Orbital Continuity and Common Fixed Point Theorems

Bharti Joshi

D. S. B. Campus, Kumaun University,
Nainital, India-263002
bhartijoshi20592@gmail.com.

Abstract

The present paper aims to show the significance of the notions of absorbing maps and orbital continuity in common fixed point considerations. We prove that orbital continuity of a pair of absorbing self-mappings of a complete metric space is equivalent to the existence of a common fixed point in various settings. We also show that orbital continuity of only one mapping is not equivalent to the existence of a common fixed point of a pair self mappings satisfying contractive conditions.

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1 Introduction

In 1971 Ciric [1] introduced the notion of orbital continuity. If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$. Every continuous self-mapping is orbitally continuous but not conversely [1]. Shastri et al [15] defined the notion of orbital continuity for a pair of mappings. If f and g are self-mappings of a metric space (X, d) and if $\{x_n\}$ is a sequence in X such that $fx_n = gx_{n+1}, n = 0, 1, 2, \ldots$ then the set $O(x_0, f, g) = \{fx_n : n = 0, 1, 2, \ldots\}$ is called the (f, g)-orbit at x_0 and g (or f) is called (f, g)-orbitally continuous if $\lim_n fx_n = u$ implies $\lim_n gfx_n = gu$ (or $\lim_n fx_n = u$ implies $\lim_n ffxn = fu$). While continuity is a nice and desirable property of functions, discontinuities occur naturally in diverse biological, industrial and economic phenomena and many of these phenomena involve threshold operations which are discontinuous. For example, a neuron in a neutral net functions in this manner. Cromme and Diener [2] and Cromme [3] have proved results on approximate fixed points for discontinuous functions and have given applications of their results to neural nets, economic equilibria and analysis. Fixed point theorems for discontinuous mappings have found wide applications, for example application of such theorems in the study of neural networks with discontinuous activation function is presently a very active area of research (e. g. [11] and references therein). Fixed point theorems for contractive mappings that admit discontinuity at the fixed point have also found applications in neural networks with discontinuous activation functions (e.g.[10], [11], [16], and references therein).

We now give some definitions relevant to the present work.

Definition 1.1. *[8] Two self-mappings f and g of a metric space (X, d) are called R- weakly commuting if there exists some real number $R > 0$ such that* $d(fgx, gfx) \leq$

$Rd(fx, gx)$ for all x in X. The mappings f and g are called point-wise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ (see [9]).

The notion of point-wise R-weak commuting implies commutativity at coincidence points and is, therefore, equivalent to the notion of weak compatibility.

Definition 1.2. *[6] Two self-mappings f and g of a metric space (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X .*

Definition 1.3. *[13] Two self-mappings f and g of a metric space (X, d) are called R-weakly commuting of type A_g if there exists some real number $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X . Similarly, the self-mappings f and g are called R- weakly commuting of type A_f if there exists some real number $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X.*

Definition 1.4. *[14] Two self-mappings f and g of a metric space (X, d) are called g-compatible or f-compatible according as or $\lim_n d(ffx_n, gfx_n) = 0$ or $\lim_n d(fgx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.*

Definition 1.5. *[12] Two self-mappings f and g of a metric space (X, d) are called compatible of type (P) if $\lim_n d(ffx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.*

In analogy with this definition, we can define two self-mappings f and g of a metric space (X, d) to be R- weakly commuting of type (P) if there exists some real number $R > 0$ such that

$$d(ffx, ggx) \leq Rd(fx, gx) \ \forall \ x \in X.$$

In a recent work, Pant and Pant [7] introduced the following definitions:

Definition 1.6. *[7] Two self-mappings f and g of a metric space (X, d) are called semi R-commuting provided there exists $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$ or $d(fgx, gfx) \leq Rd(fx, gx)$ or $d(fgx, ggx) \leq Rd(fx, gx)$ or $d(ffx, ggx) \leq Rd(fx, gx)$ is true for the set $\{x \in X : fx, gx \in f(X) \cap g(X)\}$.*

Definition 1.7. *[7] Two self-mappings f and g of a metric space (X, d) are called semi α-compatible provided every sequence $\{x_n\}$ in X satisfying $fx_n, gx_n \in f(X) \cap g(X)$ and $\lim_n fx_n = \lim_n gx_n = t$ for some t in X satisfies $\lim_n d(ffx_n, gfx_n) = 0$ or $\lim_n d(fgx_n, gfx_n) = 0$ or $\lim_n d(fgx_n, ggx_n) = 0$ or $\lim_n d(ffx_n, ggx_n) = 0$.*

It is easy to see that semi R-commuting implies semi α-compatible. It is also obvious that mappings which are compatible or f-compatible or g-compatible or compatible of type (P) are semi α-compatible.

