Maximal Numerical Range of Composition Operators on ℓ^2

Gyan Prakash Tripathi

Department of Mathematics,
SGR PG College, Dobhi, Jaunpur
gptbhu@gmail.com

Abstract

In this paper we obtain precisely when zero belongs to maximal numerical range of composition operators on ℓ^2. By using this result we characterize the norm attainability of derivations on $B(\ell^2)$.

Subject Classification: 47B33, 47B47.

Keywords:maximal numerical range, composition operators, elementary operators, norm, derivations.

1 Introduction

Let ℓ^2 be the Hilbert space of all square summable sequences of complex numbers under standard inner product on it and be a function on $\mathbb{N}$ into itself. We denote by n, characteristic function of n. Let $A_n = \phi^{-1}(n)$ and let $\bar{\bar{A}}_n$ denote the number of elements in A_n. Now we state the following result, which characterize the functions ϕ on $\mathbb{N}$ into itself which induce composition operators on ℓ^2.

Theorem 1.1. *[10] A necessary and sufficient condition that a function ϕ on $\mathbb{N}$ into itself induces a composition operator on ℓ^2 is the set $\{\bar{\bar{A}}_n : n \in \mathbb{N}\}$ is bounded.*

A composition operator on ℓ^2 has two representations, which are the following:

$$\begin{aligned} C_\phi(f) &= \sum_{j=1}^{\infty} f(\phi(j))\chi_j \\ &= \sum_{j=1}^{\infty} f(j)\chi_{\phi^{-1}(j)}, \quad \text{where } f = \sum_{j=1}^{\infty} f(j)\chi_j \in \ell^2. \end{aligned}$$

The adjoint of C_ϕ is represented by $C^*_\phi(f) = \sum\limits_{j=1}^{\infty} f(j)\chi_{\phi(j)}$

Theorem 1.2. *Let C_ϕ be a composition operator on ℓ^2 then*
(i) range of C_ϕ is always closed.
(ii) norm of C_ϕ is given by

$$\|C_\phi \mid = \sqrt{N}, \textit{where } N = \max\{\bar{\bar{A}}_n : n \in N\}.$$

Recall that numerical range of an operator $A \in B(H)$ is the set

$$W(A) = \{< Ax, x >: x \in H, \|x\| = 1\}.$$

Definition 1. *The maximal numerical range of an operator $A \in B(H)$ is defined by*

$$W_o(A) = \{\lambda \in \mathbb{C} : \textit{there exists a sequence } \{x_n\} \textit{of unit vectors in } H \textit{ such that } \|Ax_n\| \to \|A\|, \textit{ and} < Ax_n, x_n > \to \lambda\}.$$

Theorem 1.3. *[9] The set $W_0(A)$ is non-empty, closed, convex and contained in closure of numerical range of A.*

Definition 2. *Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be $n-$tuples of elements in an algebra A. The elementary operator $E_{a,b}$ on A into itself associated with a and b is defined by $E_{a,b}(x) = a_1xb_1 + a_2xb_2 + ... + a_nxb_n$*

The number n is called length of the elementary operator $E_{a,b}$.

Definition 3. *We say $a = (a_1, a_2, ..., a_n)$ is commuting family if $a_ia_j = a_ja_i$ for each $1 \leq i, j \leq n$. We denote by $M_{a,b}$ eleementary multiplication operator, defined by*

$$M_{a,b} = axb, x \in A$$

We define $U_{a,b}$ as $U_{a,b} = axb + bxa$ for all $x \in A$

Definition 4. *Derivation is a linear map δ on an algebra A into itself satisfying*

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for $a, b \in A$.

For a fixed $a \in A$, inner derivation δ_a is defined by $\delta_a(x) = ax - xa$.
For fixed $a, b \in A$, generalized derivation $\delta_{a,b}$ is defined by $\delta_{a,b}(x) = ax - xb$ for all $x \in A$. It is clear that δ_a and $\delta_{a,b}$ are elementary operators of length 2.

In 1970, J.G. Stampfli found an elegant formula for the norm of inner derivation δ_A on $B(H)$. He also got an expression for the norm of generalized derivation. Now we state some results of Stampfli on the norm of derivations and generalized derivations.

