Existence and Uniqueness Of Boundary Value Problems For Hilfer-Hadamard-Type Fractional Differential Equations

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Abstract

In this paper, we used some theorems of fixed point for studying the results of existence and uniqueness for Hilfer-Hadamard-Type fractional differential equations,

$$_HD^{\alpha,\beta}x(t) + f(t, x(t)) = 0, \qquad on \ \ the \ \ interval \ \ J := (1, e]$$

with boundary value problems

$$x(1 + \epsilon) = 0, \qquad {}_HD^{1,1}x(e) = \nu \ {}_HD^{1,1}x(\zeta)$$

Significance: In this article, we found a variety of results for the boundary value problems (1.1) by applying Leray-schauder alternative fixed point theorem in Banach's space. In section3, we introduced generalized definition of Hilfer-Hadamard fractional derivative and proved general result for a composition of Hadamard fractional integration with Hilfer- Hadamard fractional derivative in Lemma 3.2. In Theorem 3.4, we obtained existence and uniqueness of the problem (1.1). In Theorem 3.5 and Theorem 3.6, we shown that by applying Leray-schauder alternative fixed point theorem in Banach's space, there exist at least one solution for the boundary value problems (1.1). An illustrative example is included.

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1.Introduction.

The fractional differential equations give proofs of the more appropriate models for describing real world problems. Indeed, these problems cannot be described using classical integer order differential equations. In the past years the theory of fractional differential equations has received much attention from the authors, and has become an important field of investigation due to existence applications in engineering, biology, chemistry, economics and numerous branches of physics sciences [1-5]. Fractional differential equations have a several kinds of fractional differential equations. One of them is the Hadamard fractional derivative innovated by Hadamard in 1892 [6], which differs from the Riemann-Liouvill and Caputo type fractional derivative [4,7],the preceding ones in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. The properties of Hadamard Fractional integral and derivative can be found in [8,9]. Recently, the authors studied the Hadamard-type, Caputo-Hadamard-type and Hilfer-Hadamard-type fractional derivative by using the fixed point theorems with the boundary value problems and give the results of existence and uniqueness [10-24]. In this paper, we studied the existence and uniqueness result of solutions for boundary value problems for Hilfer-Hadamard-Type fractional differential equations of the form

$$
\begin{aligned}
&{}_HD^{\alpha,\beta}x(t) + f(t,x(t)) = 0, \qquad t \in J := (1,e], \qquad 1 < \alpha \leq 2\ , \qquad 0 \leq \beta \leq 1 \\
&x(1+\epsilon) = 0, \qquad {}_HD^{1,1}x(e) = \nu\ {}_HD^{1,1}x(\zeta) \qquad (1.1)
\end{aligned}
$$

where ${}_HD^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $1 < \alpha \leq 2$ and type $\beta \in [0,1]$, $0 \leq \nu < 1$, $\zeta \in (1,e)$, $0 < \epsilon < 1$, ${}_HD^{1,1} = t\frac{d}{dt}$ and $f : J \to \mathbb{R}^+$.

2.Preliminaries

In this section, we introduce some notations and definitions of Hilfer-Hadamard-Type fractional calculus.

Definition 2.1.[2,5] (Riemann-Liouville fractional integral).
The Riemann-Liouville integral of order $\alpha > 0$ of a function $\varphi : [1,\infty) \to \mathbb{R}$ is defined by

$$
(I^\alpha\varphi)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \frac{\varphi(\tau)d\tau}{(t-\tau)^{1-\alpha}} \quad , \quad (t > 1),
$$

Here $\Gamma(\alpha)$ is the Euler's Gamma function.

Definition 2.2.[2,5] (Riemann-Liouville fractional derivative).
The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $\varphi : [1,\infty) \to \mathbb{R}$ is defined by

$$
\begin{aligned}
(D^\alpha\varphi)(t) &:= (\tfrac{d}{dt})^n(I^{n-\alpha}\varphi)(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_1^t \frac{\varphi(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} \quad , \qquad (n = [\alpha]+1; t > 1),
\end{aligned}
$$

Here $[\alpha]$ is the integer part of α.

Definition 2.3.[2] (Hadamard Fractional integral).
The Hadamard Fractional integral of order $\alpha \in \mathbb{R}^+$ for a function $\varphi : [1,\infty) \to \mathbb{R}$ is defined as

$$
{}_HI^\alpha\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{\tau})^{\alpha-1} \quad \frac{\varphi(\tau)}{\tau}d\tau, \qquad (t > 1)
$$

where $\log(.) = \log_e(.)$.

Definition 2.4.[2] (Hadamard Fractional derivative).

