Further Results On Growth Of Composite Entire and Meromorphic Functions.

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Abstract

In this paper we study growth properties of composite functions formed from entire and meromorphic functions and their derivatives to generalise some earlier results.

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1 Introduction and Definitions

Let f and g be two transcendental entire functions in the open complex plane $\mathbb{C}$. Clunie [4] proved that $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$. In [13], Singh investigated some comparative growth properties of $\log T(r,f\circ g)$ and $T(r,f)$ and raised the question for comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$. Later, some results on comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ are closely investigated in [8] and also in [3].
If $f(z)$ be a non-constant meromorphic function in the finite complex plane and $a_0(z),a_1(z),...,a_l(z)$ are all small functions of $f(z)$ i.e., $T(r,a_i(z))=o\{T(r,f)\}; i=0,1,...,l.$, then in [1], Banerjee and Adhikary defined the meromorphic function $\Psi(z)$ as

$$\Psi(z)=\Sigma_{i=0}^{l}a_i(z)f^{(i)}(z) \tag{1.1}$$

where $f^{(i)}$ is the i-th derivative of $f(z)$.
Similarly one can define an entire function $\phi(z)$ as

$$\phi(z)=\Sigma_{i=0}^{m}b_i(z)g^{(i)}(z) \tag{1.2}$$

where $g(z)$ be a non-constant entire function in the finite complex plane and $b_0(z),b_1(z),...,b_m(z)$ are all small functions of $g(z)$.
Recently [1], Banerjee and Adhikary investigated some comparative growth properties of $\log T(r,\psi\circ g)$ and $T(r,g)$ to generalise some results of Lahiri and Sharma [9]. In the present paper it therefore seems reasonable to study some comparative growth properties

of $\log T(r, \psi \circ \phi)$ and $T(r, g)$ to generalise the results of Banerjee and Adhikary.
To prove our main results we need some basic definitions and standard notations of Nevanlinna theory [6] which we shall frequently use throughout the paper.

Definition 1. *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is an entire function, then since from [6] for all large values of r, $T(r,f) \leq \log M(r,f) \leq 3T(2r,f)$, so we can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2. *The hyper order $\overline{\rho_f}$ and hyper lower order $\overline{\lambda_f}$ of a meromorphic function f are defined as*

$$\overline{\rho_f} = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}$$

and

$$\overline{\lambda_f} = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\overline{\rho_f} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\overline{\lambda_f} = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

The definition of hyper order and hyper lower order $\overline{\rho_f}$ and $\overline{\lambda_f}$ are meaningful only if the order $\rho_f = \infty$.

Definition 3. *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f if*
(i)$\lambda_f(r)$ is non-negative and continuous for $r \geq r_0$ say;
(ii)$\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda'_f(r-0)$

and $\lambda_f^{'}(r+0)$ *exist;*
(iii)$\lim_{r\to\infty}\lambda_f(r)=\lambda_f<\infty$;
(iv)$\lim_{r\to\infty} r\lambda_f^{'}(r)\log r=0$; *and*
(v)$\liminf_{r\to\infty}\frac{T(r,f)}{r^{\lambda_f(r)}}=1.$

Proposition 1. *[10]. For* δ (>0) *the function* $r^{\lambda_f+\delta-\lambda_f(r)}$ *is an increasing function of* r*.*

Since $\frac{d}{dr}r^{\lambda_f+\delta-\lambda_f(r)}=\{\lambda_f+\delta-\lambda_f(r)-r\lambda_f^{'}(r)\log r\}r^{\lambda_f+\delta-\lambda_f(r)-1}>0,$
for all sufficiently large values of r.

2 Lemmas

In this section we present some known results in the form of lemmas which will be needed in the sequel.