Theorem 1.4. *[9] For $A \in B(H)$, $\|\delta_A\| = 2 \inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}$.*

Theorem 1.5. *[9] For $A \in B(H)$, $\|\delta_{A,B}\| = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}$.*

An alternative approach to computing $\|\delta_A\|$ emanating from commutator theory, is related to results of C.Apastol and L.Zsido [2]. They also proved analogue of the Theorem 1.4 for inner derivations of C^*-algebra and W^*- algebra.
The notion of maximal numerical was first introduced by J. G. Stamfli [9]. In 1970, J. G. Stampfli characterize maximum norm of derivations and generalized derivation depending on maximal numerical range as follows:

Theorem 1.6. *For $A \in B(H)$, the following are equivalent:*
(i) $0 \in W_0(A)$.
(ii) $\|\delta_A\| = 2\|A\|$.
(iii) $\|A\| \leq \|A + \lambda I\|$, $\lambda \in \mathbb{C}$
(iv) $\|A\|^2 + |\lambda|^2 \leq \|A + \lambda\|^2$, $\lambda \in \mathbb{C}$.

Definition 5. *The normalized numerical range of $A \in B(H)$ is defined by*

$$W_N(A) = W_0\big(\frac{1}{\|A\|}(A)\big)$$

The following result is analogue of Proposition 1.6 for generalized derivation.

Theorem 1.7. *For $A, B \in B(H)$, the following are equivalent:*
(i) $\|\delta_{A,B}\| = \|A\| + \|B\|$
(ii) $W_N(A) \cap W_N(-B) \neq \phi$.

Norm of Elementary Operators Let $B(H)$ be C^*- algebra of all bounded operators on a Hilbert space H. For an elementary multiplication operator $M_{A,B}$ on $B(H)$, $\|M_{A,B}\| = \|A\|.\|B\|$ [6]. For the elementary operator $U_{A,B}$ on $B(H)$, defined as $U_{A,B}(X) = AXB + BXA$, $A, B \in B(H)$, it is clear that $\|U_{A,B}\| \leq 2\|A\|\|B\|$. In this case it is natural to look for lower estimate of the form $c\|A\|\|B\|$. Martin Mathieu Conjectured [7], [8] that $c = 1$. In 2003, A. Blanco, M. Boumazgour and T.J. Ramsford [4] and Richard Timoney [11] confirmed it independently by different methods.

2 Main Results

In this section we shall prove some results on maximal numerical range of composition operator C_ϕ on ℓ^2. By using these results we find a characterization of maximum norm attainability of derivations and generalized derivations on $B(\ell^2)$ induced by composition operators on ℓ^2. We shall also prove some results on the norm of elementary operators on $B(\ell^2)$ induced by composition operators on ℓ^2.

Theorem 2.1. *Let $\phi(\neq I)$ be a one-one function on $\mathbb{N}$ into itself and C_ϕ be the composition operator on ℓ^2 induced by ϕ. Then $[0, 1] \subseteq W_0(C_\phi)$.*

Proof. Suppose ϕ is one-one function on $\mathbb{N}$ into itself. Since $\phi \neq I, \phi(n) = m$ for some $m \neq n$.
Then

$$\|C_\phi(\chi_m)\| = \|\chi_{\phi^{-1}(m)}\| = 1 = \|C_\phi\| \text{ and } < C_\phi(\chi_m), \chi_m >=< \chi_n, \chi_m >= 0.$$

Therefore $0 \in W_0(C_\phi)$. Now if $\phi^k(1) = 1$ for some least $k \in \mathbb{N}$, then let

$$f = \frac{1}{\sqrt{k}}\big(\chi_1 + \chi_{\phi(1)} + ... + \chi_{\phi^{k-1}(1)}\big).$$

Clearly $\|f\| = 1$

$$\begin{aligned}\|C_\phi f\|^2 &= < C_\phi f, C_\phi f > \\ &= < C_\phi(\frac{1}{\sqrt{k}}(\chi_1 + \chi_{\phi(1)} + ... + \chi_{\phi^{k-1}(1)})), C_\phi(\frac{1}{\sqrt{k}}(\chi_1 + \chi_{\phi(1)} + ... + \chi_{\phi^{k-1}(1)})) > \\ &= \frac{1}{k} < \chi_{\phi^{-1}(1)} + \chi_1 + ... + \chi_{\phi^{k-2}(1)}, \chi_{\phi^{-1}(1)} + \chi_1 + ... + \chi_{\phi^{k-2}(1)} > \\ &= 1 \text{ (because } \phi^{-1}(1) = \phi^{k-1}(1).\end{aligned}$$