The Hadamard Fractional derivative of order α applied to the function $\varphi : [1, \infty) \to \mathbb{R}$ is defined as

$$_H D^\alpha \varphi(t) = \delta^n ({}_H I^{n-\alpha} \varphi(t)), \quad n-1 < \alpha < n, \quad n = [\alpha] + 1,$$

where $\delta^n = (t\frac{d}{dt})^n$ and $[\alpha]$ denotes the integer part of the real number α.
Definition 2.5.[6,25] (Caputo-Hadamard Fractional derivative).
The Caputo-Hadamard Fractional derivative of order α applied to the function $\varphi \in AC^n_\delta[a, b]$ is defined as

$$_{HC} D^\alpha \varphi(t) = ({}_H I^{n-\alpha} \delta^n \varphi)(t)$$

where
$n = [\alpha] + 1$, and $\varphi \in AC^n_\delta[a, b] = \left\{\varphi : [a, b] \to \mathbb{C} : \delta^{(n-1)}\varphi \in AC[a, b], \delta = t\frac{d}{dt}\right\}$
Definition 2.6.[3,22] (Hilfer Fractional derivative).
Let $n-1 < \alpha < n$, $0 \le \beta \le 1$, $\varphi \in L^1(a, b)$. The Hilfer Fractional derivative $D^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}(D^{\alpha,\beta}\varphi)(t) &= \big(I^{\beta(n-\alpha)}(\frac{d}{dt})^n\ I^{(n-\alpha)(1-\beta)}\varphi\big)(t)\\ &= \big(I^{\beta(n-\alpha)}(\frac{d}{dt})^n I^{n-\gamma}\varphi\big)(t); \quad \gamma = \alpha + n\beta - \alpha\beta\\ &= \big(I^{\beta(n-\alpha)} D^\gamma \varphi\big)(t),\end{aligned}$$

Where $I^{(\cdot)}$ and $D^{(\cdot)}$ is the Riemann-Liouvill fractional integral and derivative defined by (2.1) and (2.2), respectively.
In particular, if $0 < \alpha < 1$, then

$$\begin{aligned}(D^{\alpha,\beta}\varphi)(t) &= \big(I^{\beta(1-\alpha)}\frac{d}{dt}\ I^{(1-\alpha)(1-\beta)}\varphi\big)(t)\\ &= \big(I^{\beta(1-\alpha)}\frac{d}{dt} I^{1-\gamma}\varphi\big)(t); \quad \gamma = \alpha + \beta - \alpha\beta\\ &= \big(I^{\beta(1-\alpha)} D^\gamma \varphi\big)(t)\end{aligned}$$

Properties 2.7.[22,23].
Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $\varphi \in L^1(a, b)$. If $D^\gamma\varphi$ exists and in $L^1(a, b)$, then

$$I^\alpha_{a+}\ (D^{\alpha,\beta}_{a+}\varphi)(t) = I^\gamma_{a+}\ (D^\gamma_{a+}\varphi)(t) = \varphi(t) - \frac{(I^{1-\gamma}_{a+}\varphi)(a)}{\Gamma(\gamma)}(t-a)^{\gamma-1}$$

Definition 2.8.[22,23](Hilfer-Hadamard Fractional derivative).

Let $0<\alpha<1$, $0\leq\beta\leq 1$, $\varphi\in L^1(a,b)$. The Hilfer-Hadamard Fractional derivative ${}_HD^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}({}_HD^{\alpha,\beta}\varphi)(t) &= \big({}_HI^{\beta(1-\alpha)}\delta\ {}_HI^{(1-\alpha)(1-\beta)}\varphi\big)(t)\\ &= \big({}_HI^{\beta(1-\alpha)}\delta\ {}_HI^{1-\gamma}\varphi\big)(t);\quad \gamma=\alpha+\beta-\alpha\beta\\ &= \big({}_HI^{\beta(1-\alpha)}\ {}_HD^{\gamma}\varphi\big)(t),\end{aligned}$$

Where ${}_HI^{(\cdot)}$ and ${}_HD^{(\cdot)}$ is the Hadamard fractional integral and derivative defined by (2.3) and (2.4), respectively.

Theorem 2.9.[2,6].
Let $\Re(\alpha)>0$, $n=[\Re(\alpha)]+1$ and $0<a<b<\infty$. if $\varphi\in L^1(a,b)$ and $({}_HI^{n-\alpha}_{a+}\varphi)(t)\in AC^n_\delta[a,b]$, then

$$({}_HI^{\alpha}_{a+}\ {}_HD^{\alpha}_{a+}\varphi)(t)=\varphi(t)-\sum_{j=0}^{n-1}\frac{(\delta^{(n-j-1)}({}_HI^{n-\alpha}_{a+}\varphi))(a)}{\Gamma(\alpha-j)}(\log\frac{t}{a})^{\alpha-j-1}$$

Theorem 2.10.[6,25]
Let $\varphi(t)\in AC^n_\delta[a,b]$ or $\varphi(t)\in C^n_\delta[a,b]$, and $\alpha\in\mathbb{C}$, then

$$({}_HI^{\alpha}_{a+}\ {}_{HC}D^{\alpha}_{a+}\varphi)(t)=\varphi(t)-\sum_{K=0}^{n-1}\frac{\delta^K\varphi(a)}{\Gamma(K+1)}(\log\frac{t}{a})^{K}$$

Theorem 2.11.[26,27] (Leray- Schauder alternative).
Let E be a Banach space. Suppose that $T:E\to E$ is completely continuous operator and the set $V=\big\{v\in E\mid v=\lambda Tv, 0<\lambda<1\big\}$ is bounded. Then T has a fixed point in E.

Theorem 2.12.[26,27]
Let E be a Banach space and V is an open bounded subset of E with $0\in V$. Suppose that $\Psi:\overline{V}\to E$ be a completely continuous operator such that $\parallel\Psi v\parallel\leq\parallel v\parallel, \forall v\in\partial V$. Then Ψ has a fixed point in $\overline{V}$.

