfrac{z}{f(z)}\right]^{1+\mu} .f'(z)$. With a little calculation, from (3.2), we have

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec q(z) + \alpha\frac{zq'(z)}{q(z)}.$$

Define the functions θ and ϕ as $\theta(w) = w$, $\phi(w) = \dfrac{\alpha}{w}$. Clearly ϕ is analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$. Set $Q_1(z) = \alpha \dfrac{zq'(z)}{q(z)}$ and $h(z) = q(z) + \alpha \dfrac{zq'(z)}{q(z)}$. On differentiation, we obtain:

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\alpha}.$$

In view of condition (3.1), we have $Q_1(z)$ is starlike and $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0$. The proof, now, follows from Lemma 2.2.

Remark 1. *Consider the dominant* $q(z) = \dfrac{1+(1-2\beta)z}{1-z}, 0 \leq \beta < 1$, *we obtain*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1+(1-2\beta)z^2}{[1+(1-2\beta)z](1-z)}\right) > 0$$

Also for $\alpha > 0$,

$$\max\left\{0, -\Re\left(\frac{q(z)}{\alpha}\right)\right\} = 0.$$

Clearly the condition in (3.1) holds and consequently, we get the following result from Theorem 3.1.

Theorem 3.2. *Let* $\alpha(\neq 0)$, μ *be real numbers and* $0 \leq \beta < 1$, *If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) + \alpha(1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha \frac{zf''(z)}{f'(z)}$$

$$\prec \frac{1 + 2z[1-2\beta+\alpha(1-\beta)] + (1-2\beta)^2 z^2}{1 - 2\beta z - (1-2\beta)z^2}, z \in \mathbb{E}$$

then

$$\left[\frac{z}{f(z)}\right]^{1+\mu} .f'(z) \prec \frac{1+(1-2\beta)z}{1-z}; z \in \mathbb{E}.$$

Select $\mu = 0$ in above theorem, we get the following result.

Corollary 3.3. *Let $\alpha(\neq 0)$ be a real number and $0 \leq \beta < 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1 + 2z[1 - 2\beta + \alpha(1-\beta)] + (1-2\beta)^2 z^2}{1 - 2\beta z - (1-2\beta)z^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1-2\beta)z}{1-z}; z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}^*(\beta)$.

selecting $\mu = -1$ in Theorem 3.2, we get the following result.

Corollary 3.4. *Let $\alpha(\neq 0)$ be real number and $0 \leq \beta < 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$f'(z) + \alpha\frac{zf''(z)}{f'(z)} \prec \frac{1 + 2z[1 - 2\beta + \alpha(1-\beta)] + (1-2\beta)^2 z^2}{1 - 2\beta z - (1-2\beta)z^2}, z \in \mathbb{E},$$

then

$$f'(z) \prec \frac{1 + (1-2\beta)z}{1-z}; z \in \mathbb{E}.$$

Remark 2. *When we select the dominant $q(z) = \left(\frac{1+z}{1-z}\right)^{\delta}, 0 < \delta \leq 1$ in Theorem 3.2, we see that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1+z^2}{1-z^2}\right) > 0$$

and for $\alpha > 0$

$$\max\left\{0, -\frac{q(z)}{\alpha}\right\} = 0.$$

Obviously the condition (3.1) of Theorem 3.1 holds and we obtain the following result.

Theorem 3.5. *Let $\alpha(\neq 0)$, μ be real numbers and $0 < \delta \leq 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) + \alpha(1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha\frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta} + 2\alpha\delta\left(\frac{z}{1-z^2}\right)$$

then

$$\left[\frac{z}{f(z)}\right]^{1+\mu} .f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\delta}; z \in \mathbb{E}.$$

Select $\mu = 0$ in the above theorem, we obtain the following result for strongly starlikeness.

Corollary 3.6. *Let $\alpha(\neq 0)$ be a real number and $0 < \delta \leq 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1+\frac{zf''(z)}{f'(z)}) \prec \left(\frac{1+z}{1-z}\right)^{\delta} + 2\alpha\delta\left(\frac{z}{1-z^2}\right), z \in \mathbb{E}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}; z \in \mathbb{E}$$

and hence $f \in \overline{\mathcal{S}}(\delta)$.

Setting $\mu = -1$ in Theorem 3.5, we get following result.

Corollary 3.7. *Let $\alpha(\neq 0)$ be real number and $0 < \delta \leq 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$f'(z) + \alpha\frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta} + 2\alpha\delta\left(\frac{z}{1-z^2}\right), z \in \mathbb{E}$$

then

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\delta}; z \in \mathbb{E}.$$

Remark 3. *When we select the dominant $q(z) = e^z$, we have*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

Also

$$\max\left\{0, -\Re\left(\frac{q(z)}{\alpha}\right)\right\} = 0$$

for $\alpha > 0$. Thus (3.1) of Theorem 3.1 holds and consequently, we get the following result.

