Examples of Simple Wavelet Sets for Matrix Dilation in $\mathbb{R}^2$

Arun Kumar[1] **and Ashish Verma** [2]

[1] Department of Mathematics,
University Of Allahabad, Prayagraj(Allahabad), India - 211002
email: arunmaths92@gmail.com

[2] Department of Mathematics,
Prof. Rajendra Singh(Rajju Bhaiya), Institute of Physical Sciences for Study and Research,
V.B.S. Purvanchal University, Jaunpur, U.P.(India) - 222003.
email: vashish.lu@gmail.com

Abstract

In this article, we tried to construct some examples supporting the results mentioned on a simple wavelet set for matrix dilation with the help of results introduced by K.D. Merrill on her article with some modification and we have also seen the difference between the simple wavelet sets for the matrix dilation in $\mathbb{R}^2$ constructed by K.D. Merrill and us.

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1. Introduction.

A set $W \subset \mathbb{R}^2$ is called a wavelet set for a wavelet $\psi \in L^2(\mathbb{R}^2)$, if the Fourier transform of ψ is a characteristic function on set W i.e. , $\hat{\psi} = \chi_W$, where $\psi \in L^2(\mathbb{R}^2)$ is an orthonormal wavelet means $\{\psi_{j,k} \equiv \sqrt[2]{|detA|^j}\psi(A^j. - k), j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$ form an orthonormal basis for $L^2(\mathbb{R}^2)$. And a Simple Wavelet set is defined as a wavelet set $W \subset \mathbb{R}^2$ for an 2-square expansive integer dilation matrix, if wavelet set W can be scripted as finite union of convex bounded polygons. K.D. Merrill has given a partial answer of questions in her article in 2012 that of which expansive integer matrix dilations in $\mathbb{R}^2$ have a wavelet set which can be scripted as the union of convex bounded polygons up to finite no.. And, she gave a new result which supports the conjectures of the matrices that can be written as a scalar matrix by using suitable positive powers of the matrices and having the determinants greater or equal to 2.

The latest examples of two-dimensional wavelet sets for dilation by 2 were constructed by Zakharov [4], Soardi and Wieland [5], and Dai and others. [6]. Most of the recent researchers believed that these geometric structures were unavoidable. In [7], Benedetto and Sumetkijanakan have described for 2 dilation wavelet set in $\mathbb{R}^n, n \geq 2$, can not be the union of n or least convex sets and stated that it could not be the finite union of convex sets. The question has since been raised to exactly which expansive integer matrix dilations have wavelet sets that are finite unions of convex sets.

In this research work, We have constructed some valuable examples based on the results of the construction of simple wavelet sets for matrix dilation provided by K.Merrill in her paper(Simple wavelet set for Matrix Dilation in $\mathbb{R}^2$) with some modifications.

2. Preliminaries in Simple Wavelet sets

In this section, we introduce some notations and definitions of Simple wavelet sets.

While a simple wavelet set was defined as the set which is a union of finite no. of convex sets but before this we should know the notion of wavelet sets i.e. the set whose characteristic function is the Fourier transform of a wavelet, The following well-known results, which are useful for constructing the examples.

Theorem 2.1.

A measurable set $W \subset \mathbb{R}^2$ is a wavelet set for dilation by an expansive integer matrix A if and only if

$$\sum_{k\in\mathbb{Z}^2} \chi_W(x+k) = 1 \qquad a.e \quad x \in \mathbb{R}^2 \tag{0.1}$$

$$\sum_{j\in\mathbb{Z}} \chi_W(A^{*j}x) = 1 \qquad a.e \quad \in \mathbb{R}^2 \tag{0.2}$$

Proof. The proof can be seen in [8].

Theorem 2.2

Let A be an expansive 2×2 integer dilation matrix. A wavelet set for dilation by A that is a finite union of convex sets of positive measure must be a finite union of bounded convex polygons.

Proof. See, [1].

For the better understanding of this article we need to know the definition of *generalized scaling set* which is motivated by the theory of multiresolution analysis.

Definition 2.3 A set $S \subset \mathbb{R}^2$ is said to be *generalized scaling set* for the matrix dilation A if $S = \cup_{j\leq 0} A^{*j}W$ for some wavelet set W, or equivalently, if $S \subset A^*S$ and A^*S/S is a wavelet set.

The following lemma gives sufficient conditions for set S to be generalized scaling set. We shall use these conditions to construct a simple wavelet set.

Lemma 2.4

Suppose that A is an expansive integer matrix and that the measurable set $S \subset \mathbb{R}^2$ satisfies $S \subset AS$ and contains a neighbourhood of the origin. Suppose further that χ_S satisfies the consistency equation

$$1 + \sum_{k\in\mathbb{Z}^2} \chi_S(x+k) = \sum_{k\in\mathbb{Z}^2} \chi_S(A^{-1}(x+k)) \qquad a.e. \tag{0.3}$$

Then S is a generaliged scaling set for dilation by A.
Proof. The proof. can be seen in [3].
Theorem 2.2 Let A be an expansive integer matrix with all of it's singular values greater than $\sqrt{2}$. If some integral power of A is multiple of the identity, then A has a simple wavelet set.
Proof. For basic proof see [1]
Definition 2.3 Matrices that have some integral power equal to a scalar matrix is reffered as a scalar-potent matrices.

