One Point Union of Paths of Cycles and their k-Numbers

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Abstract

An $L(3,2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 3$ if x and y are adjacent, $|f(x) - f(y)| \geq 2$ if x and y are at distance 2, and $|f(x) - f(y)| \geq 1$ if x and y are at distance 3, for all x and y in $V(G)$. The $L(3,2,1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3,2,1)$-labeling with k as the maximum label. In this paper, we consider one point union of paths of cycles C_n, $n > 3$ and find the k-numbers of them.

Keywords: $L(3,2,1)$-labeling, Channel assignment, One Point Union, k-number.

1 Introduction

For standard terminology and notation, we follow Bondy and Murty [1] or Murugan [2]. Unless otherwise mentioned, $G(V,E)$ denotes simple, finite, connected, undirected graph without loops or multiple edges. Following the standard terminology, we use P_n to denote a path on n vertices, C_n to denote a cycle on n vertices, T to denote a Tree, Δ to denote the maximum degree of a graph, $\lceil x \rceil$ to denote the least integer greater than or equal to x, and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

In frequency assignment problem, we assign frequencies to a given set of radio transmitters so that transmitters are assigned frequency with a minimum allowed separation. Closer stations have a stronger interference and so there should be a greater difference between their assigned channels.

Hale introduced a graph model of the channel assignment problem in 1980 [3]. Robert modified this with stations which are "close" and "very close" which correspond to stations at distance two and stations at distance one in graph theoretic terms [4]. The mathematical abstraction of this problem was introduced by Griggs and Yeh as $L(2,1)$ problem [5]. An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent, and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. But, practically, interference among channels may go beyond two levels. Liu and Shao modified the above problem, by considering stations at distance 1, 2 and 3 and it was called as $L(3,2,1)$-problem[6].

An $L(3,2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 3$ if x and y are adjacent, $|f(x) - f(y)| \geq 2$ if x and y are at distance 2, and $|f(x) - f(y)| \geq 1$ if x and y are at distance 3, for all x and y in $V(G)$. The $L(3,2,1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3,2,1)$-labeling with k as the maximum label.

Practically speaking the interference among channels may go beyond two or three levels. So we have to extend the interference level from three to the largest possible - the diameter of the corresponding graph. So the concept of $L(2,1)$ labeling was generalized to radio labeling.

Radio labeling was originally introduced by G. Chartrand et.al., [7] in 2001. A radio labeling of a graph G is an injective function $f : V(G) \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V(G)$,

$$|f(u) - f(v)| \geq \operatorname{diam}(G) - d(u,v) + 1.$$

The span of f is the difference of the largest and the smallest channels used, that is

$$\max_{u,v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of G is defined as the minimum span among all radio labelings of G and is denoted as $rn(G)$.

2 Some Existing Results

- Jean Clipperton et al.,[8] determined the $L(3,2,1)$-labeling number for paths, cycles, caterpillars, n-array trees, complete graphs and complete bipartite graphs.

- Ma-Lian Chia et al.,[9] determined the $L(3,2,1)$-labeling number for Cartesian product of paths and cycles, and the power of paths.

  Also, they presented upper bounds for the $L(3,2,1)$-labeling numbers of general graphs and trees.

- Shao[10] determined bounds for the $L(3,2,1)$-labeling numbers for Kneser graphs, extremely irregular graphs, Halin graphs.

- Liu and Shao[6] proved that for a planar graph G, $k(G) \leq 15(\Delta^2 - \Delta + 1)$, where Δ is the maximum degree of G.

- Murugan[11] determined the $L(3,2,1)$-labeling number for Fan, Double Fan, Wheel, Friendship graph in terms of the maximum degree of the graphs.

3 Concepts

Sze-Chin Shee and Yong-Song Ho[12] denoted the graph obtained from n copies of G by identifying their roots where G is a rooted graph by $G^{(n)}$ and it is named as one point union of n copies of the graph G. Kaneria and Meera Meghpara[13] used the notation P_n^t to denote the one point union of t copies of path P_n.

A graph G is obtained by replacing each vertices of P_n^t except the central vertex by the graphs $G_1, G_2, \ldots, G_{tn}$ is known as one point union of path of graphs. We shall denote such graph G by $P_n^t(G_1, G_2, \ldots, G_{tn})$, where P_n^t is the one point union of t copies of path P_n. Here, if $G_1 = G_2 = \cdots = G_{tn} = H$, then they denote this by $P_n^t(tn.H)$[13].

A k-$L(3,2,1)$-labeling is an $L(3,2,1)$-labeling whose k-number is k. Here, we consider one point union of paths of cycles C_n, $n > 3$ and determine the k-numbers of them.

4 Results

Lemma 1. *The k-number of one point union of paths of cycles C_4, $P_n^t(tn.C_4)$, $n \geq 3$, is*

$$k\left(P_n^t(tn.C_4)\right) = \begin{cases} 8 & \textit{if} \quad t = 1 \\ 9 & \textit{if} \quad t = 2, 3 \\ 10 & \textit{if} \quad t = 4 \\ 2t+1 & \textit{if} \quad t \geq 5 \end{cases}$$

Proof. Consider $P_n^t(tn.C_4)$. Let u be the central vertex (root) of $P_n^t(tn.C_4)$. Let $v_{\alpha,\beta,i}$, $\alpha = 1, 2, \ldots, t$, $\beta = 1, 2, \ldots, n$, $i = 1, 2, 3, 4$ be the vertices of $P_n^t(tn.C_4)$. We note that for a fixed α and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1, 2, 3, 4$, are the vertices of the cycle C_4 at the β^{th} position of the branch α of $P_n^t(tn.C_4)$. Now we define $f : V(P_n^t(tn.C_4)) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling.

Case 1: $t = 1$.

Join u with $v_{1,1,1}$ and let $f(u) = 8$.

If β is odd, let $f(v_{1,\beta,1}) = 5$, $f(v_{1,\beta,2}) = 2$, $f(v_{1,\beta,3}) = 7$, $f(v_{1,\beta,4}) = 0$.

If β is even, let $f(v_{1,\beta,1}) = 6$, $f(v_{1,\beta,2}) = 3$, $f(v_{1,\beta,3}) = 8$, $f(v_{1,\beta,4}) = 1$.

If β is odd join $v_{1,\beta,4}$ with $v_{1,\beta+1,2}$ and if β is even join $v_{1,\beta,3}$ with $v_{1,\beta+1,1}$.

Simple verification yields that f is an $L(3,2,1)$-labeling, and so $k(P_n^1(n.C_4)) \leq 8$. Since the maximum degree of $P_n^1(n.C_4)$ is 3 and there are 3 vertices with maximum degree 3 such that the distance between any two of them is less than or equal to 3, $k(P_n^1(n.C_4)) \geq 8$ [9]. Hence $k(P_n^1(n.C_4)) = 8$.

Case 2: $t = 2$.

Join u with $v_{1,1,1}$ and $v_{2,1,3}$. Let $f(u) = 9$.

If $\alpha = 1, 2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 0$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 2$, $f(v_{\alpha,\beta,4}) = 7$.

If $\alpha = 1, 2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 6$, $f(v_{\alpha,\beta,3}) = 3$, $f(v_{\alpha,\beta,4}) = 8$.

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,4}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,1}$.