Now

$$\begin{aligned}< C_\phi f, f > &= < \frac{1}{\sqrt{k}}(\chi_{\phi^{-1}(1)} + \chi_1 + ... + \chi_{\phi^{k-2}(1)}), \frac{1}{\sqrt{k}}(\chi_1 + \chi_{\phi(1)} + ... + \chi_{\phi^{k-1}(1)}) > \\ &= 1 \text{ (because } \phi^{-1}(1) = \phi^{k-1}(1)).\end{aligned}$$

Thus $1 \in W_0(C_\phi)$. If $\phi^k(1) \neq 1$ for any $k \in \mathbb{N}$, then let

$$f_k = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \chi_{\phi^j(1)}. \text{ Then} \|f\| = 1 \text{ and } \|C_\phi(f_k)\| = 1 = \|C_\phi\|.$$

$$\begin{aligned}< C_\phi f_k, f_k > &= < C_\phi(\frac{1}{\sqrt{k}} \sum_{j=1}^{k} \chi_{\phi^j(1)}), \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \chi_{\phi^j(1)} > \\ &= < \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} \chi_{\phi^j(1)}, \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \chi_{\phi^j(1)} > \\ &= \frac{k-1}{k}\end{aligned}$$

Now $< C_\phi f_k, f_k > = \frac{k-1}{k} \to 1$ as $k \to \infty$. Thus $1 \in W_0(C_\phi)$ in this case also.
Since $W_o(C_\phi)$ is convex, by Theorem 1.3, we have $[0, 1] \subseteq W_0(C_\phi)$. □

Theorem 2.2. *Let C_ϕ be a composition operator on ℓ^2. Then $0 \in W_0(C_\phi)$ if and only if $n \notin A_n$ for some $n \in \mathbb{N}$ such that $\|C_\phi \chi_n\| = \|C_\phi\|$.*

Proof. Suppose that for some $n \in \mathbb{N}$, $\|C_\phi(\chi_n)\| = \|C_\phi\|$ and $n \notin A_n$,
then $< C_\phi(\chi_n), \chi_n > = 0$. Therefore $0 \in W_0(C_\phi)$.
Conversely, let

$$\begin{aligned}\|C_\phi\| = \sqrt{p} \text{ and } A &= \{n \in \phi(\mathbb{N}) : \|C_\phi(\chi_n)\| = \|C_\phi\|\}, \\ B &= \{n \in \phi(\mathbb{N}) : \|C_\phi(\chi_n)\| < \|C_\phi\|\}\end{aligned}$$

Suppose $n \in A_n$ for each $n \in A$. Now we show that $0 \notin W_0(C_\phi)$.
Let $\{f_n\}$ be a sequence of unit vectors in ℓ^2 such that $\|C_\phi(f_n)\| \to \|C_\phi\|$ as $n \to \infty$

$$i.e.\ \|\sum_{j=1}^{\infty} f_n(\phi(j))\chi_j\| \to \|C_\phi\| \text{ as } n \to \infty$$

$$\implies \sum_{j=1}^{\infty} |f_n(\phi(j))|^2 \to \|C_\phi\|^2 = p \text{ as } n \to \infty$$

$$= \sum_{j\in\phi^{-1}(A)} |f_n(\phi(j))|^2 + \sum_{j\in\phi^{-1}(B)} |f_n(\phi(j))|^2 \to p \text{ as } n \to \infty$$

$$= \sum_{j\in A} p|f_n(j)|^2 + \sum_{j\in B} \bar{\bar{A}}_j|f_n(j)|^2 \to p \text{ as } n \to \infty \tag{2.1}$$

Now

$$\|f_n\|^2 = \sum_{j=1}^{\infty} |f_n(j)|^2 = 1 = \sum_{j\in A} |f_n(j)|^2 + \sum_{j\notin A} |f_n(j)|^2.$$