Lemma 2.13.[24]
For $1<\alpha\leq 2$ and $\varphi\in C([1,e]\,,\,\mathbb{R})$ the problem for Capoto-Hadamard-type,

$${}_{HC}D^{\alpha}x(t)+\varphi(t)=0,\qquad t\in[1,e]\quad 1<\alpha\leq 2,$$

$$x(1)=0,\qquad {}_{HC}Dx(e)=\nu\ {}_{HC}Dx(\zeta)$$

has a unique solution it giving in the formulae

$$\begin{aligned}x(t) &= -\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{\tau})^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau\\ &+\frac{\log t}{1-\nu}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^e(\log\frac{e}{\tau})^{\alpha-2}\frac{\varphi(\tau)}{\tau}d\end{aligned}$$

$$-\frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2}\frac{\varphi(\tau)}{\tau}d\tau\Big]$$

3.Main Results .

Definition 3.1(Hilfer-Hadamard Fractional derivative).
Let $n-1<\alpha<n,\ \ 0\leq\beta\leq 1,\ \ \varphi\in L^1(a,b)$. The Hilfer-Hadamard Fractional derivative ${}_HD^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}({}_HD^{\alpha,\beta}\varphi)(t)&=\big({}_HI^{\beta(n-\alpha)}(\delta)^n\ {}_HI^{(n-\alpha)(1-\beta)}\varphi\big)(t)\\&=\big({}_HI^{\beta(n-\alpha)}(\delta)^n\ {}_HI^{n-\gamma}\varphi\big)(t);\quad \gamma=\alpha+n\beta-\alpha\beta.\\&=\big({}_HI^{\beta(n-\alpha)}\ {}_HD^{\gamma}\varphi\big)(t),\end{aligned}$$

Where ${}_HI^{(.)}$ and ${}_HD^{(.)}$ is the Hadamard fractional integral and derivative defined by (2.3) and (2.4), respectively.
Lemma 3.2.
Let $\Re(\alpha)>0,\quad 0\leq\beta\leq 1,\quad \gamma=\alpha+n\beta-\alpha\beta,\quad n-1<\gamma\leq n,\quad n=[\Re(\alpha)]+1$ and $0<a<b<\infty$. if $\varphi\in L^1(a,b)$ and $({}_HI_{a+}^{n-\gamma}\varphi)(t)\in AC_\delta^n[a,b]$, then

$${}_HI_{a+}^{\alpha}\ ({}_HD_{a+}^{\alpha,\beta}\varphi)(t)={}_H\ I_{a+}^{\gamma}\ ({}_HD_{a+}^{\gamma}\varphi)(t)=\varphi(t)-\sum_{j=0}^{n-1}\frac{(\delta^{(n-j-1)}({}_HI_{a+}^{n-\gamma}\varphi))(a)}{\Gamma(\gamma-j)}(\log\frac{t}{a})^{\gamma-j-1}$$

From this Lemma, we notice that if $\beta=0$ the formulae reduces to the formulae in the theorem 2.9, and if the $\beta=1$ the formulae reduces to the formulae in the theorem 2.10.
Proof. We have

$${}_HI_{a+}^{\alpha}\ ({}_HD_{a+}^{\alpha,\beta}\varphi)(t)={}_H\ I_{a+}^{\gamma}\ ({}_HD_{a+}^{\gamma}\varphi)(t)$$

$$\begin{aligned}&=\frac{1}{\Gamma(\gamma)}\int_a^t(\log\frac{t}{\tau})^{\gamma-1}({}_HD_{a+}^{\gamma}\varphi(\tau))\frac{d\tau}{\tau}\\&=\frac{d}{dt}\Big\{\frac{1}{\Gamma(\gamma+1)}\int_a^t(\log\frac{t}{\tau})^{\gamma}({}_HD_{a+}^{\gamma}\varphi(\tau))\frac{d\tau}{\tau}\Big\}\end{aligned}$$

On the hand, repeatedly integrating by parts and then using

${}_HI^p_{a+}\cdot{}_HI^q_{a+} ={}_H I^q_{a+}\cdot{}_HI^p_{a+} ={}_H I^{p+q}_{a+}$, we obtain

$$\frac{1}{\Gamma(\gamma+1)}\int_a^t(\log\frac{t}{\tau})^{\gamma}({}_HD^{\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}$$

$$=\frac{1}{\Gamma(\gamma+1)}\int_a^t(\log\frac{t}{\tau})^{\gamma}\delta^n({}_HI^{n-\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}$$

$$=-\frac{1}{\Gamma(\gamma+1)}(\log\frac{t}{a})^{\gamma}\delta^{n-1}({}_HI^{n-\gamma}_{a+}\varphi(a))+\frac{1}{\Gamma(\gamma)}\int_a^t(\log\frac{t}{\tau})^{\gamma-1}\delta^{n-1}({}_HI^{n-\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}$$

$$=-\frac{1}{\Gamma(\gamma+1)}(\log\frac{t}{a})^{\gamma}\delta^{n-1}({}_HI^{n-\gamma}_{a+}\varphi(a))-\frac{1}{\Gamma(\gamma)}(\log\frac{t}{a})^{\gamma-1}\delta^{n-2}({}_HI^{n-\gamma}_{a+}\varphi(a))$$
$$+\frac{1}{\Gamma(\gamma-1)}\int_a^t(\log\frac{t}{\tau})^{\gamma-2}\delta^{n-2}({}_HI^{n-\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}$$