Definition 1.8. *[4] If f and $g(f \neq g)$ are two self-mappings of a metric space (X, d) then f is called g- absorbing if there exists some real number $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$ for all x in X. Similarly, g is called f- absorbing if there exists some real number $R > 0$ such that $d(fx, fgx) \leq Rd(fx, gx)$ for all x in X.*

Definition 1.9. *A pair (f, g) of self-mappings of a metric space (X, d) will be called a pair of R-commuting category if f and g are either $R-$weak commuting or R-weak commuting of type A_f or R-weak commuting of type A_g or R-weak commuting of type (P).*

In a recent work, Pant and Pant [7] proved the following result:

Theorem 1.10. *Let f and g be R-weakly commuting self-mappings of type A_f or of type A_g of a complete metric space (X, d) such that $fX \subseteq gX$ and*

(i) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

Then f and g have a common fixed point if and only if f and g are (f, g)-orbitally continuous.

The aim of the present work is to show that for absorbing mapping pairs the above theorem holds irrespective of the commuting conditions. It is worth noting that in the above theorem orbital continuity of one of f or g implies orbital continuity of the other mapping and also the existence of the common fixed point. However, as we will see in Example 2.6 below, this does not hold under contractive conditions weaker than condition (i) of Theorem 1.10 and the assumption of orbital continuity of both f and g is required for the existence of the common fixed point under weaker contraction conditions.

2 Main Results

Theorem 2.1. *Let f and g be self-mappings of a complete metric space (X, d) such that $fX \subseteq gX$ and*

(ii) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

Suppose f is g-absorbing. Then f and g have a (unique) common fixed point if and only if f and g are (f, g)-orbitally continuous.

Proof: Let x_0 be any point in X. Define sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, \qquad n = 0, 1, 2, \ldots. \tag{2.1}$$

This can be done since $fX \subseteq gX$. Now using a standard argument and by virtue of (ii) it follows easily that $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to t$ as $n \to \infty$. Also, $\lim_n fx_n = t$ and $\lim_n gx_n = t$. Let us assume that f and g are orbitally continuous. Then

$$\lim_n fgx_n = \lim_n ffx_n = ft, \tag{2.2}$$

$$and, \qquad \lim_n ggx_n = \lim_n gfx_n = gt. \tag{2.3}$$

Since f is g-absorbing, we get $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$. This, in view of (2.2) and (2.3) implies that $t = gt$. Using (ii) we get

$$d(fx_n, ft) \leq hd(gx_n, gt).$$

On letting $n \to \infty$ this yields $d(t, ft) \leq hd(t, gt) = 0$, that is, $t = ft = gt$. Hence t is a common fixed point of f and g. Moreover, condition (ii) implies uniqueness of the common fixed point. Conversely, let us assume that the mappings f and g satisfy (ii) and posses a unique common fixed point, say z. Then $z = fz = gz$. Also, the (f, g)-orbit of any point x_0 defined by (2.1) converges to z, that is, $\lim_n fx_n = \lim_n gx_n = z$. Since f is g-absorbing, we have

$$d(gx_n, gfx_n) \leq Rd(fx_n, gx_n).$$

Making $n \to \infty$ the last inequality yields $\lim_{n\to\infty} gfx_n = z = fz = gz$. Since $fx_n = gx_{n+1}$, we have $\lim_{n\to\infty} ggx_n = z = fz = gz$. Therefore, g is (f, g)- orbitally continuous. Now, using (ii) we get

$$d(ffx_n, fz) \leq hd(gfx_n, gz).$$

Making $n \to \infty$ this yields $\lim_{n\to\infty} ffx_n = fz = z = gz$. This also yields $fgx_n = fz$. This shows that f is (f, g)-orbitally continuous. This establishes the theorem.

The following examples illustrate the above theorem.