Thus

$$\sum_{j\in A} |f_n(j)|^2 = 1 - \sum_{j\notin A} |f_n(j)|^2.$$

By equation (2.1)

$$p(1 - \sum_{j\notin A} |f_n(j)|^2) + \sum_{j\in B} \bar{\bar{A}}_j|f_n(j)|^2 \to p$$

$$\implies -p\sum_{j\notin A} |f_n(j)|^2 + \sum_{j\in B} \bar{\bar{A}}_j|f_n(j)|^2 \to 0 \tag{2.2}$$

Since $B \subseteq \mathbb{N} - A$, it is easy to see that

$$\begin{aligned}
\sum_{j\notin A} |f_n(j)|^2 =& p\sum_{j\notin A} |f_n(j)|^2 - (p-1)\sum_{j\notin A} |f_n(j)|^2 \\
\leq& p\sum_{j\notin A} |f_n(j)|^2 - (p-1)\sum_{j\in B} |f_n(j)|^2 \\
\leq& p\sum_{j\notin A} |f_n(j)|^2 - \sum_{j\in B} \bar{\bar{A}}_j|f_n(j)|^2 \text{ (because } 1 \leq \bar{\bar{A}}_j \leq p-1 \text{ for} j \in B).
\end{aligned}$$

Therefore, by equation (2.2)

$$\sum_{j\notin A} |f_n(j)|^2 \to 0 \text{ as } n \to \infty \tag{2.3}$$

and then

$$\sum_{j\in A}|f_n(j)|^2 \to 1 \text{ as } n\to\infty \tag{2.4}$$

Now

$$\begin{aligned}
< C_\phi f_n, f_n > &= < \sum_{j=1}^{\infty} f_n(\phi(j))\chi_j, \sum_{j=1}^{\infty} f_n(j)\chi_j > \\
&= < \sum_{j\in A} f_n(\phi(j))\chi_j + \sum_{j\notin A} f_n(\phi(j))\chi_j, \sum_{j\in A} f_n(j)\chi_j + \sum_{j\notin A} f_n(j)\chi_j > \\
&= < \sum_{j\in A} f_n(\phi(j))\chi_j, \sum_{j\in A} f_n(j)\chi_j > + < \sum_{j\notin A} f_n(\phi(j))\chi_j, \sum_{j\notin A} f_n(j)\chi_j > \\
&= \sum_{j\in A}|f_n(j)|^2 + \sum_{j\notin A} f_n(\phi(j))\overline{|f_n(j)|} \quad \text{(because } j\in A_j \text{ for each } j\in A.)
\end{aligned}$$

Now

$$\begin{aligned}
|\sum_{j\notin A} f_n(\phi(j))f_n(j)| &\leq \sum_{j\notin A}|f_n(\phi(j))||f_n(j)| \\
&\leq \Big(\sum_{j\notin A}|f_n(\phi(j))|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\notin A}|f_n(j)|^2\Big)^{\frac{1}{2}} \quad \text{(by Holder's inequality)} \\
&\leq p\|f_n\|\Big(\sum_{j\notin A}|f_n(j)|^2\Big)^{\frac{1}{2}} \\
&= p\Big(\sum_{j\notin A}|f_n(j)|^2\Big)^{\frac{1}{2}} \to 0 \text{ as } n\to\infty \text{ by equation (2.3)}
\end{aligned}$$

But $\sum_{j\in A}|f_n(j)|^2 \to 1$ as $n\to\infty$ by equation (2.4).
Therefore $< C_\phi f_n, f_n > \to 1 \neq 0$. thus $0 \notin W_0(C_\phi)$. Hence the proof. □

From the proof of above Theorem, we have the following corollory.

Corollary 1. *If $0\notin W_0(C_\phi)$, then $W_0(C_\phi) = \{1\}$*

In view of Theorem 1.6 and Theorem 1.2. we have the following characterization of the norm of a derivation induced by composition operators on ℓ^2.

Theorem 2.3. *Let C_ϕ be a composition operator on ℓ^2 and $n\notin A_n$ for some n such that $\|C_\phi(\chi_n)\| = \|C_\phi\|$. Then*
(i) $0\in W_0(C_\phi)$.
(ii) $\|\delta_\phi\| = 2\|C_\phi\|$.
(iii) $\|C_\phi\| \leq \|C_\phi + \lambda I\|, \lambda\in\mathbb{C}$.
(iv) $\|C_\phi\|^2 + |\lambda|^2 \leq \|C_\phi + \lambda I\|^2, \lambda\in\mathbb{C}$.