$$=-\frac{1}{\Gamma(\gamma+1)}(\log\frac{t}{a})^{\gamma}\delta^{n-1}({}_HI^{n-\gamma}_{a+}\varphi(a))-\frac{1}{\Gamma(\gamma)}(\log\frac{t}{a})^{\gamma-1}\delta^{n-2}({}_HI^{n-\gamma}_{a+}\varphi(a))$$
$$-\frac{1}{\Gamma(\gamma-1)}(\log\frac{t}{a})^{\gamma-2}\delta^{n-3}({}_HI^{n-\gamma}_{a+}\varphi(a))+\frac{1}{\Gamma(\gamma-2)}\int_a^t(\log\frac{t}{\tau})^{\gamma-3}\delta^{n-3}({}_HI^{n-\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}$$

$$=\frac{1}{\Gamma(\gamma-2)}\int_a^t(\log\frac{t}{\tau})^{\gamma-3}\delta^{n-3}({}_HI^{n-\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}-\sum_{j=1}^{3}\frac{\delta^{n-j}({}_HI^{n-\gamma}_{a+}\varphi(a))}{\Gamma(2+\gamma-j)}(\log\frac{t}{a})^{\gamma-j+1}$$

$$=\ldots\ldots$$

$$=\frac{1}{\Gamma(\gamma-n+1)}\int_a^t(\log\frac{t}{\tau})^{\gamma-n}({}_HI^{n-\gamma}_{a+}\varphi(\tau))\frac{d\tau}{\tau}-\sum_{j=1}^{n}\frac{\delta^{n-j}({}_HI^{n-\gamma}_{a+}\varphi(a))}{\Gamma(2+\gamma-j)}(\log\frac{t}{a})^{\gamma-j+1}$$

Therefore,

$${}_HI^{\alpha}_{a+}\ ({}_HD^{\alpha,\beta}_{a+}\varphi)(t)={}_H\ I^{\gamma}_{a+}\ ({}_HD^{\gamma}_{a+}\varphi)(t)$$

$$=\frac{d}{dt}\Big\{({}_HI^1_{a+}\varphi)(t)-\sum_{j=1}^{n}\frac{\delta^{n-j}({}_HI^{n-\gamma}_{a+}\varphi(a))}{\Gamma(2+\gamma-j)}(\log\frac{t}{a})^{\gamma-j+1}\Big\}$$

$$=\varphi(t)-\sum_{j=0}^{n-1}\frac{\delta^{n-j-1}({}_HI^{n-\gamma}_{a+}\varphi(a))}{\Gamma(\gamma-j)}(\log\frac{t}{a})^{\gamma-j-1}\qquad\square$$

Lemma 3.3.
For $1<\alpha\leq 2$, $0\leq\beta\leq 1$ and $\varphi\in C([1,e],\mathbb{R})$,
$\gamma=\alpha+2\beta-\alpha\beta$, $\gamma\in(1,2]$
the problem

$$\begin{aligned}&{}_HD^{\alpha,\beta}x(t)+\varphi(t)=0, \qquad t\in J\ ,\ 1<\alpha\leq 2\ ,\ 0\leq\beta\leq 1\\&x(1+\epsilon)=0,\qquad {}_HD^{1,1}x(e)=\nu\ {}_HD^{1,1}x(\zeta)\end{aligned}\tag{3.1}$$

has a unique solution it giving in the formulae

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{\tau})^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau$$
$$+(\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau$$
$$+[\frac{1}{\log(1+\epsilon)}-\frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^{e}(\log\frac{e}{\tau})^{\alpha-2}\frac{\varphi(\tau)}{\tau}d\tau$$
$$-\frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2}\frac{\varphi(\tau)}{\tau}d\tau+\varepsilon[1-\nu(\log\zeta)^{\gamma-2}]\Bigg]$$

Where

$$\sum_{i=0}^{1}\eta_i=\sum_{i=0}^{1}(-1)^i(\log(1+\epsilon))^{i-1}(\gamma-i-1)[1-\nu(\log\zeta)^{\gamma-i-2}],\quad with\quad \sum_{i=0}^{1}\eta_i\neq 0$$
$$\varepsilon=(1-\gamma)(\log(1+\epsilon))^{1-\gamma}\ {}_HI^{\alpha}\varphi(1+\epsilon)$$

Proof. In the view of the Lemma(3.2), the solution of the Hilfer-Hadamard differential equation (3.1) can be written as

$$x(t) = -\ {}_HI^{\alpha}\varphi(t)+c_0(\log t)^{\gamma-1}+c_1(\log t)^{\gamma-2} \tag{3.2}$$

and

$${}_HD^{1,1}x(t) = -\ {}_HI^{\alpha-1}\varphi(t)+(\gamma-1)c_0(\log t)^{\gamma-2}+(\gamma-2)c_1(\log t)^{\gamma-3} \tag{3.3}$$

The boundary condition $x(1+\epsilon)=0$ gives

$$c_0=(\log(1+\epsilon))^{1-\gamma}\ {}_HI^{\alpha}\varphi(1+\epsilon)-\frac{c_1}{\log(1+\epsilon)} \tag{3.4}$$