Example 2.2. *Let $X = [0, \infty)$ and d be the usual metric. Define $f, g : X \to X$ by*

$$fx = \frac{x}{2} \quad \forall\, x \in X; \qquad gx = x \quad \forall\, x \in X.$$

Then it is easily seen that f and g satisfy all the conditions of the above theorem and have a unique common fixed point $x = 0$. The mappings f and g are commuting, R-weak commuting, R-weak commuting of type A_f, R-weak commuting of type A_g, as well as R-weak commuting of type (P).

Example 2.3. *Let $X = [2, 20]$ and d be the Euclidean metric. Define $f, g : X \to X$ by*

$$\begin{aligned} fx &= 2 \quad if\ x = 2\ or\ > 5, \qquad fx = 6 \quad if\ 2 < x \leq 5, \\ g2 &= 2, \qquad gx = 12 \quad if\ 2 < x \leq 5, \qquad gx = \frac{(x+1)}{3} \quad if\ x > 5. \end{aligned}$$

Then f and g satisfy all the conditions of Theorem 2.1 and have a unique common fixed point $x = 2$. The mappings f and g are R-weak commuting of type A_g since $d(ffx, gfx) \leq d(fx, gx)$ for all x in X. However, f and g are neither commuting nor R-weak commuting.

In the next theorem we give necessary and sufficient conditions for the existence of a unique common fixed point under an (ϵ, δ)- contractive condition. We also give an example to show that the theorem does not hold unless both f and g are orbitally continuous.

Theorem 2.4. *Let (f, g) be a pair self-mappings of R-commuting category of a complete metric space (X, d) such that $fX \subseteq gX$ and*

(iii) $d(fx, fy) < \max\{d(fx, gx), d(fy, gy)\}$, *whenever* $\max\{d(fx, gx), d(fy, gy)\} > 0$,

(iv) *given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that* $\epsilon < \max\{d(fx, gx), d(fy, gy)\} \leq \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$.

Suppose f is g- absorbing or g is f- absorbing. Then f and g have a (unique) common fixed point if and only if f and g are (f, g)- orbitally continuous.

Proof: Condition (iii) implies that

$$d(fx, fy) < \max\{d(fx, gx), d(fy, gy)\},$$

whenever $\max\{d(x, fx), d(y, fy)\} > 0$. Let x_0 be any point in X. As done in (2.1), define sequences $\{x_n\}$ and $\{y_n\}$ in X recursively by

$$y_n = fx_n = gx_{n+1},\ n = 0, 1, 2, \ldots.$$

If $y_n = y_{n+1}$ for some n then fx_n is easily seen to be a common fixed point of f and g. We can, therefore, assume that $y_n \neq y_{n+1}$ for each n. Then using (iii) we get

$$\begin{aligned} d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) &< \max\{d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1})\} \\ &= \max\{d(y_n, y_{n-1}), d(y_{n+1}, y_n)\} = d(y_{n-1}, y_n). \end{aligned}$$

Thus $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \geq 0$. Suppose $r > 0$. Then there exists a positive integer N such that

$$n \geq N \Rightarrow r < d(y_n, y_{n+1}) < r + \delta(r). \tag{2.4}$$

This yields $r < \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} = \max\{d(fx_{n+1}, gx_{n+1}), d(fx_{n+2}, gx_{n+2})\} < r+\delta(r)$ which by virtue of (iv) yields $d(fx_{n+1}, fx_{n+2}) = d(y_{n+1}, y_{n+2}) \leq r$. This contradicts (2.4). Hence $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. Now if p is any positive integer then

$$\begin{aligned} d(y_n, y_{n+p}) &= d(fx_n, fx_{n+p}) < \max\{d(fx_n, gx_n), d(fx_{n+p}, gx_{n+p})\} \\ &= \max\{d(y_{n-1}, y_n), d(y_{n+p-1}, y_{n+p})\} = d(y_{n-1}, y_n). \end{aligned}$$

This implies that $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists t in X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t. \tag{2.5}$$

Suppose that f and g are (f, g)-orbitally continuous. Then (2.5) implies that

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = ft, \tag{2.6}$$

$$and,\ \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gt. \tag{2.7}$$

Suppose f is g-absorbing. Then, there exists some real number $R > 0$ such that $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, in view of (2.5), (2.6) and (2.7), this implies that $t = gt$. We assert that $t = ft$. Since the pair of mappings (f, g) is of R-commuting category, there exists a real number $R_1 > 0$ such that for all x in X we have $d(ffx, gfx) \leq R_1d(fx, gx)$ or $d(fgx, gfx) \leq R_1d(fx, gx)$ or $d(fgx, ggx) \leq R_1d(fx, gx)$ or $d(ffx, ggx) \leq R_1d(fx, gx)$. Suppose $d(ffx, gfx) \leq R_1d(fx, gx)$ for all x in X. On letting $n \to \infty$ and by using (2.5), (2.6) and (2.7), this yields $d(ft, gt) \leq R_1d(t, t) = 0$. Hence $t = ft = gt$ and t is a common fixed point of f and g.