Lemma 1. *[11]. Let* $f(z)$ *be an entire function of finite lower order. If there exist entire functions* $b_i(i=1,2,...,n;n\leq\infty)$ *satisfying* $T(r,b_i)=o\{T(r,f)\}$ *and* $\sum_{i=1}^{n}\delta(b_i,f)=1$*, then*

$$\lim_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=\frac{1}{\pi}.$$

Lemma 2. *[2]. If* $f(z)$ *is meromorphic and* $g(z)$ *is entire then for all large values of* r

$$T(r,f\circ g)\leq\{(1+o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f).$$

Lemma 3. *[9]. Let* $f(z)$ *be meromorphic function and* $g(z)$ *be entire with* $0<\mu<\rho_g<\infty$ *and* $\lambda_f>0$*. Then for a sequence of values of* r *tending to infinity*

$$T(r,f\circ g)\geq T(\exp(r)^{\mu},g).$$

Lemma 4. *[5]. Let* $f(z)$ *be a meromorphic function of finite order and* $g(z)$ *be an entire function with* $0<\lambda_g<\mu<\infty$ *. Then for a sequence of values of* r *tending to infinity*

$$T(r,f\circ g)\leq T(\exp(r)^{\mu},g).$$

Lemma 5. *[7]. If* $f(z)$ *be an entire function then for* $r>0$

$$\frac{M(r,f)}{2r}\leq M(r,f^{'})\leq\frac{M(2r,f)}{r}.$$

In particular for all large values of r

$$T(r,f^{'})\leq\log M(r,f^{'})\leq\log M(2r,f)\leq 3T(4r,f).$$

3 Main Results

In this section we present the main results of the paper where throughout we assume $\Psi(z)$ and $\phi(z)$ are functions defined by (1.1) and (1.2) respectively.

Theorem 3.1. *Let $f(z)$ be a non-constant meromorphic function and $g(z)$ be an entire function. Then*

$$\liminf_{r\to\infty} \frac{logT(r,\psi\circ\phi)}{T(r,g)} \leq 3\rho_f(3m+1)(2^{m+2})^{\lambda_g}.$$

Proof. If $\rho_f = \infty$ then the theorem is obvious. So we suppose that $\rho_f < \infty$. We have for all large values of r and arbitrary $\epsilon\ (>0)$ from Lemma 2

$$\begin{aligned} T(r,\Psi\circ\phi) &\leq \{1+o(1)\}\, T(M(r,\phi),\Psi) \\ &\leq \{1+o(1)\}\,[T(M(r,\phi),a_0 f)+T(M(r,\phi),a_1 f^{(1)})+...+T(M(r,\phi),a_l f^{(l)})]+O(1) \\ &\leq \{1+o(1)\}\,[\{T(M(r,\phi),a_0)+T(M(r,\phi),f)\}+\{T(M(r,\phi),a_1)+T(M(r,\phi),f^{(1)})\} \\ &+ ...+\{T(M(r,\phi),a_l)+T(M(r,\phi),f^{(l)})\}]+O(1) \\ &\leq \{1+o(1)\}\,[o\,\{T(M(r,\phi),f)\}+T(M(r,\phi),f)+o\,\{T(M(r,\phi),f)\}+T(M(r,\phi),f^{(1)}) \\ &+ ...+o\,\{T(M(r,\phi),f)\}+T(M(r,\phi),f^{(l)})]+O(1) \\ (3.1) &\leq \{1+o(1)\}\,[T(M(r,\phi),f)+T(M(r,\phi),f^{(1)})+...+T(M(r,\phi),f^{(l)})]+O(1). \end{aligned}$$

Now

$$\begin{aligned} M(r,\phi) &\leq M(r,b_0 g)+M(r,b_1 g^{(1)})+\cdots+M(r,b_m g^{(m)}) \\ &\leq M(r,b_0)M(r,g)+M(r,b_1)M(r,g^{(1)})+\cdots+M(r,b_m)M(r,g^{(m)}). \end{aligned}$$