2. Algorithm for the Construction of Simple Wavelet Sets

The algorithm for the construction of simple wavelet set for integer matrix dilation is given as follows :

Let us consider a 2×2 scalar potent matrix A, such that for some $n \in \mathbb{N}$, $A^n = B$ is a scalar matrix with integer eigenvalue λ , then the generalized scaling set for scalar matrix dilation can be written as $S^B = D_1 \cup D_2$, where D_1 and D_2 are diamonds whose centres depend on eigen value λ is either even no. or odd no., if it is even no. then the center(c) is origin $(0, 0)$ and if eigenvalue is odd then the center c is $(\frac{-1}{2(\lambda^2-1)}, \frac{-1}{2(\lambda^2-1)})$.

Thus the corners of D_1 will be $\{c \pm (\frac{\lambda}{2(\lambda^2-1)}, \frac{\lambda}{2(\lambda^2-1)}), c \pm (\frac{\lambda}{2(\lambda^2-1)}, \frac{-\lambda}{2(\lambda^2-1)})\}$ and the corners of $D_2 = \frac{1}{\lambda} D_1 + ([\frac{\lambda}{2}], [\frac{\lambda}{2}]), where [.]$ represents greatest integer value. Therefore we can get the corners of $D_2 = \{\frac{c}{\lambda} \pm (\frac{1}{2(\lambda^2-1)}, \frac{1}{2(\lambda^2-1)}) + ([\frac{\lambda}{2}], [\frac{\lambda}{2}]), \frac{c}{\lambda} \pm (\frac{1}{2(\lambda^2-1)}, \frac{-1}{2(\lambda^2-1)}) + ([\frac{\lambda}{2}], [\frac{\lambda}{2}])\}$. So if $S^B = D_1 \cup D_2$ is a generalized scaling set then the corresponding simple wavelet set can be written as $W^B = \lambda S^B \backslash S^b$.

3. Examples

Example 3.1 Let $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ be a matrix with integer entries then $B = A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ thus, here the eigen value is λ=4 which is an even no. so, the generalized scaling set for B is $S^B = D_1 \cup D_2$ where $D_1 = \{c \pm (\frac{\lambda}{2(\lambda^2-1)}, \frac{\lambda}{2(\lambda^2-1)}), c \pm (\frac{\lambda}{2(\lambda^2+1)}, \frac{\lambda}{2(\lambda^2+1)})\}$ and $D_2 = \frac{1}{\lambda} D_1 + ([\frac{\lambda}{2}], [\frac{\lambda}{2}]), where [\frac{\lambda}{2}]$ is even no. so, here the center will be $c = (0, 0)$. Hence, we have the corners of diamonds D_1 and D_2 are:

$$D_1 = \{(\frac{4}{2(4^2-1)}, \frac{4}{2(4^2-1)}), c \pm (\frac{4}{2(4^2+1)}, \frac{4}{2(4^2+1)})\}$$
$$= \{(\frac{2}{15}, \frac{2}{15}), (\frac{-2}{15}, \frac{-2}{15}), (\frac{2}{17}, \frac{-2}{17}), (\frac{-2}{17}, \frac{2}{17})\}$$

And

$$\begin{aligned} D_2 &= \frac{1}{4}D_1 + (2,2) \\ &= \{(\frac{1}{30},\frac{1}{30}),(\frac{-1}{30},\frac{-1}{30}),(\frac{-1}{34},\frac{1}{34}),(\frac{1}{34},\frac{-1}{34})\} + (2,2) \\ &= \{(\frac{61}{30},\frac{61}{30}),(\frac{59}{30},\frac{59}{30}),(\frac{67}{34},\frac{69}{34}),(\frac{69}{34},\frac{67}{34})\}. \end{aligned}$$

So, the generalized scaling set

$$\begin{aligned} S^B &= D_1 \cup D_2 \\ &= \{(\frac{2}{15},\frac{2}{15}),(\frac{-2}{15},\frac{-2}{15}),(\frac{2}{17},\frac{-2}{17}),(\frac{-2}{17},\frac{2}{17}),(\frac{61}{30},\frac{61}{30}),(\frac{59}{30},\frac{59}{30}),(\frac{67}{34},\frac{69}{34}),(\frac{69}{34},\frac{67}{34})\}. \end{aligned}$$