Next suppose that g is f- absorbing. Then, there exists some real number $R > 0$ such that $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, in view of (2.5), (2.6) and (2.7), this implies that $t = ft$. Since the pair of mappings (f, g) is of R-commuting category, there exists a real number $R_1 > 0$ such that for all x in X we have $d(ffx, gfx) \leq R_1 d(fx, gx)$ or $d(fgx, gfx) \leq R_1 d(fx, gx)$ or $d(fgx, ggx) \leq R_1 d(fx, gx)$ or $d(ffx, ggx) \leq R_1 d(fx, gx)$. Suppose $d(ffx, gfx) \leq R_1 d(fx, gx)$ for all x in X. By virtue of (2.5), (2.6) and (2.7) this implies $ft = gt$, that is, $t = ft = gt$. Therefore, t is a common fixed point of f and g. We arrive at the same conclusion if $d(fgx, gfx) \leq R_1 d(fx, gx)$ or $d(fgx, ggx) \leq R_1 d(fx, gx)$ or $d(ffx, ggx) \leq R_1 d(fx, gx)$. Moreover, condition (iii) implies uniqueness of the common fixed point.

Conversely, suppose that f and g satisfy (iii) and (iv) and possesses a unique common fixed point z, that is, $z = fz = gz$. Then for any x_0 in X, the sequence $\{y_n\}$ of (f, g)-iterates of x_0 defined by $y_n = fx_n = gx_{n+1}$ is a Cauchy sequence and

$$d(z, fx_n) = d(fz, fx_n) < max\{d(fz, gz), d(fx_n, gx_n)\} = d(fx_n, gx_n).$$

Taking limit as $n \to \infty$ the last inequality yields

$$\lim_n fx_n = z = \lim_n gx_n.$$

Thus, every sequence of iterates of the form $\{y_n : fx_n = gx_{n+1}\}$ converges to the fixed point z. Now, since the pair of mappings (f, g) is of R-commuting category, there exists a real number $R_1 > 0$ such that for all x in X we have $d(ffx, gfx) \leq R_1 d(fx, gx)$ or $d(fgx, gfx) \leq R_1 d(fx, gx)$ or $d(fgx, ggx) \leq R_1 d(fx, gx)$ or $d(ffx, ggx) \leq R_1 d(fx, gx)$. Suppose $d(ffx, gfx) \leq R_1 d(fx, gx)$ for all x in X. This implies

$$d(ffx_n, gfx_n) \leq R_1 d(fx_n, gx_n). \tag{2.8}$$

On letting $n \to \infty$ this implies $\lim_{n\to\infty} d(ffx_n, gfx_n) = 0$. Further, if f is g-absorbing then there exists a real number $R > 0$ such that $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$. On taking limit as $n \to \infty$ this yields $\lim_{n\to\infty} gfx_n = z$. By virtue of (2.8) we get $\lim_{n\to\infty} ffx_n = z$. Since $fx_n = gx_{n+1}$, these limits yield

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = z = gz; \lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = z = fz.$$

Therefore, both f and g are (f, g)-orbitally continuous. Similarly, it can be proved that f and g are (f, g)-orbitally continuous if g is assumed f-absorbing. This establishes the theorem.

The next two examples illustrate Theorem 2.4.

Example 2.5. *Let $X = [0, \infty)$ equipped with the usual metric and let $f, g : X \to X$ be defined by*

$$fx = \frac{x}{3}, \qquad gx = x$$

for each x in X. Then it easy to verify that X is complete, f and g are continuous commuting mappings that satisfy (iii) and (iv) and have a unique common fixed point $x = 0$. The mapping f is g-absorbing and g is f-absorbing.