In view of the boundary condition ${}_HD^{1,1}x(e)=\nu\ {}_HD^{1,1}x(\zeta)$, and by (3.3), and (3.4), we have

$$c_1=\frac{1}{\sum_{i=0}^{1}\eta_i}\Bigg[-{}_H\ I^{\alpha-1}\varphi(e)+\nu\ {}_HI^{\alpha-1}\varphi(\zeta)+\varepsilon[1-\nu(\log\zeta)^{\gamma-2}]\Bigg]$$

Where

$$\eta_i=(-1)^i(\log(1+\epsilon))^{i-1}(\gamma-i-1)[1-\nu(\log\zeta)^{\gamma-i-2}],$$
$$\varepsilon=(1-\gamma)(\log(1+\epsilon))^{1-\gamma}\ {}_HI^{\alpha}\varphi(1+\epsilon)$$

Substituting the value of c_1 in (3.4) we have

$$c_0 = (\log(1+\epsilon))^{1-\gamma}\ {}_HI^{\alpha}\varphi(1+\epsilon) - \frac{1}{\sum_{i=0}^{1}\eta_i \log(1+\epsilon)}\Big[-{}_H I^{\alpha-1}\varphi(e) + \nu\ {}_HI^{\alpha-1}\varphi(\zeta) + \varepsilon[1-\nu(\log\zeta)^{\gamma-2}]\Big]$$

Now substituting the values of c_0 and c_1 in (3.2) we obtain the solution of the problem(3.1).

Results of Existence.

Suppose that

$$K = C([1,e],\mathbb{R}) \tag{3.5}$$

is a Banach space of all continuous functions from $[1,e]$ into $\mathbb{R}$ given with the norm $\| x \| = \sup_{t\in J} \mid x(t) \mid$.

From the Lemma3.1,we getting an operator $\rho : K \to K$ defined as

$$\begin{aligned}(\rho x)(t) = &-\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{\tau})^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau \\ &+ (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1}\frac{\varphi(\tau)}{\tau}d\tau \\ &+ [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Big[\frac{1}{\Gamma(\alpha-1)}\int_1^e (\log\frac{e}{\tau})^{\alpha-2}\frac{\varphi(\tau)}{\tau}d\tau \\ &- \frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2}\frac{\varphi(\tau)}{\tau}d\tau + \varepsilon[1-\nu(\log\zeta)^{\gamma-2}]\Big], \quad t\in J\end{aligned} \tag{3.6}$$

It must be noticed that the problem (1.1) has solutions if and only if the operator ρ has fixed points.The result of existence and uniqueness is based on the Banach Principle of contraction.

Theorem 3.4 suppose that there exists a constant $C > 0$ such that $\mid f(t,x(t)) - f(t,y(t)) \mid \leq C \mid x-y \mid, \quad \forall t\in J, \quad C>0, x,y\in\mathbb{R}$.
If Φ satisfied the condition $C\Phi < 1$, where

$$\Phi = \Big\{\frac{[1+(\log(1+\epsilon))^{1-\gamma+\alpha}]}{\Gamma(\alpha+1)} + \frac{[1-(\log(1+\epsilon))]}{(\log(1+\epsilon))\Gamma(\alpha)\sum_{i=0}^{1}\eta_i}\Big[1+\nu(\log\zeta)^{\alpha-1} + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}}{\alpha}[1-\nu(\log\zeta)^{\gamma-2}]\Big]\Big\} \tag{3.7}$$

Then the problem (1.1) has a unique solution.
Proof. We put $sup_{t\in J} \mid f(\tau,0) \mid = P < \infty$ and choose $r \geq \frac{\Phi P}{1-\Phi C}$.

Now,assume that $B_r = \{x \in K :\parallel x \parallel \leq r\}$, then we show that $\rho B_r \subset B_r$. For any $x \in B_r$, we have

$$\begin{aligned}
&\parallel (\rho x)(t) \parallel \\
&= sup_{t\in J}\Bigg\{\Bigg| -\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^{e}(\log\frac{e}{\tau})^{\alpha-2} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad - \frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau}\Bigg]\Bigg|\Bigg\} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{\tau})^{\alpha-1}\Big(\mid f(\tau,x(\tau)) - f(\tau,0)\mid + \mid f(\tau,0)\mid\Big)\frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1}\Big(\mid f(\tau,x(\tau)) - f(\tau,0)\mid + \mid f(\tau,0)\mid\Big)\frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^{e}(\log\frac{e}{\tau})^{\alpha-2}\Big(\mid f(\tau,x(\tau)) - f(\tau,0)\mid + \mid f(\tau,0)\mid\Big)\frac{d\tau}{\tau} \\
&\quad + \frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2}\Big(\mid f(\tau,x(\tau)) - f(\tau,0)\mid + \mid f(\tau,0)\mid\Big)\frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1}\Big(\mid f(\tau,x(\tau)) - f(\tau,0)\mid + \mid f(\tau,0)\mid\Big)\frac{d\tau}{\tau}\Bigg] \\
&\leq (Cr+P)\Bigg\{\frac{1}{\Gamma(\alpha+1)} + \frac{(\log(1+\epsilon))^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{[1-\log(1+\epsilon)]}{\log(1+\epsilon)\sum_{i=0}^{1}\eta_i\Gamma(\alpha)}\Bigg[1+\nu(\log\zeta)^{\alpha-1} \\
&\qquad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}[1-\nu(\log\zeta)^{\gamma-2}]}{\alpha}\Bigg]\Bigg\} \\
&\leq (Cr+P)\Bigg\{\frac{[1+(\log(1+\epsilon))^{1-\gamma+\alpha}]}{\Gamma(\alpha+1)} \\
&\qquad + \frac{[1-(\log(1+\epsilon))]}{(\log(1+\epsilon))\Gamma(\alpha)\sum_{i=0}^{1}\eta_i}\Bigg[1+\nu(\log\zeta)^{\alpha-1} \\
&\qquad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}}{\alpha}[1-\nu(\log\zeta)^{\gamma-2}]\Bigg]\Bigg\} \\
&\leq (Cr+P)\Phi \leq r \qquad (3.8)
\end{aligned}$$