Simple verification yields that f is an $L(3,2,1)$-labeling, and so $k(P_n^2(2n.C_4)) \leq 9$. Since the maximum degree of $P_n^2(2n.C_4)$ is 3 and there are 3 vertices with maximum degree 3 such that the distance between any two of them is less than or equal to 3, $k(P_n^2(2n.C_4)) \geq 8$ [9]. Suppose $k(P_n^2(2n.C_4)) = 8$, then there is an optimal $L(3,2,1)$-labeling f_1 for $P_n^2(2n.C_4)$ with labels $\{0,1,2,\ldots,8\}$. Without loss of generality, we assume that $f(u) = 8$ and let u is adjacent with $v_{1,1,1}$ and $v_{2,1,1}$.Then $v_{1,1,1}$ cannot get labels 8,7,6. If $v_{1,1,1}$ receives label 5 then $v_{2,1,1}$ cannot get labels 8,7,6,5,4. We note that the label 8 cannot be used for $v_{2,1,1}$, $v_{2,1,2}$, $v_{2,1,3}$ and $v_{2,1,4}$. Also we note that the label 7 can be used only at $v_{2,1,3}$. If $v_{2,1,1}$ receives label 3 then and $v_{2,1,3}$ receives label 7 then the remaining 2 vertices (at least one) cannot be labeled with any label of f_1, since f_1 has to satisfy $L(3,2,1)$-labeling condition. This is a contradiction to f_1. Similar cases can be disposed similarly. Therefore, $k(P_n^2(2n.C_4)) \geq 9$. Hence $k(P_n^2(2n.C_4)) = 9$.

Case 3: $t = 3$.

Join u with $v_{1,1,1}$, $v_{2,1,3}$ and $v_{3,1,2}$. Let $f(u) = 9$.

If $\alpha = 1,2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 0$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 2$, $f(v_{\alpha,\beta,4}) = 7$.

If $\alpha = 1,2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 6$, $f(v_{\alpha,\beta,3}) = 3$, $f(v_{\alpha,\beta,4}) = 8$.

If $\alpha = 3$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 6$, $f(v_{\alpha,\beta,3}) = 3$, $f(v_{\alpha,\beta,4}) = 8$.

If $\alpha = 3$ and β is even, then let $f(v_{\alpha,\beta,1}) = 0$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 2$, $f(v_{\alpha,\beta,4}) = 7$.

For $\alpha = 1,2$, if β is odd join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,4}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,1}$.

For $\alpha = 3$, if β is odd join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,1}$ and if β is even join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,4}$.

Simple verification yields that f is an $L(3,2,1)$-labeling and so $k(P_n^3(3n.C_4)) \leq 9$. Since $P_n^2(2n.C_4)$ is a sub graph of $P_n^3(3n.C_4)$ and $k(P_n^2(2n.C_4)) = 9$, we have, $k(P_n^3(3n.C_4)) \geq 9$. Hence $k(P_n^3(3n.C_4)) = 9$.

Case 4: $t = 4$.

Join u with $v_{1,1,1}$, $v_{2,1,3}$, $v_{3,1,2}$ and $v_{4,1,4}$. Let $f(u) = 0$.

If $\alpha = 1,2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 3$, $f(v_{\alpha,\beta,2}) = 8$, $f(v_{\alpha,\beta,3}) = 5$, $f(v_{\alpha,\beta,4}) = 10$.

If $\alpha = 1,2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 7$, $f(v_{\alpha,\beta,3}) = 4$, $f(v_{\alpha,\beta,4}) = 9$.

If $\alpha = 3,4$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 7$, $f(v_{\alpha,\beta,3}) = 4$, $f(v_{\alpha,\beta,4}) = 9$.

If $\alpha = 3,4$ and β is even, then let $f(v_{\alpha,\beta,1}) = 3$, $f(v_{\alpha,\beta,2}) = 8$, $f(v_{\alpha,\beta,3}) = 5$, $f(v_{\alpha,\beta,4}) = 10$.

For $\alpha = 1,2$, if β is odd join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,3}$.

For $\alpha = 3,4$, if β is odd join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,3}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,2}$.

Simple verification yields that f is an $L(3,2,1)$-labeling and so $k(P_n^4(4n.C_4)) \leq 10$. Since the maximum degree of $P_n^4(4n.C_4)$ is 4, $k(P_n^4(4n.C_4)) \geq 2\Delta + 1 = 9$ [9].

Suppose $k(P_n^4(4n.C_4)) = 9$, then there is an optimal $L(3,2,1)$-labeling f_1 for $P_n^4(4n.C_4)$ with labels $\{0,1,2,\ldots,9\}$. Since f_1 is an optimal labeling with $k(P_n^4(4n.C_4)) = 9$, the vertex with maximum degree u receives label 0 or 9 [9].

Without loss of generality, we assume that u receives the label 9 and let u is adjacent with $v_{1,1,1}$, $v_{2,1,1}$, $v_{3,1,1}$ and $v_{4,1,1}$. Since f_1 is an optimal $L(3,2,1)$-labeling, the vertices $v_{1,1,1}$, $v_{2,1,1}$, $v_{3,1,1}$ and $v_{4,1,1}$ will receive labels at least 0, 2, 4, 6 respectively. Now consider the C_4 at $\alpha = 3$ and $\beta = 1$. We note that if the label of $v_{3,1,1}$ is 4, then $v_{3,1,2}$ and $v_{3,1,4}$ cannot receive labels 9, 8, 6, 5, 4, 3, 2, 0. So, the remaining labels 7, 1 can be given to $v_{3,1,2}$ and $v_{3,1,4}$. Then there is no suitable label available for $v_{3,1,3}$. This is a contradiction to f_1. Therefore, $k(P_n^4(4n.C_4)) \geq 10$. $k(P_n^4(4n.C_4)) = 10$.

Case 5: $t \geq 5$.

Join u with $v_{1,1,1}$, $v_{2,1,3}$, $v_{3,1,2}$ and $v_{4,1,4}$. Also join u with $v_{\alpha,1,4}$ for $\alpha \geq 5$. Let $f(u) = 0$.

If $\alpha = 1,2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 3$, $f(v_{\alpha,\beta,2}) = 8$, $f(v_{\alpha,\beta,3}) = 5$, $f(v_{\alpha,\beta,4}) = 10$.

If $\alpha = 1,2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 7$, $f(v_{\alpha,\beta,3}) = 4$, $f(v_{\alpha,\beta,4}) = 9$.

If $\alpha = 3,4$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 7$, $f(v_{\alpha,\beta,3}) = 4$, $f(v_{\alpha,\beta,4}) = 9$.

If $\alpha \geq 5$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 7$, $f(v_{\alpha,\beta,3}) = 4$, $f(v_{\alpha,\beta,4}) = 2\alpha + 1$.

If $\alpha \geq 3$ and β is even, then let $f(v_{\alpha,\beta,1}) = 3$, $f(v_{\alpha,\beta,2}) = 8$, $f(v_{\alpha,\beta,3}) = 5$, $f(v_{\alpha,\beta,4}) = 10$.

For $\alpha = 1,2$, if β is odd join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,3}$.

For $\alpha \geq 3$, if β is odd join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,3}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,2}$.

Now we show that f is an $L(3,2,1)$-labeling. Since the labels of C_4 are either $(3,8,5,10)$ or $(2,7,4,9)$ or $(2,7,4,2\alpha{+}1)$, $\alpha \geq 5$, cyclically, all C_4s satisfy $L(3,2,1)$-labeling conditions.

Now we consider other adjacent vertices.