Example 2.6. *Let $X = [0, 2]$ and d be the usual metric. Define $f, g : X \to X$ by*

$$fx = 1 \quad if\ 0 \le x \le 1, \qquad fx = 0 \quad if\ 1 < x \le 2,$$

$$gx = x \quad for\, each\ x.$$

Then f and g satisfy all the conditions of the above theorem and have a unique common fixed point $x = 1$. The mapping f is discontinuous at the common fixed point. However, f is orbitally continuous. It is easy to verify that f is g- absorbing and g if f- absorbing. It can also be easily verified that

$$\begin{aligned} d(fx, fy) &= 0, 0 < \max\{d(fx, gx), d(fy, gy)\} \le 1\ if\ x, y \le 1, \\ d(fx, fy) &= 0, 1 < \max\{d(fx, gx), d(fy, gy)\} \le 2\ if\ x, y > 1, \\ and,\ d(fx, fy) &= 1, 1 < \max\{d(fx, gx), d(fy, gy)\} \le 2\ if\ x \le 1, y > 1. \end{aligned}$$

Therefore, f and g satisfy condition (iv) with $\delta(\epsilon) = 1 - \epsilon$ if $\epsilon < 1$ and $\delta(\epsilon) = 1$ for $\epsilon \ge 1$.

The next example shows that the condition of orbital continuity of both f and g cannot be dropped in the above theorem.

Example 2.7. *Let $X = [0, 2]$ and d be the usual metric. Define $f : X \to X$ by*

$$\begin{aligned} fx &= \frac{(1 + x)}{2}\ if\ 0 \le x < 1, \qquad fx = 0\ if\ 1 \le x \le 2, \\ gx &= x\ for\, each\ x. \end{aligned}$$

Then X is complete and f and g satisfy the contractive conditions (iii) and (iv) with $\delta(\epsilon) = \frac{(1-\epsilon)}{2}$ for $\epsilon < 1$ and $\delta(\epsilon) = 1$ for $\epsilon \ge 1$ but f does not have a fixed point. It is easy to show that f is g- absorbing and g is f- absorbing. The mapping f is not orbitally continuous since for any x_0 in X and the sequence of iterates $\{fx_n = gx_{n+1}\}$ we have $\lim_{n\to\infty} fx_n = 1$ and $\lim_{n\to\infty} ffx_n = 1 \ne f1$.

Remark 1. *The contraction condition (ii) pertaining to a pair of mappings employed in Theorem 2.1 above was introduced by Jungck [5] and is often referred to as Jungck contraction condition.*

In Theorems 2.1 and 2.4 f is assumed g- absorbing or g is assumed f-absorbing. However, there exist pairs of self-mappings (f, g) of a metric space (X, d) such that f is g-absorbing on a subset Y of X and g is f-absorbing on the complement of Y. Examples of such mappings include semi R- commuting mapping pairs. We now generalise the notion of absorbing mappings to study such cases. The new notion is very useful for studying necessary and sufficient conditions for the existence of common fixed points of semi R-commuting mappings. We prove that under the new notion orbital continuity of f and g is equivalent to the existence of a common fixed point of semi R-commuting mappings f and g.

Definition 2.8. *Two self-mappings f and g of a metric space (X, d) will be called weak absorbing if there exists a real number $R > 0$ such that given $x \in X, d(gx, gfx) \le Rd(fx, gx)$ or $d(fx, fgx) \le Rd(fx, gx)$.*

Recently, Pant and Pant [7] proved the following:

Theorem 2.9. *[7] Let f and g be orbitally continuous self-mappings of a complete metric space (X, d) such that $fX \subseteq gX$ and*

(v) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

If f and g are semi R-commuting then f and g have a coincidence point which is their unique common fixed point.

We now prove that in Theorem 2.9 if (f, g) is a weak absorbing pair then existence of common fixed point implies orbital continuity of f and g, that is, the converse of Theorem 2.9 holds for weak absorbing pairs.

Theorem 2.10. *Let f and g be semi R-commuting self-mappings of a complete metric space (X, d) such that $fX \subseteq gX$ and*

(vi) $d(fx, fy) \leq hd(gx, gy), 0 \leq h < 1.$

If f and g are (f, g)-orbitally continuous then f and g have a (unique) common fixed point. Further, if f and g are weak absorbing and possess a fixed point then f and g are (f, g)-orbitally continuous.