Thus we shown $\rho B_r \subset B_r$.
Now,For $x, y \in K$ and $\forall t \in J$, we have

$$\begin{aligned}
&\mid (\rho x)(t) - (\rho y)(t) \mid \\
&= \Bigg| -\frac{1}{\Gamma(\alpha)}{}_1\int^t (\log\frac{t}{\tau})^{\alpha-1}\Big(f(\tau,x(\tau)) - f(\tau,y(\tau))\Big)\frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}{}_1\int^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1}\Big(f(\tau,x(\tau)) - f(\tau,y(\tau))\Big)\frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}{}_1\int^e(\log\frac{e}{\tau})^{\alpha-2}\Big(f(\tau,x(\tau)) - f(\tau,y(\tau))\Big)\frac{d\tau}{\tau} \\
&\qquad - \frac{\nu}{\Gamma(\alpha-1)}{}_1\int^\zeta(\log\frac{\zeta}{\tau})^{\alpha-2}\Big(f(\tau,x(\tau)) - f(\tau,y(\tau))\Big)\frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}{}_1\int^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1}\Big(f(\tau,x(\tau)) - f(\tau,y(\tau))\Big)\frac{d\tau}{\tau}\Bigg]\Bigg| \\
&\le \frac{1}{\Gamma(\alpha)}{}_1\int^t(\log\frac{t}{\tau})^{\alpha-1} \mid f(\tau,x(\tau)) - f(\tau,y(\tau)) \mid \frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}{}_1\int^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} \mid f(\tau,x(\tau)) - f(\tau,y(\tau)) \mid \frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}{}_1\int^e(\log\frac{e}{\tau})^{\alpha-2} \mid f(\tau,x(\tau)) - f(\tau,y(\tau)) \mid \frac{d\tau}{\tau} \\
&\qquad + \frac{\nu}{\Gamma(\alpha-1)}{}_1\int^\zeta(\log\frac{\zeta}{\tau})^{\alpha-2} \mid f(\tau,x(\tau)) - f(\tau,y(\tau)) \mid \frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}{}_1\int^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} \mid f(\tau,x(\tau)) - f(\tau,y(\tau)) \mid \frac{d\tau}{\tau}\Bigg] \\
&\le C \parallel x - y \parallel \Bigg\{\frac{[1 + (\log(1+\epsilon))^{1-\gamma+\alpha}]}{\Gamma(\alpha+1)} \\
&\qquad + \frac{[1 - (\log(1+\epsilon))]}{(\log(1+\epsilon))\Gamma(\alpha)\sum_{i=0}^{1}\eta_i}\Bigg[1 + \nu(\log\zeta)^{\alpha-1} \\
&\qquad\qquad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}}{\alpha}[1-\nu(\log\zeta)^{\gamma-2}]\Bigg]\Bigg\} \\
&\le C\Phi \parallel x - y \parallel
\end{aligned} \tag{3.9}$$

Therefore it shown that $\parallel (\rho x)(t) - (\rho y)(t) \parallel \le C\Phi \parallel x - y \parallel$, where $C\Phi < 1$. Hence ρ is a contraction. Thus by the mapping of contraction principle the problem (1.1) has a uniqueness solution.

Theorem 3.5 suppose that there exists a constant $C_1 > 0$ such that $\mid f(t, x(t)) \mid \le C_1$,for each $t \in J, x \in \mathbb{R}$. Then the problem (1.1) has at least one solution.

Proof. The proof of this theorem will be given in several steps, firstly, we will show

that the operator ρ is completely continuous for this, in the view of the continuity of f,we note that the operator ρ is continuous.
Now, Assume that $\rho \subset \rho$ be a bounded set.
By the supposition that $\mid f(t,x(t)) \mid \leq C_1$,for each $t \in J, x \in \mathbb{R}$, we get