Clearly, $|f(u) - f(v_{1,1,1})| = 3$, $|f(u) - f(v_{2,1,3})| = 5$, $|f(u) - f(v_{3,2,1})| = 7$, $|f(u) - f(v_{4,1,4})| = 9$ and for all $\alpha \geq 5$, $|f(u) - f(v_{\alpha,1,4})| = 2\alpha + 1 \geq 3$. For all α and β, the difference of the vertex labels joining any two C_4s is equal to 3. Now we consider vertices at distance 2. Since the labels of the vertices of C_4s which are joined to u are in the form $2\alpha + 1$, the label difference of any two such vertices which are at distance 2 are always greater than or equal to 2. The label difference of the vertices which are at distance 2 on the α^{th} branch is greater than or equal to 2, by construction of f. Now we consider vertices at distance 3. For each α, cycles at odd β and even β have two different set of labels. Thus the labels at the vertices at distance 3 are distinct and so their label difference is greater than or equal to 1. Therefore, f is an $L(3,2,1)$-labeling and hence $k(P_n^t(tn.C_4)) \leq 2t + 1$. Since the maximum degree of $P_n^t(tn.C_4)$ is t, $k(P_n^t(tn.C_4)) \geq 2t + 1$. Hence, $k(P_n^t(tn.C_4)) = 2t + 1$. □

Lemma 2. *k-$L(3,2,1)$-labeling of C_5 is unique.*

Proof. Since $k(C_5) = 8$, in any k-$L(3,2,1)$-labeling, labels 0 and 8 should present on C_5. $(0,4,8,2,6)$ is one such labeling. In C_5, since any two non-adjacent vertices are at distance 2, the labels of C_5 are all distinct even integers between 0 and 8 in any k-$L(3,2,1)$-labeling such that adjacent labels keeps difference 3. Hence the labels 0, 2, 4, 6, 8 alone can occur on C5 such that adjacent labels keeps difference 3. Hence, k-$L(3,2,1)$-labeling of C_5, $(0,4,8,2,6)$ is unique. □

Lemma 3. *The k-number of one point union of paths of cycles C_5, $P_n^t(tn.C_5)$, $n \geq 3$, is*

$$k(P_n^t(tn.C_5)) = \begin{cases} 11 & \text{if} \quad t \leq 4 \\ 2t+2 & \text{if} \quad t \geq 5 \end{cases}$$

Proof. Consider $P_n^t(tn.C_5)$. Let u be the central vertex (root) of $P_n^t(tn.C_5)$. Let $v_{\alpha,\beta,i}$, $\alpha = 1,2,\ldots,t$, $\beta = 1,2,\ldots,n$, $i = 1,2,\ldots,5$ be the vertices of $P_n^t(tn.C_5)$. We note that for a fixed α, and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1,2,\ldots,5$, are the vertices of the cycle C_5 at the β^{th} position of the branch α of $P_n^t(tn.C_5)$. Now we define $f : V(P_n^t(tn.C_5)) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling.

Case 1: $t \leq 4$.

Join u with $v_{1,1,2}$, $v_{2,1,4}$, $v_{3,1,3}$, $v_{4,1,5}$ and let $f(u) = 0$.

If $\alpha = 1,2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 7$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 9$, $f(v_{\alpha,\beta,4}) = 5$, $f(v_{\alpha,\beta,5}) = 11$.

If $\alpha = 1,2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 4$, $f(v_{\alpha,\beta,2}) = 0$, $f(v_{\alpha,\beta,3}) = 6$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 8$.

If $\alpha = 3,4$ and $\beta = 1$, then let $f(v_{\alpha,\beta,1}) = 6$, $f(v_{\alpha,\beta,2}) = 2$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 4$, $f(v_{\alpha,\beta,5}) = 10$.

If $\alpha = 3,4$, β odd and $\beta \neq 1$, then let $f(v_{\alpha,\beta,1}) = 6$, $f(v_{\alpha,\beta,2}) = 2$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 4$, $f(v_{\alpha,\beta,5}) = 0$.

If $\alpha = 3,4$ and β is even, then let $f(v_{\alpha,\beta,1}) = 7$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 9$, $f(v_{\alpha,\beta,4}) = 5$, $f(v_{\alpha,\beta,5}) = 11$.

For $\alpha = 1,2$, if β is odd join $v_{\alpha,\beta,5}$ with $v_{\alpha,\beta+1,4}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha = 3,4$, if β is odd join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,5}$.

Simple verification shows that f is an $L(3,2,1)$-labeling and so $k(P_n^t(tn.C_5)) \leq 11$. Let G be two disjoint C_5s joined by an edge. Let there is an optimal $L(3,2,1)$-labeling for G. We know that $k(C_5) = 8$ and such a k-$L(3,2,1)$-labeling is unique. So, suppose the first C_5 has the labeling $(0,4,8,2,6)$ in the cyclical order, then the second C_5 cannot have $(1,5,9,3,7)$ or $(2,6,10,4,8)$, since any two vertices on C_5 are at distance 2. Therefore, the second C_5 should have $(3,7,11,5,9)$ as its labeling in the cyclical order. Thus, $k(G) \geq 11$. Since G is a sub graph of $P_n^t(tn.C_5)$, we have $k(P_n^t(tn.C_5)) \geq 11$. Hence $k(P_n^t(tn.C_5)) = 11$.

Case 2: $t \geq 5$.

Join u with $v_{1,1,2}$, $v_{2,1,4}$, $v_{3,1,3}$, $v_{4,1,5}$ and $v_{\alpha,1,5}$, $\alpha \geq 5$. Let $f(u) = 0$.

If $\alpha = 1,2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 7$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 9$, $f(v_{\alpha,\beta,4}) = 5$, $f(v_{\alpha,\beta,5}) = 11$.

If $\alpha = 1,2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 4$, $f(v_{\alpha,\beta,2}) = 0$, $f(v_{\alpha,\beta,3}) = 6$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 8$.

If $\alpha = 3,4$ and $\beta = 1$, then let $f(v_{\alpha,\beta,1}) = 6$, $f(v_{\alpha,\beta,2}) = 2$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 4$, $f(v_{\alpha,\beta,5}) = 10$.

If $\alpha = 3,4$ and $\beta \neq 1$ is odd, then let $f(v_{\alpha,\beta,1}) = 6$, $f(v_{\alpha,\beta,2}) = 2$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 4$, $f(v_{\alpha,\beta,5}) = 0$.

If $\alpha = 3, 4$ and β is even, then let $f(v_{\alpha,\beta,1}) = 7$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 9$, $f(v_{\alpha,\beta,4}) = 5$, $f(v_{\alpha,\beta,5}) = 11$.

If $\alpha \geq 5$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 6$, $f(v_{\alpha,\beta,2}) = 2$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 4$, $f(v_{\alpha,\beta,5}) = 2\alpha + 2$.

If $\alpha \geq 5$ and β is even, then let $f(v_{\alpha,\beta,1}) = 7$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 9$, $f(v_{\alpha,\beta,4}) = 5$, $f(v_{\alpha,\beta,5}) = 11$.

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,5}$ with $v_{\alpha,\beta+1,4}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha \geq 3$, if β is odd join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,5}$.

Now we show that f is an $L(3, 2, 1)$-labeling. Since the labels of C_5 are either $(7, 3, 9, 5, 11)$ cyclically or $(4, 0, 6, 2, 8)$ or $(6, 2, 8, 4, 10)$ or $(6, 2, 8, 4, 2\alpha + 2)$, $\alpha \geq 5$, cyclically, all C_5s satisfy $L(3, 2, 1)$-labeling conditions.

Now we consider adjacent vertices.