Therefore for large values of r

$$\begin{aligned} \log M(r,\phi) &\leq \log M(r,b_0)+\log M(r,g)+\log M(r,b_1)+\log M(r,g^{(1)})+\cdots \\ &+ \log M(r,b_m)+\log M(r,g^{(m)}) \\ &\leq 3T(2r,b_0)+3T(2r,g)+3T(2r,b_1)+3T(2r,g^{(1)})+\cdots+3T(2r,b_m)+3T(2r,g^{(m)}) \\ (3.2) &\leq 3\{1+o(1)\}[T(2r,g)+T(2r,g^{(1)})+\cdots+T(2r,g^{(m)})]. \end{aligned}$$

Now, from (3.1) we get

$$\begin{aligned} T(r,\Psi\circ\phi) &\leq \{1+o(1)\}\,[\{M(r,\phi)\}^{\rho_f+\epsilon}+\{M(r,\phi)\}^{\rho_f+\epsilon}+...+\{M(r,\phi)\}^{\rho_f+\epsilon}]+O(1) \\ &\leq \{M(r,\phi)\}^{\rho_f+\epsilon}O(1). \end{aligned}$$

Taking logarithm on both sides we get

$$\begin{aligned} \log T(r,\Psi\circ\phi) &\leq (\rho_f+\epsilon)\log M(r,\phi)+O(1) \\ (3.3) \qquad &\leq 3(\rho_f+\epsilon)\{1+o(1)\}[T(2r,g)+T(2r,g^{(1)})+\cdots+T(2r,g^{(m)})]+O(1) \end{aligned}$$

for large of values of r.
Since $\liminf_{r\to\infty} \frac{T(r,g)}{r^{\lambda_g(r)}} = 1$, so for given ϵ $(0 < \epsilon < 1)$ we get for a sequence of values of r tending to infinity

$$T(r,g) < (1+\epsilon) r^{\lambda_g(r)}$$

and for all large values of r

$$T(r,g) > (1-\epsilon) r^{\lambda_g(r)}.$$

So,

$$\begin{aligned}
\frac{\log T(r,\Psi\circ\phi)}{T(r,g)} &\le \frac{3(\rho_f+\epsilon)\{1+o(1)\}[T(2r,g)+T(2r,g^{(1)})+\cdots+T(2r,g^{(m)})]+O(1)}{T(r,g)}\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}[\frac{T(2r,g)}{T(r,g)}+\frac{T(2r,g^{(1)})}{T(r,g)}+\cdots+\frac{T(2r,g^{(m)})}{T(r,g)}]+O(1)\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}[\frac{T(2r,g)}{T(r,g)}+\frac{3T(8r,g)}{T(r,g)}+\cdots+\frac{3T(2^{m+2}r,g)}{T(r,g)}]+O(1) \quad \text{(from Lemma 5)}\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}(3m+1)\frac{T(2^{m+2}r,g)}{T(r,g)}+O(1)\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}(3m+1)\frac{T(2^{m+2}r,g)}{T(r,g)}+O(1)\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}(3m+1)\frac{(1+\epsilon)(2^{m+2}r)^{\lambda_g(2^{m+2}r)}}{(1-\epsilon)r^{\lambda_g(r)}}+O(1)\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}(3m+1)\frac{(1+\epsilon)(2^{m+2}r)^{\lambda_g+\delta}}{(1-\epsilon)(2^{m+2}r)^{\lambda_g+\delta-\lambda_g(2^{m+2}r)}r^{\lambda_g(r)}}+O(1)\\
&\le 3(\rho_f+\epsilon)\{1+o(1)\}(3m+1)\frac{(1+\epsilon)}{(1-\epsilon)}(2^{m+2})^{\lambda_g+\delta}+O(1)
\end{aligned}$$

since $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r, by Proposition 1.1.
So,

$$\liminf_{r\to\infty}\frac{\log T(r,\Psi\circ\phi)}{T(r,g)} \le 3\rho_f(3m+1)(2^{m+2})^{\lambda_g}.$$