$$\begin{aligned}
&\mid (\rho x)(t) \mid \\
&= \Bigg| -\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^e(\log\frac{e}{\tau})^{\alpha-2} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\qquad - \frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau}\Bigg]\Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{\tau})^{\alpha-1} \mid f(\tau,x(\tau)) \mid \frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} \mid f(\tau,x(\tau)) \mid \frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^e(\log\frac{e}{\tau})^{\alpha-2} \mid f(\tau,x(\tau)) \mid \frac{d\tau}{\tau} \\
&\qquad + \frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2} \mid f(\tau,x(\tau)) \mid \frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} \mid f(\tau,x(\tau)) \mid \frac{d\tau}{\tau}\Bigg] \\
&\leq C_1\Bigg\{\frac{[1+(\log(1+\epsilon))^{1-\gamma+\alpha}]}{\Gamma(\alpha+1)} \\
&\qquad + \frac{[1-(\log(1+\epsilon))]}{(\log(1+\epsilon))\Gamma(\alpha)\sum_{i=0}^{1}\eta_i}\Bigg[1+\nu(\log\zeta)^{\alpha-1} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}}{\alpha}[1-\nu(\log\zeta)^{\gamma-2}]\Bigg]\Bigg\} = C_2
\end{aligned} \tag{3.10}$$

this implies that $\| (\rho x)(t) \| \leq C_2$. Moreover,

$$
\begin{aligned}
&| \ {}_H D^{1,1}(\rho x)(t) \ | \\
&= \Bigg| -\frac{1}{\Gamma(\alpha-1)} \int_1^t (\log \frac{t}{\tau})^{\alpha-2} f(\tau,x(\tau)) \frac{d\tau}{\tau} \\
&\quad - \frac{(1-\gamma)}{\log(1+\epsilon)} (\frac{\log t}{\log(1+\epsilon)})^{\gamma-2} \ \frac{1}{\Gamma(\alpha)} \int_1^{1+\epsilon} (\log \frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau)) \frac{d\tau}{\tau} \\
&\quad + [\frac{(\gamma-1)}{\log(1+\epsilon)} - \frac{(\gamma-2)}{\log t}] \frac{(\log t)^{\gamma-2}}{\sum_{i=0}^{1} \eta_i} \Bigg[\frac{1}{\Gamma(\alpha-1)} \int_1^e (\log \frac{e}{\tau})^{\alpha-2} f(\tau,x(\tau)) \frac{d\tau}{\tau} \\
&\qquad - \frac{\nu}{\Gamma(\alpha-1)} \int_1^\zeta (\log \frac{\zeta}{\tau})^{\alpha-2} f(\tau,x(\tau)) \frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)} \int_1^{1+\epsilon} (\log \frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau)) \frac{d\tau}{\tau} \Bigg] \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_1^t (\log \frac{t}{\tau})^{\alpha-2} \ | \ f(\tau,x(\tau)) \ | \ \frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}}{\Gamma(\alpha)} \int_1^{1+\epsilon} (\log \frac{1+\epsilon}{\tau})^{\alpha-1} \ | \ f(\tau,x(\tau)) \ | \ \frac{d\tau}{\tau} \\
&\quad + \frac{(\gamma-1)[1-\log(1+\epsilon)]+\log(1+\epsilon)}{\log(1+\epsilon)\sum_{i=0}^{1}\eta_i} \Bigg[\frac{1}{\Gamma(\alpha-1)} \int_1^e (\log \frac{e}{\tau})^{\alpha-2} \ | \ f(\tau,x(\tau)) \ | \ \frac{d\tau}{\tau} \\
&\qquad + \frac{\nu}{\Gamma(\alpha-1)} \int_1^\zeta (\log \frac{\zeta}{\tau})^{\alpha-2} \ | \ f(\tau,x(\tau)) \ | \ \frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)} \int_1^{1+\epsilon} (\log \frac{1+\epsilon}{\tau})^{\alpha-1} \ | \ f(\tau,x(\tau)) \ | \ \frac{d\tau}{\tau} \Bigg] \\
&\leq C_1 \Bigg\{ \frac{1}{\Gamma(\alpha)} + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} \\
&\qquad + \frac{(\gamma-1)[1-\log(1+\epsilon)]+\log(1+\epsilon)}{\log(1+\epsilon)\sum_{i=0}^{1}\eta_i} \Bigg[\frac{1+\nu(\log\zeta)^{\alpha-1}}{\Gamma(\alpha)} \\
&\qquad\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha+1)} \Bigg] \Bigg\} = C_3
\end{aligned}
\tag{3.11}
$$

Thus,for each $t_1, t_2 \in J$, we get

$$
| \ (\rho x)(t_1) - (\rho x)(t_2) \ | \leq \int_{t_1}^{t_2} {}_H D^{1,1}(\rho x)(\tau) \frac{d\tau}{\tau} \leq C_3(t_2 - t_1) \tag{3.12}
$$

which implies that ρ is continuous over J. Hence,the operator $\rho : K \to K$ is completely continuous,(by the Arzela-Ascoli theorem).
Finally,consider the set $U = \{ v \in K \mid x = \lambda T x, 0 < \lambda < 1 \}$,we show that the set U is bounded. Assume that $x \in U$,then $x = \lambda \rho x, \quad 0 < \lambda < 1$.