Clearly, $|f(u) - f(v_{1,1,2})| = 3$, $|f(u) - f(v_{2,1,4})| = 5$, $|f(u) - f(v_{3,1,3})| = 8$, $|f(u) - f(v_{4,1,5})| = 10$ and for all $\alpha \geq 5$, $|f(u) - f(v_{\alpha,1,5})| = 2\alpha + 2 \geq 3$. For all α and β, the difference of the vertex labels joining any two C_5s is equal to 9 or $2\alpha - 7 \geq 3$, $(\alpha \geq 5)$. Now we consider vertices at distance 2. For $\alpha \geq 5$, since the labels of the vertices of C_5s which are joined to u are in the form $2\alpha + 2$, the label difference of any two such vertices which are at distance 2 are always greater than or equal to 2. The label difference of the vertices which are at distance 2 on the α^{th} branch is greater than or equal to 2, by construction of f. Now we consider vertices at distance 3. For each α, cycles at odd β and even β have two different sets of labels. Thus the labels at the vertices at distance 3 are distinct and so their label difference is greater than or equal to 1. Therefore, f is an $L(3, 2, 1)$-labeling and hence $k(P_n^t(tn.C_5)) \leq 2t + 2$. Without loss of generality, let $v_1, v_2, \ldots, v_t$ be the vertices of $P_n^t(tn.C_5)$ which are adjacent to u such that the labels in an optimal labeling are in the increasing order. Since any two v_is are at distance 2, v_t should have at least the label $2(t - 1)$. But u cannot be adjacent with the two labels which are adjacent to the label at v_1. That is, v_t should have at least the label $2(t - 1 + 2) = 2(t + 1) = 2t + 2$. That is, $k(P_n^t(tn.C_5)) \geq 2t + 2$. Hence, $k(P_n^t(tn.C_5)) = 2t + 2$. □

Lemma 4. *The k-number of one point union of paths of cycles C_6, $P_n^t(tn.C_6)$, $n \geq 3$, is*

$$k(P_n^t(tn.C_6)) = \begin{cases} 8 & \text{if } t = 1 \\ 9 & \text{if } t = 2, 3 \\ 2t + 1 & \text{if } t \geq 4 \end{cases}$$

Proof. Consider $P_n^t(tn.C_6)$. Let u be the central vertex (root) of $P_n^t(tn.C_6)$. Let $v_{\alpha,\beta,i}$, $\alpha = 1, 2, \ldots, t$, $\beta = 1, 2, \ldots, n$, $i = 1, 2, \ldots, 6$ be the vertices of $P_n^t(tn.C_6)$. We note that for a fixed α, and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1, 2, \ldots, 6$, are the vertices of the cycle C_6 at the β^{th} position of the branch α of $P_n^t(tn.C_6)$. Now we define $f : V(P_n^t(tn.C_6)) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3, 2, 1)$-labeling.

Case 1: $t = 1$.

Join u with $v_{1,1,4}$ and let $f(u) = 8$.

If β is odd, then let $f(v_{1,\beta,1}) = 0$, $f(v_{1,\beta,2}) = 3$, $f(v_{1,\beta,3}) = 6$, $f(v_{1,\beta,4}) = 1$, $f(v_{1,\beta,5}) = 4$, $f(v_{1,\beta,6}) = 7$.

If β is even, then let $f(v_{1,\beta,1}) = 1$, $f(v_{1,\beta,2}) = 4$, $f(v_{1,\beta,3}) = 7$, $f(v_{1,\beta,4}) = 2$, $f(v_{1,\beta,5}) = 5$, $f(v_{1,\beta,6}) = 8$.

If β is odd, join $v_{1,\beta,2}$ with $v_{1,\beta+1,6}$ and if β is even, join $v_{1,\beta,5}$ with $v_{1,\beta+1,1}$.

Simple verification yields that f is an $L(3,2,1)$-labeling and so $k(P_n^1(n.C_6)) \leq 8$. Since the maximum degree of $P_n^1(n.C_6)$ is 3 and there are 3 vertices with maximum degree 3 such that the distance between any two of them is less than or equal to 3, $k(P_n^1(n.C_6)) \geq 8$ [9]. Hence $k(P_n^1(n.C_6)) = 8$.

Case 2: $t = 2$.

Join u with $v_{1,1,4}$, and $v_{2,1,3}$. Let $f(u) = 0$.

If $\alpha = 1, 2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 9$.

If $\alpha = 1, 2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 4$, $f(v_{\alpha,\beta,3}) = 7$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 5$, $f(v_{\alpha,\beta,6}) = 8$.

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$.

Simple verification yields that f is an $L(3,2,1)$-labeling and so $k(P_n^2(2n.C_6)) \leq 9$.

Suppose $k(P_n^2(2n.C_6)) \leq 8$, since $P_n^1(n.C_6)$ is a sub graph of $P_n^2(2n.C_6)$ and $k(P_n^1(n.C_6)) = 8$, $k(P_n^2(2n.C_6)) \geq 8$ and so $k(P_n^2(2n.C_6)) = 8$. That is, there is an optimal $L(3,2,1)$-labeling f_1 with labels $\{0, 1, 2, \ldots, 8\}$ to $P_n^2(2n.C_6)$. Without loss of generality, we assume that $f_1(u) = 8$. Then the labels 8,7,6 cannot be given to the vertices of C_6s which are adjacent to u. If 5 or 3 are given to these vertices then let the vertex on the C_6 which receive the label 5 be say v. Since any two vertices on this C_6 are at a maximum distance 3, all the labels of this C_6 are distinct. Then the vertices at distance 1 from v may receive labels 2 or 0 and the vertices at distance 2 from v may receive labels 7 or 3. Then there is no suitable label of f_1 available for the vertex at distance 3 from v. This is a contradiction to f_1. Other cases can be disposed similarly. Therefore, $k(P_n^2(2n.C_6)) = 9$.

Case 3: $t = 3$.

Join u with $v_{1,1,4}$, $v_{2,1,2}$ and $v_{3,1,3}$. Let $f(u) = 0$.

If $\alpha = 1, 2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 9$.

If $\alpha = 1, 2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 4$, $f(v_{\alpha,\beta,3}) = 7$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 5$, $f(v_{\alpha,\beta,6}) = 8$.

If $\alpha = 3$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 4$, $f(v_{\alpha,\beta,3}) = 7$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 5$, $f(v_{\alpha,\beta,6}) = 8$.

If $\alpha = 3$ and β is even, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 9$.

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha = 3$, if β is odd join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$.

Simple verification yields that f is an $L(3,2,1)$-labeling and so $k(P_n^3(3n.C_6)) \leq 9$.

Since $P_n^2(2n.C_6)$ is a sub graph of $P_n^3(3n.C_6)$ and $k(P_n^2(2n.C_6)) = 9$, we have, $k(P_n^3(3n.C_6)) \geq 9$. Hence, $k(P_n^3(3n.C_6)) = 9$.

Case 4: $t \geq 4$.

Join u with $v_{1,1,4}$, $v_{2,1,2}$, $v_{3,1,3}$, $v_{4,1,6}$ and $v_{5,1,3}$. Also join u with $v_{\alpha,1,1}$ for $\alpha \geq 6$. Let $f(u) = 0$.

If $\alpha = 1, 2, 4$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 9$.

If $\alpha = 1, 2, 4$ and β is even, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 4$, $f(v_{\alpha,\beta,3}) = 7$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 5$, $f(v_{\alpha,\beta,6}) = 8$.

If $\alpha = 3, 5$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 4$, $f(v_{\alpha,\beta,3}) = 2\alpha+1$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 5$, $f(v_{\alpha,\beta,6}) = 8$.

If $\alpha = 3, 5$ and β is even, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 9$.

If $\alpha \geq 6$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2\alpha + 1$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 9$.

If $\alpha \geq 6$ and β is even, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 4$, $f(v_{\alpha,\beta,3}) = 7$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 5$, $f(v_{\alpha,\beta,6}) = 8$.

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha = 3, 5$, if β is odd join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$.

For $\alpha = 4$, if β is odd join $v_{\alpha,\beta,5}$ with $v_{\alpha,\beta+1,1}$ and if β is even join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,6}$.

For $\alpha \geq 6$, if β is odd join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$.