Theorem 2.4. *Let C_ϕ and C_ψ be two composition operators on ℓ^2, , where ϕ and ψ $(\phi \neq I, \psi \neq I)$ are one-one functions on $\mathbb{N}$ into itself. Then*

$$\|\delta_{C_\phi,C_\psi}\| = \|\delta_{C_\phi}\| + \|\delta_{C_\psi}\|.$$

Proof. First note that normalized maximal numerical range $W_N(C_\phi) = W_0(C_\phi)$ when ϕ is one-one i.e. $\|C_\phi\| = 1$. Since ϕ is one-one $[0,1] \subseteq W_0(C_\phi) = W_N(C_\phi)$, by Theorem 2.1. Similarly $[0,1] \subseteq W_N(C_\psi)$. For $\lambda \in W_N(C_\psi)$, it is easy to see that $-\lambda \in W_N(-C_\psi)$. Therefore $[-1,0] \subseteq W_N(-C_\psi)$. Thus $W_N(C_\phi) \cap W_N(-C_\psi)$ contains zero so non-empty. Hence

$$\|\delta_{C_\phi,C_\psi}\| = \|\delta_{C_\phi}\| + \|\delta_{C_\psi}\|$$

by Theorem 1.5. □

Now we shall state a result of M. Barraa and M. Boumazgour [3] which is useful in our context.

Theorem 2.5. *[3] Let $A, B \in B(H)$. Then $\|A + B\| = \|A\| + \|B\|$ if and only if $\|A\|\|B\| \in \overline{W(A^*B)}$, where $W(A)$ denotes numerical range of A.*

Theorem 2.6. *Let C_ϕ and C_ψ be two composition operators on ℓ^2 where both ϕ and ψ are one-one and onto functions on $\mathbb{N}$. Then $\|C_\phi + C_\psi\| = \|C_\phi\| + \|C_\psi\|$.*

Proof. If $\phi = \psi$, then above equality is clearly satisfied. Assume $\phi \neq \psi$ Since ϕ and ψ are one-one and onto, C_ϕ and C_ψ are invertible composition operators on ℓ^2. Also C_ϕ^* is an invertible composition operator on ℓ^2 induced by ϕ^{-1}. Since composition of two composition operators on ℓ^2 is again a composition operator on ℓ^2, $C_\phi^* C_\psi$ is a composition operator on ℓ^2 induced by $\phi o \psi^{-1}$, which is one-one and onto function on $\mathbb{N}$. Since $\zeta = \phi o \psi^{-1}$ is one-one and $\zeta \neq I$, $[0,1] \subset W_0(C_\zeta)$ by theorem 2.1. But $W_0(C_\zeta) \subseteq \overline{W(C_\zeta)}$ by Theorem 1.3. Thus $\|C_\phi\|\|C_\psi\| = 1 \in \overline{W(C_\zeta)} = \overline{W(C_\phi^* C_\psi)}$. Therefore $\|C_\phi + C_\psi\| = \|C_\phi\| + \|C_\psi\|$ by Theorem 2.5. □

Theorem 2.7. *Let $C_\phi = (C_{\phi_1}, C_{\phi_2})$ and $C_\psi = (C_{\psi_1}, C_{\psi_2})$ be 2-tuples of composition operators in $B(\ell^2)$, where ϕ_1, ϕ_2, ψ_1 and ψ_2 are one-one and onto functions on $\mathbb{N}$. Then*

$$\|E_{C_\phi,C_\psi}\| = \sum_{i=1}^{2} \|C_{\phi_i}\|\|C_{\psi_i}\|$$

Proof. We have $E_{C_\phi,C_\psi}(X) = C_{\phi_1} X C_{\psi_1} + C_{\phi_2} X C_{\psi_2}$. Since ϕ_1, ϕ_2, ψ_1 and ψ_2 are one-one and onto, $\|C_{\phi_1}\| = \|C_{\phi_2}\| = \|C_{\psi_1}\| = \|C_{\psi_2}\| = 1$. Clearly $\|E_{C_\phi,C_\psi}\| \leq 2$. We have to prove that $\|E_{C_\phi,C_\psi}\| = \sum\limits_{i=1}^{2} \|C_{\phi_i}\|\|C_{\psi_i}\| = 2$.