Now for any $t \in J$, we get

$$\begin{aligned}
| x(t) | &= \lambda | (\rho x)(t) | \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{\tau})^{\alpha-1} | f(\tau, x(\tau)) | \frac{d\tau}{\tau} \\
&\quad + (\frac{1}{\log(1+\epsilon)})^{\gamma-1} \frac{1}{\Gamma(\alpha)} \int_1^{1+\epsilon} (\log \frac{1+\epsilon}{\tau})^{\alpha-1} | f(\tau, x(\tau)) | \frac{d\tau}{\tau} \\
&\quad + \frac{[1-(\log(1+\epsilon))]}{(\log(1+\epsilon))\Gamma(\alpha-1)\sum_{i=0}^{1}\eta_i} \Big[\int_1^e (\log \frac{e}{\tau})^{\alpha-2} | f(\tau, x(\tau)) | \frac{d\tau}{\tau} \\
&\quad + \nu_1 \int^{\zeta} (\log \frac{\zeta}{\tau})^{\alpha-2} | f(\tau, x(\tau)) | \frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\alpha-1} {}_1\int^{1+\epsilon} (\log \frac{1+\epsilon}{\tau})^{\alpha-1} | f(\tau, x(\tau)) | \frac{d\tau}{\tau} \Big] \\
&\leq C_1 \Big\{ \frac{1}{\Gamma(\alpha+1)} + \frac{(\log(1+\epsilon))^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{[1-(\log(1+\epsilon))]}{(\log(1+\epsilon))\Gamma(\alpha)\sum_{i=0}^{1}\eta_i} \Big[1 + \nu(\log\zeta)^{\alpha-1} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma+\alpha}}{\alpha}[1-\nu(\log\zeta)^{\gamma-2}] \Big] \Big\} = M
\end{aligned} \tag{3.13}$$

Therefore, $\| x(t) \| \leq M$ for any $t \in J$. Hence, the set U is bounded. So, from the above and by the Theorem 2.11, the operator ρ has at least one fixed point,that implies to the problem(1.1) has at least one solution.

Theorem 3.6 Assume that there exist a small positive number $\tilde{r}$ and $0 < \mu < \frac{1}{\Phi}$ such that

$$| f(t,x) | \leq \mu | x | \quad for\ 0 < | x | < \tilde{r}, \quad where\ \ \Phi\ \ is\ \ defined\ \ by\ (3.7).$$

Then the problem (1.1) has at least one solution.

Proof. Firstly, let K be a Banach space defined by (3.5), and define $B_{\tilde{r}} = \{x \in K : \| x \| \leq \tilde{r}\}$ and put $x \in K$ such that $\| x \| = \tilde{r}$, that is, $x \in \partial B_{\tilde{r}}$.

Now, with the same argument of proof in the previous theorem, we can shown that ρ is completely continuous and we have,

$$
\begin{aligned}
&\| (\rho x)(t) \| \\
&= sup_{t\in J}\Bigg\{\Bigg| -\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + (\frac{\log t}{\log(1+\epsilon)})^{\gamma-1}\ \frac{1}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + [\frac{1}{\log(1+\epsilon)} - \frac{1}{\log t}]\frac{(\log t)^{\gamma-1}}{\sum_{i=0}^{1}\eta_i}\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_1^{e}(\log\frac{e}{\tau})^{\alpha-2} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad - \frac{\nu}{\Gamma(\alpha-1)}\int_1^{\zeta}(\log\frac{\zeta}{\tau})^{\alpha-2} f(\tau,x(\tau))\frac{d\tau}{\tau} \\
&\quad + \frac{(1-\gamma)(\log(1+\epsilon))^{1-\gamma}[1-\nu(\log\zeta)^{\gamma-2}]}{\Gamma(\alpha)}\int_1^{1+\epsilon}(\log\frac{1+\epsilon}{\tau})^{\alpha-1} f(\tau,x(\tau))\frac{d\tau}{\tau}\Bigg]\Bigg|\Bigg\} \\
&\le \mu\Phi \| x \|
\end{aligned}
\tag{3.14}
$$

Hence, $\| (\rho x)(t) \| \le \| x \|$, with $x \in \partial B_{\tilde{r}}$. Then by applied the theorem 2.12, ρ has at least one fixed point. Therefore, the problem(1.1) has at least one solution on J.

Example.

Consider the following boundary value problem for Hilfer-Hadamard-type fractional differential equation:

$$
{}_H D^{3/2,1/2} x(t) + f(t,x(t)) = 0, \qquad t \in J := (1,e] \tag{3.15}
$$

$$
x(1.2) = 0, \qquad {}_H D^{1,1} x(e) = (1/2)\ {}_H D^{1,1} x(3/2).
$$

Here,

$$
\alpha = 3/2, \quad \beta = 1/2, \quad \gamma = 7/4, \quad \nu = 1/2, \quad \zeta = 3/2, \quad \epsilon = 0.2, \quad 1+\epsilon = 1.2
$$

and

$$
f(t,x(t)) = \frac{1}{32}(\sqrt{t} + \log t)(\frac{|x|}{2+|x|}).
$$

Clearly,

$$
| f(t,x) | \le \frac{1}{32}(\sqrt{t}+1)(|x|+1)
$$

and

$$
| f(t,x) - f(t,y) | \le \frac{1}{32}(\sqrt{t}+1)(|x-y|) \le \frac{1}{16}|x-y|
$$

Therefore, by Theorem3.4, the boundary value problem (3.15) has a unique solution on $(1,e]$ with $C = \frac{1}{16} = 0.0625$. We can show that $\Phi = 1.404, \quad C\Phi = 0.0876 < 1$.

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