Now we show that f is an $L(3, 2, 1)$-labeling. Since the labels of C_6 are either $(2, 5, 8, 3, 6, 9)$ or $(1, 4, 7, 2, 5, 8)$ or $(2\alpha + 1, 5, 8, 3, 6, 9)$, $\alpha \geq 5$, cyclically, all C_6s satisfy $L(3, 2, 1)$-labeling conditions.

Now we consider adjacent vertices.

Clearly, $|f(u) - f(v_{1,1,4})| = 3$, $|f(u) - f(v_{2,1,2})| = 5$, $|f(u) - f(v_{3,1,3})| = 7$, $|f(u) - f(v_{4,1,4})| = 9$ and for all $\alpha \geq 5$, $|f(u) - f(v_{\alpha,1,1})| = 2\alpha + 1 \geq 3$. For all α and β, the difference of the vertex labels joining any two C_6s is equal to 5. Now we consider vertices at distance 2. Since the labels of the vertices of C_6s which are joined to u are in the form $2\alpha + 1$, the label difference of any two such vertices which are at distance 2 are always greater than or equal to 2. The label difference of the vertices which are at distance 2 on the α^{th} branch is greater than or equal to 2, by construction of f. Now we consider vertices at distance 3. For each α, cycles at odd β and even β have two different sets of labels. Thus the labels at the vertices at distance 3 are distinct and so their label difference is greater than or equal to 1. Therefore, f is an $L(3, 2, 1)$-labeling and hence $k(P_n^t(tn.C_6)) \leq 2t + 1$. Since the maximum degree of $P_n^t(tn.C_6)$ is t, $k(P_n^t(tn.C_6)) \geq 2t + 1$. Hence, $k(P_n^t(tn.C_6)) = 2t + 1$. □

Lemma 5. *The k-number of one point union of paths of cycles C_7, $P_n^t(tn.C_7)$, $n \geq 3$, is $k(P_n^t(tn.C_7)) = 2t + 1$ if $t \geq 5$.*

Proof. Consider $P_n^t(tn.C_7)$. Let u be the central vertex (root) of $P_n^t(tn.C_7)$. Let $v_{\alpha,\beta,i}$, $\alpha = 1, 2, \ldots, t$, $\beta = 1, 2, \ldots, n$, $i = 1, 2, \ldots, 7$ be the vertices of $P_n^t(tn.C_7)$. We note that for a fixed α, and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1, 2, \ldots, 7$, are the vertices of the cycle C_7 at the β^{th} position of the branch α of $P_n^t(tn.C_7)$. Now we define $f : V(P_n^t(tn.C_7)) \rightarrow \mathbb{N} \cup \{0\}$ such that f is an $L(3, 2, 1)$-labeling.

Join u with $v_{1,1,4}$, $v_{2,1,2}$, $v_{3,1,6}$ and $v_{4,1,6}$. Also join u with $v_{\alpha,1,3}$ for $\alpha \geq 5$. Let $f(u) = 0$.

If $\alpha = 1, 2$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 10$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 11$, $f(v_{\alpha,\beta,7}) = 8$.

If $\alpha = 1, 2$ and β is even, then let $f(v_{\alpha,\beta,1}) = 0$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 1$, $f(v_{\alpha,\beta,5}) = 4$, $f(v_{\alpha,\beta,6}) = 9$, $f(v_{\alpha,\beta,7}) = 6$.

If $\alpha = 3$ and β is odd, then let $f(v_{\alpha,\beta,1}) = 1$, $f(v_{\alpha,\beta,2}) = 8$, $f(v_{\alpha,\beta,3}) = 5$, $f(v_{\alpha,\beta,4}) = 2$, $f(v_{\alpha,\beta,5}) = 10$, $f(v_{\alpha,\beta,6}) = 7$, $f(v_{\alpha,\beta,7}) = 4$.

If $\alpha = 3$ and β is even, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 10$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 11$, $f(v_{\alpha,\beta,7}) = 8$.

If $\alpha = 4$ and $\beta = 1$, then let $f(v_{\alpha,\beta,1}) = 11$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 1$, $f(v_{\alpha,\beta,5}) = 4$, $f(v_{\alpha,\beta,6}) = 9$, $f(v_{\alpha,\beta,7}) = 6$.

If $\alpha = 4$ and β is odd $(\neq 1)$, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 10$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 11$, $f(v_{\alpha,\beta,7}) = 8$.

If $\alpha = 4$ and β is even, then let $f(v_{\alpha,\beta,1}) = 0$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 1$, $f(v_{\alpha,\beta,5}) = 4$, $f(v_{\alpha,\beta,6}) = 9$, $f(v_{\alpha,\beta,7}) = 6$.

If $\alpha \geq 5$ and $\beta = 1$, then let $f(v_{\alpha,\beta,1}) = 3$, $f(v_{\alpha,\beta,2}) = 6$, $f(v_{\alpha,\beta,3}) = 2\alpha + 1$, $f(v_{\alpha,\beta,4}) = 4$, $f(v_{\alpha,\beta,5}) = 7$, $f(v_{\alpha,\beta,6}) = 1$, $f(v_{\alpha,\beta,7}) = 9$.

If $\alpha \geq 5$ and β is odd $(\neq 1)$, then let $f(v_{\alpha,\beta,1}) = 2$, $f(v_{\alpha,\beta,2}) = 5$, $f(v_{\alpha,\beta,3}) = 10$, $f(v_{\alpha,\beta,4}) = 3$, $f(v_{\alpha,\beta,5}) = 6$, $f(v_{\alpha,\beta,6}) = 11$, $f(v_{\alpha,\beta,7}) = 8$.

If $\alpha \geq 5$ and β is even, then let $f(v_{\alpha,\beta,1}) = 0$, $f(v_{\alpha,\beta,2}) = 3$, $f(v_{\alpha,\beta,3}) = 8$, $f(v_{\alpha,\beta,4}) = 1$, $f(v_{\alpha,\beta,5}) = 4$, $f(v_{\alpha,\beta,6}) = 9$, $f(v_{\alpha,\beta,7}) = 6$.

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,3}$.

For $\alpha = 3$, if β is odd join $v_{\alpha,\beta,7}$ with $v_{\alpha,\beta+1,6}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,2}$.

For $\alpha = 4$, if $\beta = 1$, join $v_{\alpha,\beta,7}$ with $v_{\alpha,\beta+1,4}$.

For $\alpha = 4$, if β is odd $(\neq 1)$, join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,3}$.

For $\alpha \geq 5$, if $\beta = 1$, join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,4}$.

For $\alpha \geq 5$, if β is odd $(\neq 1)$, join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,3}$.

Now we show that f is an $L(3, 2, 1)$-labeling. Since the labels of C_7 are either $(2, 5, 10, 3, 6, 11, 8)$ or $(0, 3, 8, 1, 4, 9, 6)$ or $(1, 8, 5, 2, 10, 7, 4)$ or $(11, 3, 8, 1, 4, 9, 6)$ or $(3, 6, 2\alpha+1, 4, 7, 1, 9)$, $\alpha \geq 5$, cyclically, all C_7s satisfy $L(3, 2, 1)$-labeling conditions.

Now we consider adjacent vertices of $P_n^t(tn.C_7)$.