Now $E_{C_\phi,C_\psi}(I) = C_{\phi_1} C_{\psi_1} + C_{\phi_2} C_{\psi_2}$. It is easy to see that $C_{\phi_1} C_{\psi_1} = C_{\phi_1 o \psi_1}$ and $C_{\phi_2} C_{\psi_2} = C_{\phi_2 o \psi_2}$, where $\phi_1 o \psi_1$ and $\phi_2 o \psi_2$ are one-one onto. Now $\|C_{\phi_1} C_{\psi_1} + C_{\phi_2} C_{\psi_2}\| = \|C_{\phi_1 o \psi_1} + C_{\phi_2 o \psi_2}\|$.

But by Theorem 2.6 $\|C_{\phi_1 o \psi_1} + C_{\phi_2 o \psi_2}\| = \|C_{\phi_1 o \psi_1}\| + \|C_{\phi_2 o \psi_2}\| = 2$

Thus $\|E_{C_\phi,C_\psi}\| = \|E_{C_\phi,C_\psi}(I)\| = 2$. Hence the proof. □

The next result was proved by Mathieu Martin [6] on prime C^*-algebra. We give a simple proof in case of elementary operators on $B(\ell^2)$ induced by composition operators on ℓ^2.

Theorem 2.8. *Let M_{C_ϕ,C_ψ} be elementary multiplication operator on $B(\ell^2)$ then $\|M_{C_\phi,C_\psi}\| = \|C_\phi\|\|C_\psi\|$ for all $C_\phi, C_\psi \in B(\ell^2)$.*

Proof. We have $M_{C_\phi,C_\psi}(X) = C_\phi X C_\psi$. Clearly $\|M_{C_\phi,C_\psi}\| \leq \|C_\phi\|\|C_\psi\|$.
Take $X = f \otimes g$, where f and g are unit vectors in ℓ^2.
Then
$M_{C_\phi,C_\psi}(f \otimes g) = C_\phi(f \otimes g)C_\psi$ and $C_\phi(f \otimes g)C_\psi(h) =< C_\psi h, g >, C_\phi f, h \in \ell^2$.
Choose $h = \chi_n$ such that $\|C_\psi(\chi_n)\| = \|C_\psi\|$, $g = \frac{C_\psi(\chi_n)}{\|C_\psi\|}$ and $f = \chi_m$ such that $\|C_\phi(\chi_m)\| = \|C_\phi\|$.
Now

$$\begin{aligned} M_{C_\phi,C_\psi}(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|})(\chi_n) &= < C_\psi(\chi_n), \frac{C_\psi(\chi_n)}{\|C_\psi\|} > C_\phi(\chi_m) \\ &= \frac{1}{\|C_\psi\|}\|C_\psi(\chi_n)\|^2 C_\phi(\chi_m) = \|C_\psi\| C_\phi(\chi_m). \end{aligned}$$

Thus

$$\begin{aligned} \|M_{C_\phi,C_\psi}(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|})(\chi_n)\| &= \|C_\psi\|\|C_\phi(\chi_m)\| \\ &= \|C_\phi\|\|C_\psi\|. \end{aligned}$$

□

Theorem 2.9. *Let U_{C_ϕ,C_ψ} be an elementary operator on $B(\ell^2)$ defined by $U_{C_\phi,C_\psi}(X) = C_\phi X C_\psi + C_\psi X C_\phi$ then*

$$\|U_{C_\phi,C_\psi}\| \geq \|C_\phi\|\|C_\psi\|.$$

Proof.

$$\begin{aligned} \|U_{C_\phi,C_\psi}\| &= \sup_{\|X\|=1} \{\|C_\phi X C_\psi + C_\psi X C_\phi\| : X \in B(\ell^2)\} \\ &= \sup_{\|X\|=1} \{ \sup_{\|f\|=1} \|(C_\phi X C_\psi + C_\psi X C_\phi f)\| : f \in \ell^2, X \in B(\ell^2)\} \end{aligned}$$

Clearly $\|U_{C_\phi,C_\psi}\| \geq \|(C_\phi X C_\psi + C_\psi X C_\phi f)\|$ for unit vector $f \in \ell^2$,
Supoose $h = \chi_n$ such that $\|C_\psi(\chi_n)\| = \|C_\psi\|$, $g = \frac{C_\psi(\chi_n)}{\|C_\psi\|}$ and $f = \chi_m$ such that