Remark 1. If for $\Psi(z)$ in (1.1), $a_i(z)$ $(i=0,1,2,\ldots,l)$ are meromorphic functions of order zero instead of $T(r,a_i(z)) = o\{T(r,f)\}$ then we have

$$\begin{aligned}
T(r,\Psi\circ\phi) &\le \{1+o(1)\}[\{T(M(r,\phi),a_0)+T(M(r,\phi),f)\}+\{T(M(r,\phi),a_1)+T(M(r,g),f^{(1)})\}+\ldots\\
&+\{T(M(r,\phi),a_l)+T(M(r,\phi),f^{(l)})\}]+O(1)\\
&\le \{1+o(1)\}[\{(M(r,\phi))^{\rho_f+\epsilon}+(M(r,\phi))^{\epsilon}\}+\{(M(r,\phi))^{\rho_{f^{(1)}}+\epsilon}+(M(r,\phi))^{\epsilon}\}+\ldots\\
&+\{(M(r,\phi))^{\rho_{f^{(l)}}+\epsilon}+(M(r,\phi))^{\epsilon}\}]+O(1)\\
&= \{1+o(1)\}(l+1)[\{M(r,\phi)\}^{\rho_f+\epsilon}+\{M(r,\phi)\}^{\epsilon}]+O(1)\\
&\le \{1+o(1)\}(l+1)[\{M(r,\phi)\}^{\rho_f+2\epsilon}]+O(1),
\end{aligned}$$

for all large values of r.
Therefore, $\log T(r,\Psi\circ\phi)\le(\rho_f+2\epsilon)\log M(r,\phi)+O(1)$
and finally we have the same result.

Theorem 3.2. *Let $f(z)$ be a non-constant meromorphic function in the finite complex plane and $g(z)$ be an entire function. Also let there exist entire functions $c_i(i=1,2\ldots,n;n\le\infty)$ such that $T(r,c_i)=o\{T(r,g)\}$ with $\Sigma_{i=1}^{n}\delta(c_i,g)=1$. Then*

$$\limsup_{r\to\infty}\frac{\log T(r,\Psi\circ\phi)}{T(2^{m+1}r,g)}\le 3(m+1)\pi\rho_f.$$

Proof. For large values of r and arbitrary ϵ (>0) we get from (3.2)

$$\begin{aligned}\log M(r,\phi)&\le 3\{1+o(1)\}[T(2r,g)+T(2r,g^{(1)})+\cdots+T(2r,g^{(m)})]\\&\le 3\{1+o(1)\}[\log M(2r,g)+\log M(2r,g^{(1)})+\cdots+\log M(2r,g^{(m)})]\\&\le 3\{1+o(1)\}[\log M(2r,g)+\log M(4r,g)+\cdots+\log M(2^{(m+1)}r,g)]+O(1)\quad\text{(using Lemma 5)}\\&\le 3\{1+o(1)\}(m+1)\log M(2^{(m+1)}r,g)+O(1).\end{aligned}$$

Now from (3.3) we get

$$\log T(r,\Psi\circ\phi)\le 3(\rho_f+\epsilon)\{1+o(1)\}(m+1)\log M(2^{(m+1)}r,g)+O(1).$$

i.e.,

$$\frac{\log T(r,\Psi\circ\phi)}{T(2^{(m+1)}r,g)}\le\frac{3(\rho_f+\epsilon)\{1+o(1)\}(m+1)\log M(2^{(m+1)}r,g)+O(1)}{T(2^{(m+1)}r,g)}.$$

Since ϵ (>0) was arbitrary, so we have by using Lemma 1

$$\limsup_{r\to\infty}\frac{\log T(r,\Psi\circ\phi)}{T(2^{(m+1)}r,g)}\le 3(m+1)\pi\rho_f.$$

Remark 2. If for $\Psi(z)$ in (1.1), $a_i(z)$ $(i=0,1,2,\ldots,l)$ are meromorphic functions of order zero instead of $T(r,a_i(z))=o\{T(r,f)\}$ then we have the same result.