Clearly, $|f(u) - f(v_{1,1,4})| = 3$, $|f(u) - f(v_{2,1,2})| = 5$, $|f(u) - f(v_{3,1,6})| = 7$, $|f(u) - f(v_{4,1,6})| = 9$ and for all $\alpha \geq 5$, $|f(u) - f(v_{\alpha,1,3})| = 2\alpha+1 \geq 3$. For all α and β, the difference of the vertex labels joining any two C_7s is equal to 5 or 7 or 9. Now we consider vertices at distance 2. Since the labels of the vertices of C_7s which are joined to u are in the form $2\alpha+1$, the label difference of any two such vertices which are at distance 2 are always greater than or equal to 2. The label difference of the vertices which are at distance 2 on the α^{th} branch is greater than or equal to 2, by construction of f. Now we consider vertices at distance 3. By f, for each α, cycles at odd β and even β have two different sets of labels and the labels at the vertices at distance 3 are distinct, their label difference is greater than or equal to 1. Therefore,

f is an $L(3,2,1)$-labeling and hence $k(P_n^t(tn.C_7)) \leq 2t+1$. Since the maximum degree of $P_n^t(tn.C_7)$ is t, $k(P_n^t(tn.C_7)) \geq 2t+1$. Hence, $k(P_n^t(tn.C_7)) = 2t+1$. □

Lemma 6. *The k-number of one point union of paths of cycles C_{11}, $P_n^t(tn.C_{11})$, $n \geq 3$, is $k(P_n^t(tn.C_{11})) = 2t+1$ if $t \geq 5$.*

Proof. Consider $P_n^t(tn.C_{11})$. Let u be the central vertex (root) of $P_n^t(tn.C_{11})$. Let $v_{\alpha,\beta,i}$, $\alpha = 1,2,\ldots,t$, $\beta = 1,2,\ldots,n$, $i = 1,2,\ldots,11$ be the vertices of $P_n^t(tn.C_{11})$. We note that for a fixed α, and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1,2,\ldots,11$, are the vertices of the cycle C_{11} at the β^{th} position of the branch α of $P_n^t(tn.C_{11})$. Now we define $f : V(P_n^t(tn.C_{11})) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling.

Join u with $v_{1,1,3}$, $v_{2,1,5}$, $v_{3,1,2}$ and $v_{4,1,4}$. Also join u with $v_{\alpha,1,4}$ for $\alpha \geq 5$. Let $f(u) = 0$.

If $\alpha = 1,2$ and β is odd, then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 1, & f(v_{\alpha,\beta,2}) = 6, & f(v_{\alpha,\beta,3}) = 3, & f(v_{\alpha,\beta,4}) = 8, \\ f(v_{\alpha,\beta,5}) = 5, & f(v_{\alpha,\beta,6}) = 1, & f(v_{\alpha,\beta,7}) = 9, & f(v_{\alpha,\beta,8}) = 6, \\ f(v_{\alpha,\beta,9}) = 2, & f(v_{\alpha,\beta,10}) = 8, & f(v_{\alpha,\beta,11}) = 4. & \end{array}$$

If $\alpha = 1,2$ and β is even, then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 0, & f(v_{\alpha,\beta,2}) = 5, & f(v_{\alpha,\beta,3}) = 2, & f(v_{\alpha,\beta,4}) = 7, \\ f(v_{\alpha,\beta,5}) = 4, & f(v_{\alpha,\beta,6}) = 0, & f(v_{\alpha,\beta,7}) = 8, & f(v_{\alpha,\beta,8}) = 5, \\ f(v_{\alpha,\beta,9}) = 1, & f(v_{\alpha,\beta,10}) = 7, & f(v_{\alpha,\beta,11}) = 3. & \end{array}$$

If $\alpha = 3$ and β is odd, then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 2, & f(v_{\alpha,\beta,2}) = 7, & f(v_{\alpha,\beta,3}) = 4, & f(v_{\alpha,\beta,4}) = 9, \\ f(v_{\alpha,\beta,5}) = 6, & f(v_{\alpha,\beta,6}) = 2, & f(v_{\alpha,\beta,7}) = 10, & f(v_{\alpha,\beta,8}) = 7, \\ f(v_{\alpha,\beta,9}) = 3, & f(v_{\alpha,\beta,10}) = 9, & f(v_{\alpha,\beta,11}) = 5. & \end{array}$$

If $\alpha = 3$ and β is even, then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 0, & f(v_{\alpha,\beta,2}) = 5, & f(v_{\alpha,\beta,3}) = 2, & f(v_{\alpha,\beta,4}) = 7, \\ f(v_{\alpha,\beta,5}) = 4, & f(v_{\alpha,\beta,6}) = 0, & f(v_{\alpha,\beta,7}) = 8, & f(v_{\alpha,\beta,8}) = 5, \\ f(v_{\alpha,\beta,9}) = 1, & f(v_{\alpha,\beta,10}) = 7, & f(v_{\alpha,\beta,11}) = 3. & \end{array}$$

If $\alpha \geq 4$ and $\beta = 1$, then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 2, & f(v_{\alpha,\beta,2}) = 7, & f(v_{\alpha,\beta,3}) = 4, & f(v_{\alpha,\beta,4}) = 2\alpha+1, \\ f(v_{\alpha,\beta,5}) = 6, & f(v_{\alpha,\beta,6}) = 2, & f(v_{\alpha,\beta,7}) = 10, & f(v_{\alpha,\beta,8}) = 7, \\ f(v_{\alpha,\beta,9}) = 3, & f(v_{\alpha,\beta,10}) = 9, & f(v_{\alpha,\beta,11}) = 5. & \end{array}$$

If $\alpha \geq 4$ and β is odd($\neq 1$), then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 2, & f(v_{\alpha,\beta,2}) = 7, & f(v_{\alpha,\beta,3}) = 4, & f(v_{\alpha,\beta,4}) = 9, \\ f(v_{\alpha,\beta,5}) = 6, & f(v_{\alpha,\beta,6}) = 2, & f(v_{\alpha,\beta,7}) = 10, & f(v_{\alpha,\beta,8}) = 7, \\ f(v_{\alpha,\beta,9}) = 3, & f(v_{\alpha,\beta,10}) = 9, & f(v_{\alpha,\beta,11}) = 5. & \end{array}$$

If $\alpha \geq 4$ and β is even, then let

$$\begin{array}{llll} f(v_{\alpha,\beta,1}) = 0, & f(v_{\alpha,\beta,2}) = 5, & f(v_{\alpha,\beta,3}) = 2, & f(v_{\alpha,\beta,4}) = 7, \\ f(v_{\alpha,\beta,5}) = 4, & f(v_{\alpha,\beta,6}) = 0, & f(v_{\alpha,\beta,7}) = 8, & f(v_{\alpha,\beta,8}) = 5, \\ f(v_{\alpha,\beta,9}) = 1, & f(v_{\alpha,\beta,10}) = 7, & f(v_{\alpha,\beta,11}) = 3. & \end{array}$$

For $\alpha = 1, 2$, if β is odd join $v_{\alpha,\beta,10}$ with $v_{\alpha,\beta+1,1}$ and if β is even join $v_{\alpha,\beta,11}$ with $v_{\alpha,\beta+1,7}$.

For $\alpha \geq 3$, if β is odd join $v_{\alpha,\beta,10}$ with $v_{\alpha,\beta+1,9}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,4}$.

Now we show that f is an $L(3,2,1)$-labeling. Since the labels of C_{11} in f are either (0,5,2,7,4,0,8,5,1,7,3) or (1,6,3,8,5,1,9,6,2,8,4) or (2,7,4,9,6,2,10,7,3,9,5) or $(2,7,4,2\alpha+1,6,2,10,7,3,9,5)$, $\alpha \geq 5$, cyclically, all C_{11}s satisfy $L(3,2,1)$-labeling conditions.

Now we consider adjacent vertices of $P_n^t(tn.C_{11})$.