Proof: The first part of this theorem is the same as Theorem 2.9. Therefore, we prove only the second part of the theorem. Suppose that f and g are weak absorbing self-mappings which satisfy (vi) and have a unique common fixed point, say z. Then $z = fz = gz$. Let x_0 be any point in X. Define sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, n = 0, 1, 2, \ldots,$$

as defined in (2.1). Then using the argument employed in Theorem 2.10 it follows that $\{y_n\}$ is a Cauchy sequence which converges to some point, say t, in X. Using (vi) we get

$$d(z, fx_n) = d(fz, fx_n) \leq hd(gz, gx_n) = hd(z, gx_n).$$

On taking limit as $n \to \infty$ the above inequality yields $t = z$. Thus for any x_0 in X we have

$$\lim_n fx_n = \lim_n gx_n = z = fz = gz.$$

Since f is semi R-commuting and $fx_n, gx_n \in f(X) \cap g(X)$, there exists $R_1 > 0$ such that $d(ffx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$ or $d(fgx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$ or $d(fgx_n, ggx_n) \leq R_1 d(fx_n, gx_n)$ or $d(ffx_n, ggx_n) \leq R_1 d(fx_n, gx_n)$. Suppose $d(ffx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$ holds. Taking limit as $n \to \infty$ this implies

$$\lim_n d(ffx_n, fgx_n) = 0. \tag{2.9}$$

Since f and g are weak absorbing, the sequence of iterates $\{x_n\}$ splits up in two subsequence $\{u_m\}$ and $\{v_k\}$ such that $d(fu_m, fgu_m) \leq Rd(fu_m, gu_m)$ and $d(gv_k, gfv_k) \leq Rd(fv_k, gv_k)$. On taking limit as $m \to \infty, k \to \infty$ these inequalities yield

$$\lim_m fgu_m = \lim_k gfv_k = z = fz = gz.$$

Since $fx_n = gx_{n+1}$, the above limits are equivalent to

$$\lim_m ffu_m = \lim_k ggv_k = z = fz = gz.$$

Combining these limits with (2.9) we get

$$\lim_n ffx_n = \lim_n fgx_n = z = fz = gz,$$

$$and \quad \lim_n gfx_n = \lim_n ggx_n = z = fz = gz.$$

Therefore, f and g are (f, g)-orbitally continuous. We get the same conclusion if we assume $d(fgx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$ or $d(fgx_n, ggx_n) \leq R_1 d(fx_n, gx_n)$ or $d(ffx_n, ggx_n) \leq R_1 d(fx_n, gx_n)$ in place of assuming
$d(ffx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$. This completes the proof.

The next example illustrates Theorem 2.10.

Example 2.11. *Let $X = [0, 11]$ and d be the Euclidean metric. Define $f, g : X \to X$ by*

$$fx = \frac{(6-x)}{2} \; if \; x \leq 2, \qquad fx = 3 \; if \; 2 < x \leq 5, \qquad fx = 2 \; if \; x > 5,$$

$$gx = x \; if \; x \leq 2, \qquad gx = 10 \; if \; 2 < x \leq 5, \qquad gx = \frac{(x+1)}{3} \; if \; x > 5.$$

Then f and g satisfy all the conditions of Theorem 2.10 and have a unique common fixed point $x = 2$. It can be seen in this example that $f(X) \cap g(X) = [2, 3], \{x : fx, gx \in f(X) \cap g(X)\} = \{2\} \cup (5, 8]$ and $d(ffx, gfx) \leq d(fx, gx)$ whenever $fx, gx \in f(X) \cap g(X)$. Therefore the mappings f and g are semi R- commuting with $R = 1$. Further, (f, g) is a weak absorbing pair since $d(fx, fgx) \leq d(fx, gx)$ if $x < 2$ and $d(gx, gfx) \leq d(fx, gx)$ if $x \geq 2$. It can also be verified that f and g satisfy the contractive condition $d(fx, fy) \leq \frac{1}{2} d(gx, gy)$ for all x, y in X. f and g are orbitally continuous mappings though neither f is continuous nor g is continuous. It may be seen in this example that f and g are neither R-weakly commuting, nor R-weakly commuting of type A_f nor R-weakly commuting of type A_g. Also, f and g are neither compatible, nor f-compatible, nor g-compatible nor compatible of type (P).

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