Theorem 3.3. *Let $f(z)$ be a non-constant meromorphic function of finite order in the finite complex plane and $\Psi(z)$ is of positive lower order and $g(z)$ be an entire function such that $0<\lambda_g\le\rho_g<\infty$. Then*

$$(i)\ \liminf_{r\to\infty}\frac{\log^{[2]}T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)}\le 2^{\mu(m+1)}\frac{\overline{\rho_g}}{\overline{\lambda_g}}\ \textit{when}\ 0<\lambda_\phi<\mu<\infty;$$
$$\textit{and}\ (ii)\ \limsup_{r\to\infty}\frac{\log^{[2]}T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)}\ge\frac{\overline{\lambda_g}}{\rho_g}\ \textit{when}\ 0<\mu<\rho_\phi<\infty.$$

Proof. (i) Suppose that $0 < \lambda_\phi < \mu < \infty$. Then for a sequence of values of r tending to infinity we get from Lemma 4

$$\begin{aligned}
T(r, \Psi \circ \phi) &\leq T(\exp(r)^\mu, \phi) \\
&\leq T(\exp(r)^\mu, b_0) + T(\exp(r)^\mu, g) + T(\exp(r)^\mu, b_1) + T(\exp(r)^\mu, g^{(1)}) + \cdots \\
&+ T(\exp(r)^\mu, b_m) + T(\exp(r)^\mu, g^{(m)}) \\
&\leq \{1 + o(1)\}[T(\exp(r)^\mu, g) + T(\exp(r)^\mu, g^{(1)}) + \cdots + T(\exp(r)^\mu, g^{(m)})] \\
&\leq \{1 + o(1)\}[\log M(\exp(r)^\mu, g) + \log M(\exp(r)^\mu, g^{(1)}) + \cdots + \log M(\exp(r)^\mu, g^{(m)})] \\
&\leq \{1 + o(1)\}[\log M(\exp(r)^\mu, g) + \log M(\exp(2r)^\mu, g) + \cdots \\
&+ \log M(\exp(2^{(m)} r)^\mu, g)] + O(1) \\
&\leq \{1 + o(1)\}(m+1)[\log M(\exp(2^{(m)} r)^\mu, g)] + O(1) \\
&\leq \{1 + o(1)\}(m+1)[T(\exp(2^{(m+1)} r)^\mu, g)] + O(1)
\end{aligned}$$

$$\begin{aligned}
i.e., \quad \frac{\log^{[2]} T(r, \Psi \circ \phi)}{\log T(\exp(r)^\mu, g)} &\leq \frac{\log^{[2]} T((\exp(2^{(m+1)} r)^\mu, g)}{\log T(\exp(r)^\mu, g)} + O(1) \\
&\leq \left[\frac{\log^{[2]} T(\exp(2^{m+1} r)^\mu, g)}{\log(\exp(2^{m+1} r)^\mu)} \frac{\log(\exp(2^{m+1} r)^\mu)}{\log T(\exp(r)^\mu, g)} \right] + O(1) \\
&\leq \left[2^{(m+1)\mu} \frac{\log^{[2]} T(\exp(2^{m+1} r)^\mu, g)}{\log(\exp(2^{m+1} r)^\mu)} \frac{\log(\exp(r)^\mu)}{\log T(\exp(r)^\mu, g)} \right] + O(1).
\end{aligned}$$

Also for ϵ (> 0), we see that for all large values of r

$$log^{[2]} T(\exp(2^{m+1} r)^\mu, g) < (\overline{\rho}_g + \epsilon) log(\exp(2^{m+1} r)^\mu)$$

and

$$\log T(\exp(r)^\mu, g) > (\lambda_g - \epsilon) \log(\exp(r)^\mu)$$

So,

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, \Psi \circ \phi)}{\log T(\exp(r)^\mu, g)} \leq 2^{(m+1)\mu} \frac{\overline{\rho}_g + \epsilon}{\lambda_g - \epsilon}.$$

Since ϵ (> 0) is arbitrary, so (i) holds.