Clearly, $|f(u) - f(v_{1,1,3})| = 3$, $|f(u) - f(v_{2,1,5})| = 5$, $|f(u) - f(v_{3,1,2})| = 7$, $|f(u) - f(v_{4,1,4})| = 9$ and for all $\alpha \geq 5$, $|f(u) - f(v_{\alpha,1,4})| = 2\alpha + 1 \geq 3$. For all α and β, the difference of the vertex labels joining any two C_{11}s is equal to 6 or 8 or 9. Now we consider vertices at distance 2. Since the labels of the vertices of C_{11}s which are joined to u are in the form $2\alpha + 1$, the label difference of any two such vertices which are at distance 2 are always greater than or equal to 2. The label difference of the vertices which are at distance 2 on the α^{th} branch is greater than or equal to 2, by construction of f. Now we consider vertices at distance 3. By f, for each α, cycles at odd β and even β have two different sets of labels and the labels at the vertices at distance 3 are distinct, their label difference is greater than or equal to 1. Therefore, f is an $L(3,2,1)$-labeling and hence $k(P_n^t(tn.C_{11})) \leq 2t + 1$. Since the maximum degree of $P_n^t(tn.C_{11})$ is t, $k(P_n^t(tn.C_{11})) \geq 2t + 1$. Hence, $k(P_n^t(tn.C_{11})) = 2t + 1$. □

Theorem 1. *The k-number of one point union of paths of even cycles C_m, $P_n^t(tn.C_m)$, $n \geq 3$, $t \geq 5$ is $k(P_n^t(tn.C_m)) = 2t + 1$.*

Proof. Consider $P_n^t(tn.C_m)$. Let u be the central vertex (root) of $P_n^t(tn.C_m)$. Let $v_{\alpha,\beta,i}$, $\alpha = 1, 2, \ldots, t$, $\beta = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$ be the vertices of $P_n^t(tn.C_m)$. We note that for a fixed α, and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1, 2, \ldots, m$, are the vertices of the cycle C_m at the β^{th} position of the branch α of $P_n^t(tn.C_m)$. Now we define $f : V(P_n^t(tn.C_m)) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling. We note that any even integer $m \geq 4$ can be written as $m = 4a_1 + 6a_2$, where a_1 and a_2 are non-negative integers.

Case 1: $a_1 \neq 0$.

Let $a_2 \neq 0$.

Join u with $v_{1,1,1}$, $v_{2,1,3}$, $v_{3,1,6}$ and $v_{4,1,9}$. Also join u with $v_{\alpha,1,4}$ for $\alpha \geq 5$. Let $f(u) = 0$.

Consider C_m.

If $\alpha = 1, 2, 3, 4$ and β is odd, then label the first $4a_1$ vertices of C_m with $3, 8, 5, 10$ repeatedly and then label the remaining $6a_2$ vertices of C_m with $3, 7, 11, 2, 9, 6$ repeatedly.

If $\alpha = 1, 2, 3, 4$ and β is even, then label the first $4a_1$ vertices of C_m with $2, 7, 4, 11$ repeatedly and then label the remaining $6a_2$ vertices of C_m with $2, 5, 8, 1, 4, 11$ repeatedly.

If $\alpha \geq 5$ and β is odd, then label the first $4a_1$ vertices of C_m with $2, 7, 4, 2\alpha+1$ repeatedly and then label the remaining $6a_2$ vertices of C_m with $2, 5, 8, 1, 4, 2\alpha+1$ repeatedly.

If $\alpha \geq 5$ and β is even, then label the first $4a_1$ vertices of C_m with $3, 8, 5, 10$ repeatedly and then label the remaining $6a_2$ vertices of C_m with $3, 6, 9, 2, 5, 11$ repeatedly.

For $\alpha = 1, 2, 3, 4$ if β is odd join $v_{\alpha,\beta,m}$ with $v_{\alpha,\beta+1,m}$ and if β is even join $v_{\alpha,\beta,4a_1}$ with $v_{\alpha,\beta+1,m}$.

For $\alpha \geq 5$, if β is odd join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,3}$ and if β is even join $v_{\alpha,\beta,4a_1}$ with $v_{\alpha,\beta+1,2}$.

Clearly, f is an $L(3, 2, 1)$-labeling, as the verification is very similar to Lemma 4.1 and 4.4. Therefore, $k(P_n^t(tn.C_m)) \leq 2t+1$. Since the maximum degree of $P_n^t(tn.C_m)$ is t, $k(P_n^t(tn.C_m)) \geq 2t+1$. Hence, $k(P_n^t(tn.C_m)) = 2t+1$.

If $a_2 = 0$, then the 4 labels of the cycles C_4 can be used repeatedly (in that order) for the $4a_1$ vertices, then the proof is very similar to Lemma 4.1.

Case 2: $a_1 = 0$.

Join u with $v_{1,1,4}$, $v_{2,1,2}$, $v_{3,1,3}$ and $v_{4,1,6}$. Also join u with $v_{\alpha,1,1}$ for $\alpha \geq 5$. Let $f(u) = 0$.

Consider C_m.

If $\alpha = 1, 2, 4$ and β is odd, then label the vertices of C_m with $2, 5, 8, 3, 6, 9$ repeatedly.

If $\alpha = 1, 2, 4$ and β is even, then label the vertices of C_m with $1, 4, 7, 2, 5, 8$ repeatedly.

If $\alpha = 3$ and β is odd, then label the vertices of C_m with $1, 4, 7, 2, 5, 8$ repeatedly.

If $\alpha = 3$ and β is even, then label the vertices of C_m with $2, 5, 8, 3, 6, 9$ repeatedly.

If $\alpha \geq 5$ and β is odd, then label the vertices of C_m with $1, 4, 2\alpha+1, 2, 5, 8$ repeatedly.

If $\alpha \geq 5$ and β is even, then label the vertices of C_m with $2, 5, 8, 3, 6, 9$ repeatedly.

For $\alpha = 1, 2$ and if β is odd, join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$ and if β is even join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha = 3$ and if β is odd, join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$.

For $\alpha = 4$ and if β is odd join $v_{\alpha,\beta,5}$ with $v_{\alpha,\beta+1,1}$ and if β is even join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,6}$.

For $\alpha \geq 5$ and if β is odd join $v_{\alpha,\beta,1}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,6}$ with $v_{\alpha,\beta+1,2}$.

Clearly, f is an $L(3, 2, 1)$-labeling, as the verification is very similar to Lemma 4.4. Therefore, $k(P_n^t(tn.C_m)) \leq 2t+1$. Since the maximum degree of $P_n^t(tn.C_m)$ is t, $k(P_n^t(tn.C_m)) \geq 2t+1$. Hence, $k(P_n^t(tn.C_m)) = 2t+1$. □

Theorem 2. *The k-number of one point union of paths of odd cycles C_m, $P_n^t(tn.C_m)$, $n \geq 3$, $m \geq 5$, $t \geq 5$ is*

$$k(P_n^t(tn.C_m)) = \begin{cases} 2t+2 & \text{if} \quad m \neq 7, 11 \\ 2t+1 & \text{if} \quad m = 7, 11 \end{cases}$$

Proof. Consider $P_n^t(tn.C_m)$. Let u be the central vertex (root) of $P_n^t(tn.C_m)$. Let $v_{\alpha,\beta,i}$, $\alpha = 1, 2, \ldots, t$, $\beta = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$ be the vertices of $P_n^t(tn.C_m)$. We note that for a fixed α and for a fixed β, $v_{\alpha,\beta,i}$, $i = 1, 2, \ldots, m$, are the vertices of the cycle C_m at the β^{th} position of the branch α of $P_n^t(tn.C_m)$. It is enough if we prove the theorem when $m \neq 7, 11$, by Lemma 4.5 and 4.6. Now we define $f : V(P_n^t(tn.C_m)) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling. We note that any odd integer $m \geq 5$, $m \neq 7, 11$, can be written as $m = 4a_1 + 5a_2$, where a_1 and a_2 are non-negative integers.

Case 1: $a_1 \neq 0$.

Let $a_2 \neq 0$.