Theorem 3.8. *Let $\alpha(\neq 0)$ and μ be real numbers. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) + \alpha(1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha\frac{zf''(z)}{f'(z)} \prec e^z + \alpha z, z \in \mathbb{E}$$

then

$$\left[\frac{z}{f(z)}\right]^{1+\mu} .f'(z) \prec e^z; z \in \mathbb{E}.$$

Selecting $\mu = 0$ in above theorem, we obtain:

Corollary 3.9. *Let α be a non-zero real number. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^z + \alpha z, z \in \mathbb{E}$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, z \in \mathbb{E} \quad i.e. \quad f \in \mathcal{S}^*.$$

Select $\mu = -1$ in Theorem 3.8, we get following result.

Corollary 3.10. *Let α be a non-zero real number. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$f'(z) + \alpha\frac{zf''(z)}{f'(z)} \prec e^z + \alpha z, z \in \mathbb{E}$$

then

$$f'(z) \prec e^z, z \in \mathbb{E}.$$

Remark 4. *For the dominant $q(z) = 1 + az, 0 \leq a \leq 1$, we obtain*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1}{1+az}\right) > 0.$$

Moreover for $\alpha > 0$, $\max\left\{0, -\frac{q(z)}{\alpha}\right\} = 0$. *Clearly the condition (3.1) of Theorem 3.1 holds and consequently we have the following result.*

Theorem 3.11. *Let $\alpha(\neq 0)$, μ be real numbers and $0 \leq a \leq 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) + \alpha(1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha\frac{zf''(z)}{f'(z)} \prec 1 + az + \alpha\left(\frac{az}{1+az}\right)$$

then

$$\left[\frac{z}{f(z)}\right]^{1+\mu} .f'(z) \prec 1 + az; z \in \mathbb{E}.$$

Setting $\mu = 1$ in above theorem, we get the following result.

Corollary 3.12. *Let $\alpha(\neq 0)$ be a real number and $0 \leq a \leq 1$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^2 f'(z) + 2\alpha\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha\frac{zf''(z)}{f'(z)} \prec 1 + az + \alpha\left(\frac{az}{1+az}\right)$$

then

$$\left[\frac{z}{f(z)}\right]^2 .f'(z) \prec 1 + az; z \in \mathbb{E}$$

or equivalently $\left|\frac{z^2 f'(z)}{f(z)} - 1\right| \leq a, z \in \mathbb{E}.$

Remark 5. *When we select the dominant* $q(z) = 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}$, *we have*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

Also

$$\max\left\{0, -\Re\left(\frac{q(z)}{\alpha}\right)\right\} = 0$$

for $\alpha > 0$. *Thus (3.1) of Theorem 3.1 holds and consequently, we get the following result.*

Theorem 3.13. *Let* $\alpha(\neq 0)$ *and* μ *be real numbers. If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) + \alpha(1+\mu)\left(1 - \frac{zf'(z)}{f(z)}\right) + \alpha\frac{zf''(z)}{f'(z)}$$
$$\prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}} + \alpha\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\frac{log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}}, z \in \mathbb{E}$$

then

$$\left[\frac{z}{f(z)}\right]^{1+\mu} .f'(z) \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}; z \in \mathbb{E}.$$

Taking $\mu = 0$ in above theorem, we obtain:

Corollary 3.14. *Let* $\alpha(\neq 0)$ *be real number. If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$
$$\prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}} + \alpha\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\frac{log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}; z \in \mathbb{E},$$

or equivalently $f \in \mathcal{S}_p$.

Select $\mu = -1$ in Theorem 3.8, we get the following result.

Corollary 3.15. *Let $\alpha(\neq 0)$ be real number, If $f \in \mathcal{A}$ satisfies the differential subordination*

$$f'(z) + \alpha\frac{zf''(z)}{f'(z)} \prec 1 + \frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}} + \alpha\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\frac{log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{1+\frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}}, z \in \mathbb{E},$$

then

$$f'(z) \prec 1 + \frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}; z \in \mathbb{E}$$

References

[1] D. A. Brannan and W. E. Kirwan, On some classes of bounded univalent functions, *J. London Math. Soc.*, **2**(1)(1969), 431-443.

[2] R. Fournier and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, *Complex Var. Elliptic Eqn.*, **52**(1) (2007), 1-8.

[3] S. S. Miller, P. T. Mocano and M. O. Reade, , All alpha-covex functions are univalent and starlike, *Proc. Amer. Math. Soc.*, **37**(1973), 553-554.

[4] S. S. Miller and P. T. Mocanu, Differential Subordination: theory and Applications, *Marcel Dekker, new York and Basel*, (2000).

[5] J. Stankiewicz, Some remarks concerning starlike functions, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phy.*, **18**(1970), 143-146.

[6] W.C. Ma and D. Minda, Uniformly convex functions, *Ann. Polon. Math*, **57**(2) (1992), 165-175

[7] F. Ronning, Uniformly convex functions and a corresponding class of Starlike Functions, *Proc. Amer. Math. Soc.*, **118** (1993) (1), 189-196

[8] S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, *Proceedings of the American Mathematical Society*, **33** (2) (1972), 392-394

[9] L. A. Aksentiev, Sufficient conditions for univalence of regular functions, *Izv. Vysv. Ucebn. Zaved. mathematika*, **3** (1958), 3-7 (Russian).

[10] S. Ponnusamy and P. Vasundhara, Starlikeness of nonlinear integral transforms, *Proceedings of the International Conference on Geometric Function Theory, Special Functions and Applications, J. Analysis*, **15** (2007), 195-210