$\|C_\phi(\chi_m)\| = \|C_\phi\|$,

$$\begin{aligned} C_\phi\big(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|}\big)C_\psi(\chi_n) =& \|C_\psi\|C_\phi(\chi_m) \\ C_\psi\big(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|}\big)C_\phi(\chi_n) =& < C_\phi(\chi_m), C_\phi(\chi_n) > \frac{C_\psi(\chi_m)}{\|C_\psi\|} \\ =& \frac{1}{\|C_\psi\|} < C_\phi(\chi_m), C_\psi(\chi_n) > C_\psi(\chi_m). \end{aligned}$$

Now

$$\begin{aligned} &\|\big(C_\phi\big(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|}\big)C_\psi + C_\psi\big(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|}\big)C_\phi\big)(\chi_n)\|^2 \\ =& \|C_\psi\|C_\phi(\chi_m) + \frac{1}{\|C_\psi\|} < C_\phi(\chi_m), C_\psi(\chi_n) > C_\psi(\chi_m), \end{aligned}$$

$$\begin{aligned} &\|C_\psi\|C_\phi(\chi_m) + \frac{1}{\|C_\psi\|} < C_\phi(\chi_m), C_\psi(\chi_n) > C_\psi(\chi_m) \\ =& \|C_\psi\|^2 < C_\phi\chi_m, C_\phi\chi_m > + \overline{< C_\phi\chi_m, C_\psi\chi_n >} < C_\phi\chi_m, C_\psi\chi_m > \\ &+ < C_\phi(\chi_m), C_\psi(\chi_n) > < C_\psi(\chi_m), C_\phi(\chi_m) > \\ &+ \frac{1}{\|C_\psi\|^2} | < C_\phi(\chi_m), C_\psi(\chi_n) > |^2 \ < C_\psi(\chi_m), C_\psi(\chi_n) > \\ =& \|C_\psi\|^2\|C_\phi\|^2 + \overline{\overline{(A_m \cap B_n)}}\ (\overline{\overline{A_m \cap B_m}}) \\ &+ \overline{\overline{(A_m \cap B_n)}}\ (\overline{\overline{A_m \cap B_m}}) + \frac{1}{\|C_\psi\|^2}(\overline{\overline{A_m \cap B_n}})^2(\overline{\overline{B_m \cap B_n}}) \end{aligned}$$

here $A_m = \phi^{-1}(m), B_m = \psi^{-1}(m)$.

Clearly $\overline{\overline{(A_m \cap B_n)}}\ \overline{\overline{(A_m \cap B_m)}} + \overline{\overline{(A_m \cap B_n)}}\ \overline{\overline{(A_m \cap B_n)}} + \frac{1}{\|C_\psi\|^2}(\overline{\overline{A_m \cap B_n}})(\overline{\overline{B_m \cap B_n}}) \geq 0$.

Thus $\|U_{C_\phi,C_\psi}(\chi_m \otimes \frac{C_\psi(\chi_n)}{\|C_\psi\|})(\chi_n)\| \geq \|C_\phi\|\|C_\psi\|$.

Therefore $\|U_{C_\phi,C_\psi}\| \geq \|C_\phi\|\|C_\psi\|$ □

Examples

2.1 Let ϕ be a function on $\mathbb{N}$ into itself defined by

$$\phi(n) = \begin{cases} 3 & n = 1, 2 \\ n+3 & n \neq 1, 2 \end{cases}$$

Then $\|C_\phi(\chi_3)\| = \|C_\phi\| = \sqrt{2}$ but $3 \notin A_3$. Therefore $0 \in W_0(C_\phi)$.

2.2 2 Let ϕbe a function on $\mathbb{N}$ into itself defined by $\phi(n) = n + 1$. Then ϕ is one-one and $\|C_\phi\| = 1$. In this case $[0, 1] \subseteq W_0(C_\phi)$.

2.3 Let ϕ be a function on $\mathbb{N}$ into itself defined by

$$\phi(n) = \begin{cases} 1 & n = 1, 2 \\ n + 3 & n \neq 1, 2 \end{cases}$$

Then $\|C_\phi(\chi_1)\| = \|C_\phi\| = \sqrt{2}$ but $1 \in A_1$. Therefore $0 \notin W_0(C_\phi)$.

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