(ii) Next suppose that $0 < \mu < \rho_\phi < \infty$. Then we have from Lemma 3 for a sequence of values of r tending to infinity

$$\begin{aligned}
T(r, \Psi \circ \phi) &\geq T(\exp(r)^\mu, \phi) \\
&= T(\exp(r)^\mu, b_0 g + b_1 g^{(1)} + \cdots + b_m g^{(m)}) \\
&\geq T(\exp(r)^\mu, b_0 g) - T(\exp(r)^\mu, b_1 g^{(1)} + \cdots + b_m g^{(m)}) \\
&\geq T(\exp(r)^\mu, b_0 g) - T(\exp(r)^\mu, b_1 g^{(1)}) - \cdots - T(\exp(r)^\mu, b_m g^{(m)}) \\
&\geq T(\exp(r)^\mu, b_0 g) + O(1) \\
&\geq T(\exp(r)^\mu, g) + O(1).
\end{aligned}$$

$$i.e., \quad \frac{\log^{[2]} T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \geq \frac{\log^{[2]} T(\exp(r)^{\mu},g)}{\log T(\exp(r)^{\mu},g)}$$
$$\geq \left[\frac{\log^{[2]} T(\exp(r)^{\mu},g)}{\log(\exp(r)^{\mu})}\frac{\log(\exp(r)^{\mu})}{\log T(\exp(r)^{\mu},g)}\right].$$

Also for ϵ (>0), we see that for all large values of r

$$\log^{[2]} T(\exp(r)^{\mu},g) > (\overline{\lambda}_g-\epsilon)\log(\exp(r)^{\mu})$$

and

$$\log T(\exp(r)^{\mu},g) < (\rho_g+\epsilon)\log(\exp(r)^{\mu})$$

So,

$$\limsup_{r\to\infty}\frac{log^{[2]}T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \geq \frac{\overline{\lambda}_g-\epsilon}{\rho_g+\epsilon}.$$

Since ϵ (>0) is arbitrary, so (ii) holds.

Theorem 3.4. *Under the hypothesis of Theorem 3.3 we have*

$$(i)\liminf_{r\to\infty}\frac{\log T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \leq 2^{(m+1)\mu}\frac{\rho_g}{\lambda_g},\ when\ 0<\lambda_{\phi}<\mu<\infty\ ;$$
$$and\ (ii)\limsup_{r\to\infty}\frac{\log T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \geq \frac{\lambda_g}{\rho_g},\ when\ 0<\mu<\rho_{\phi}<\infty.$$

Proof. (i) Suppose that $0<\lambda_{\phi}<\mu<\infty$. Then for a sequence of values of r tending to infinity we get from Lemma 4 and from above result

$$T(r,\Psi\circ\phi) \leq T(\exp(r)^{\mu},\phi) \leq \{1+0(1)\}(m+1)[T(\exp(2^{(m+1)}r)^{\mu},g)]+O(1).$$

$$i.e., \quad \frac{\log T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \leq \frac{\log T((\exp(2^{(m+1)}r)^{\mu},g)}{\log T(\exp(r)^{\mu},g)}+O(1)$$
$$\leq \left[\frac{\log T(\exp(2^{m+1}r)^{\mu},g)}{\log(\exp(2^{m+1}r)^{\mu})}\frac{\log(\exp(2^{m+1}r)^{\mu})}{\log T(\exp(r)^{\mu},g)}\right]+O(1)$$
$$\leq \left[2^{(m+1)\mu}\frac{\log T(\exp(2^{m+1}r)^{\mu},g)}{\log(\exp(2^{m+1}r)^{\mu})}\frac{\log(\exp(r)^{\mu})}{\log T(\exp(r)^{\mu},g)}\right]+O(1).$$

Since ϵ (>0) is arbitrary, so (i) holds.
(ii) Next suppose that $0<\mu<\rho_{\phi}<\infty$. Then we have from Lemma 3 and from above Theorem for a sequence of values of r tending to infinity

$$T(r,\Psi\circ\phi) \geq T(\exp(r)^{\mu},\phi) \geq T(\exp(r)^{\mu},g)+O(1).$$

$$i.e., \quad \frac{\log T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \geq \frac{\log T(\exp(r)^{\mu},g)}{\log T(\exp(r)^{\mu},g)}+O(1)$$
$$\geq \left[\frac{\log T(\exp(r)^{\mu},g)}{\log(\exp(r)^{\mu})}\frac{\log(\exp(r)^{\mu})}{\log T(\exp(r)^{\mu},g)}\right]+O(1).$$