Join u with $v_{1,1,1}$, $v_{2,1,3}$, $v_{3,1,2}$ and $v_{4,1,4}$. Also join u with $v_{\alpha,1,4}$ for $\alpha \geq 5$. Let $f(u) = 0$.

Consider C_m.

If $\alpha = 1, 2, 3, 4$ and β is odd, then label the first $4a_1$ vertices of C_m with $3, 8, 5, 10$ repeatedly and then label the remaining $5a_2$ vertices of C_m with $7, 3, 9, 5, 11$ repeatedly.

If $\alpha = 1, 2, 3, 4$ and β is even, then label the first $4a_1$ vertices of C_m with $2, 7, 4, 9$ repeatedly and then label the remaining $5a_2$ vertices of C_m with $6, 2, 8, 4, 12$ repeatedly.

If $\alpha \geq 5$ and β is odd, then label the first $4a_1$ vertices of C_m with $2, 7, 4, 9$ repeatedly and then label the remaining $5a_2$ vertices of C_m with $6, 2, 8, 4, 2\alpha + 2$ repeatedly.

If $\alpha \geq 5$ and β is even, then label the first $4a_1$ vertices of C_m with $3, 8, 5, 10$ repeatedly and then label the remaining $5a_2$ vertices of C_m with $7, 3, 9, 5, 11$ repeatedly.

For $\alpha = 1, 2, 3, 4$ if β is odd join $v_{\alpha,\beta,4a_1+3}$ with $v_{\alpha,\beta+1,m}$ and if β is even join $v_{\alpha,\beta,4a_1+2}$ with $v_{\alpha,\beta,4a_1}$.

For $\alpha \geq 5$, if β is odd join $v_{\alpha,\beta,4a_1+2}$ with $v_{\alpha,\beta+1,4a_1}$ and if β is even join $v_{\alpha,\beta,4a_1+3}$ with $v_{\alpha,\beta+1,m}$.

Clearly, f is an $L(3,2,1)$-labeling, as the verification is very similar to Lemma 4.1 and 4.3. Therefore, $k(P_n^t(tn.C_m)) \leq 2t + 2$. Without loss of generality, let $v_1, v_2, \ldots, v_t$ be the vertices of $P_n^t(tn.C_m)$ which are adjacent to u such that the labels in an optimal labeling are in the increasing order. Since any two v_is are at distance 2, v_t should have at least the label $2(t-1)$. But u cannot be adjacent with the two labels which are adjacent to the label at v_1. That is, v_t should have at least the label $2(t - 1 + 2) = 2(t + 1) = 2t + 2$. That is, $k(P_n^t(tn.C_m)) \geq 2t + 2$. Hence, $k(P_n^t(tn.C_m)) = 2t + 2$.

If $a_2 = 0$, then the 4 labels of the cycles C_4 can be used repeatedly (in that order) for the $4a_1$ vertices, then the proof is very similar to Lemma 4.1.

Case 2: $a_1 = 0$.

Join u with $v_{1,1,2}$, $v_{2,1,4}$, $v_{3,1,3}$ and $v_{4,1,5}$. Also join u with $v_{\alpha,1,5}$ for $\alpha \geq 5$. Let $f(u) = 0$.

Consider C_m.

If $\alpha = 1, 2$ and β is odd, then label the vertices of C_m with $7, 3, 9, 5, 11$ repeatedly.

If $\alpha = 1, 2$ and β is even, then label the vertices of C_m with $4, 0, 6, 2, 8$ repeatedly.

If $\alpha = 3, 4$ and $\beta = 1$, then let

$$f(v_{\alpha,\beta,1}) = 6, \quad f(v_{\alpha,\beta,2}) = 2, \quad f(v_{\alpha,\beta,3}) = 8, \quad f(v_{\alpha,\beta,4}) = 4, \quad f(v_{\alpha,\beta,5}) = 10.$$

If $\alpha = 3, 4$ and $\beta \neq 1$ is odd, then let

$$f(v_{\alpha,\beta,1}) = 6, \quad f(v_{\alpha,\beta,2}) = 2, \quad f(v_{\alpha,\beta,3}) = 8, \quad f(v_{\alpha,\beta,4}) = 4, \quad f(v_{\alpha,\beta,5}) = 0.$$

If $\alpha = 3, 4$ and β is even, then let

$$f(v_{\alpha,\beta,1}) = 7, \quad f(v_{\alpha,\beta,2}) = 3, \quad f(v_{\alpha,\beta,3}) = 9, \quad f(v_{\alpha,\beta,4}) = 5,$$
$$f(v_{\alpha,\beta,5}) = 11.$$

If $\alpha \geq 5$ and β is odd, then label the vertices of C_m with $6, 2, 8, 4, 2\alpha + 2$ repeatedly.

If $\alpha \geq 5$ and β is even, then label the vertices of C_m with $7, 3, 9, 5, 11$ repeatedly.

For $\alpha = 1, 2$, if β is odd, join $v_{\alpha,\beta,5}$ with $v_{\alpha,\beta+1,4}$ and if β is even join $v_{\alpha,\beta,4}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha = 3, 4$, if β is odd, join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,5}$.

For $\alpha \geq 5$, if β is odd join $v_{\alpha,\beta,2}$ with $v_{\alpha,\beta+1,5}$ and if β is even join $v_{\alpha,\beta,3}$ with $v_{\alpha,\beta+1,5}$.

Clearly, f is an $L(3, 2, 1)$-labeling, as the verification is very similar to Lemma 4.3. Therefore, $k(P_n^t(tn.C_m)) \leq 2t + 2$. As in Case 1, $k(P_n^t(tn.C_m)) \geq 2t + 2$. Hence, $k(P_n^t(tn.C_m)) = 2t + 2$. □

5 Conclusion

We have determined k-number of one point union of paths of cycles. We believe that this work will create an interest on researchers about $L(3, 2, 1)$-labeling.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, 2008.

[2] M. Murugan, Topics in Graph Theory and Algorithms, Muthali Publishing House, Chennai, Revised Edition, 2018.

[3] W.K.Hale, Frequency assignment: theory and application, Proc.IEEE, 68(1980),1497-1514.

[4] F.S.Roberts, T-colorings of graphs: recent results and open problems, Discrete Math, 93(1991), 229-245.

[5] J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance 2, SIAM J. Discrete Math. 5 (1992), 586-595.

[6] Jia-Zhuang Liu and Zhen-dong Shao, The $L(3, 2, 1)$-labeling problem on graphs, Mathematica Applicata, 17(4)(2004), 596-602.

[7] Gary Chartrand, David Erwin, Frank Harary and Phing Zhang, Radio labeling of graphs, Bulletin of Inst. Combin. Appl., 33, (2001), 77-85.

[8] Jean Clippeton, Jessica Gehrtz, Zsuzsanna Szaniszlo and Desmond Torkornoo, $L(3, 2, 1)$-labeling of Simple Graphs, VERUM, Valparaiso University, 2006.

[9] Ma-Lian Chia, David Kuo, Hong-ya Liao, Cian-Hui Yang and Roger K.Yeh, $L(3, 2, 1)$-labeling of Graphs, Taiwanese Journal of Mathematics, Vol.15, No.6 (2011), pp 2439-2457.

[10] Zhen-dong Shao, The $L(3,2,1)$-labeling problem on graphs, J. Qufu Normal University, 30(3) (2004), 24-28.

[11] M. Murugan and M. Suriya, k-number of special family of graphs, International J. Math. Combin., Vol. 1, (2020), 77-85.

[12] Sze-Chin Shee and Yong-Song Ho, The cordiality of the path-union of n copies of a graph, Discrete Mathematics 151 (1996), 221-229.

[13] V.J.Kaneria and Meera Meghpara, Graceful labeling for one point union for path of graphs, International journal of mathematics and its applications, Volume 3, 1(2015),49-54.