So,

$$\limsup_{r\to\infty} \frac{log^{[2]}T(r,\Psi\circ\phi)}{\log T(\exp(r)^{\mu},g)} \geq \frac{\overline{\lambda}_g-\epsilon}{\rho_g+\epsilon}.$$

Since ϵ (>0) is arbitrary, so (ii) holds.

Theorem 3.5. *Let $f(z)$ be a non-constant meromorphic function of finite order in the finite complex plane and $g(z)$ be an entire function such that $0<\lambda_g\leq\rho_g<\infty$. Then*

$$\lim_{r\to\infty} \frac{\log T(r,\Psi\circ\phi)}{T(\exp r,g^{(k)})}=0,$$

for $k=0,1,2,\ldots$.

Proof. For all large values of r and arbitrary ϵ (>0) we have from (3.3)

$$\begin{aligned} \log T(r,\Psi\circ\phi) &\leq 3(\rho_f+\epsilon)\{1+o(1)\}[T(2r,g)+T(2r,g^{(1)})+\cdots+T(2r,g^{(m)})]+O(1)\\ &\leq 3(m+1)(\rho_f+\epsilon)\{1+o(1)\}(2r)^{\rho_g+\epsilon} \qquad \text{for large values of } r. \end{aligned} \tag{3.4}$$

Also for all large values of r,

$$T(\exp r,g^{(k)})>(\exp r)^{\lambda_g-\epsilon}. \tag{3.5}$$

Therefore from (3.4) and (3.5) we have

$$\frac{\log T(r,\Psi\circ\phi)}{T(\exp r,g^{(k)})}<\frac{3(m+1)(\rho_f+\epsilon)\{1+o(1)\}(2r)^{\rho_g+\epsilon}}{e^{r(\lambda_g-\epsilon)}}.$$

Hence

$$\lim_{r\to\infty} \frac{\log T(r,\Psi\circ\phi)}{T(\exp r,g^{(k)})}=0$$

for $k=0,1,2,\ldots$.

Note. The condition $\rho_f<\infty$ is necessary in Theorem 3.5. For this we consider the following example.

EXAMPLE 1. Let $f(z)=\exp^{[2]}(z)$ and $g(z)=\exp(z)$ and also we take $a_0(z)=z$ and $a_1(z)=3z$. $b_0(z)=1$ and $b_1(z)=7$. Then $\rho_f=\infty$,

$$\begin{aligned} \Psi(z)=&a_0(z)f(z)+a_1(z)f^{(1)}(z)\\ =&z\exp^{[2]}(z)[1+3\exp(z)] \end{aligned}$$

$$\begin{aligned} \text{and}\quad \Phi(z)=&b_0(z)g(z)+b_1(z)g^{(1)}(z)\\ =&8\exp(z). \end{aligned}$$

Now

$$\Psi\circ\phi(z)=8\exp(z)\exp^{[2]}(8\exp z)\left[1+3\exp(8\exp z)\right].$$

Again we know that

$$\begin{aligned} 3T(2r, \Psi og) &\geq \log M(r, \Psi \circ g) \\ &= \log 8 + r + \exp(8\exp r) + \log[1 + 3\exp(8\exp r)] \\ &\geq \exp(8\exp r). \end{aligned}$$

Therefore,

$$\log T(r, \Psi \circ \phi) \geq \exp(\frac{r}{2}) + O(1).$$

Since, $T(r, g) = \frac{r}{\pi}$, so $T(\exp r, g^{(k)}) = \frac{\exp r}{\pi}$, for $k = 0, 1, 2 \ldots$.
Hence

$$\lim_{r\to\infty} \frac{\log T(r, \Psi \circ \phi)}{T(\exp r, g^{(k)})} = \infty.$$

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