And the corresponding simple wavelet set can be written as

$$\begin{aligned} W^B &= \lambda S^{\lambda} \backslash S^{\lambda} \\ &= 4 * \{(\frac{2}{15},\frac{2}{15}),(\frac{-2}{15},\frac{-2}{15}),(\frac{2}{17},\frac{-2}{17}),(\frac{-2}{17},\frac{2}{17}),(\frac{61}{30},\frac{61}{30}),(\frac{59}{30},\frac{59}{30}),(\frac{67}{34},\frac{69}{34}),(\frac{69}{34},\frac{67}{34})\} \backslash \\ &\quad \{(\frac{2}{15},\frac{2}{15}),(\frac{-2}{15},\frac{-2}{15}),(\frac{2}{17},\frac{-2}{17}),(\frac{-2}{17},\frac{2}{17}),(\frac{61}{30},\frac{61}{30}),(\frac{59}{30},\frac{59}{30}),(\frac{67}{34},\frac{69}{34}),(\frac{69}{34},\frac{67}{34})\} \end{aligned}$$

$$\begin{aligned} W^B = &\{(\frac{8}{15},\frac{8}{15}),(\frac{-8}{15},\frac{-8}{15}),(\frac{8}{17},\frac{-8}{17}),(\frac{-8}{17},\frac{8}{17}),(\frac{122}{15},\frac{122}{15}),(\frac{118}{15},\frac{118}{15}),(\frac{134}{17},\frac{138}{17}),(\frac{138}{17},\frac{134}{17})\} \backslash \\ &\{(\frac{2}{15},\frac{2}{15}),(\frac{-2}{15},\frac{-2}{15}),(\frac{2}{17},\frac{-2}{17}),(\frac{-2}{17},\frac{2}{17}),(\frac{61}{30},\frac{61}{30}),(\frac{59}{30},\frac{59}{30}),(\frac{67}{34},\frac{69}{34}),(\frac{69}{34},\frac{67}{34})\} \end{aligned}$$

Now, the Scaling set S^B and corresponding Simple Wavelet set W^B can be seen in the figure:2 and figure:3 respectively:
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While in the example 3.2 of article[1] have used $(\frac{1}{2},\frac{1}{2})$ instead of using $([\frac{\lambda}{2}],[\frac{\lambda}{2}])$.

And hence comparatively we see that there is few difference in the simple wavelet set that is at the edge of D_1 and the position of the second diamond is translated too for comparative to simple wavelet constructed in [1]. Thus we have seen that there is some change arises if we take $D_2 = \frac{1}{\lambda}D_1 + ([\frac{1}{2}],[\frac{1}{2}])$ in place of $D_2 = \frac{1}{\lambda}D_1 + ([\frac{\lambda}{2}],[\frac{\lambda}{2}])$.

Example 3.2 Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ be a matrix with integer entries then $A^2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix}$ and $B = A^4 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ thus we have $|\lambda| = 4 > 0$ thus B is an expanssive matrix thus the eigen value of b is -4 which is an even integer so we can obtain a scaling set S and corresponding a wavelet set W with matrix dialation B. So in order to obtain scaling set first we find diamonds D_1 and D_2 whose centre is c=(0,0) with Corners defining by using algorithm. D_1

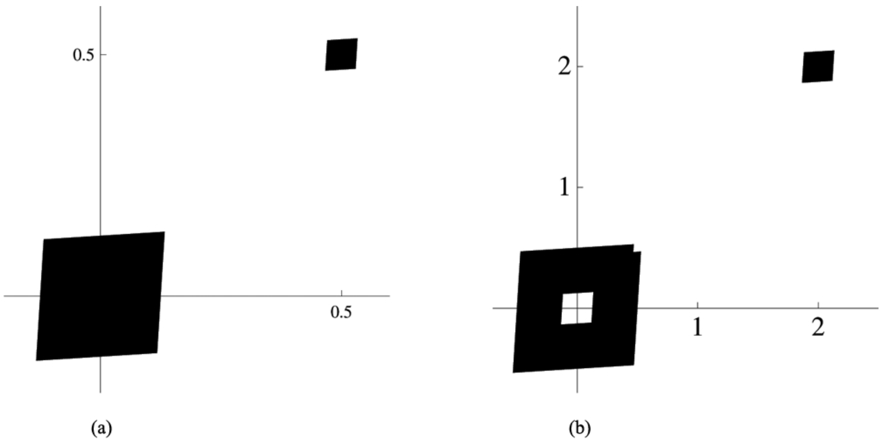

Fig. 1
(a) scaling set and (b) simple wavelet set for matrix dilation[1] if $D_2 = \frac{1}{\lambda}D_1 + ([\frac{1}{2}], [\frac{1}{2}])$.

and D_2 are-

$$D_1 = \{c \pm (\frac{-4}{2(4^2-1)}, \frac{-4}{2(-4^2-1)}), c \pm (\frac{-4}{2(-4^2+1)}, \frac{-4}{2(-4^2+1)})\}$$
$$= \{(\frac{2}{15}, \frac{2}{15}), (\frac{-2}{15}, \frac{-2}{15}), (\frac{2}{17}, \frac{-2}{17}), (\frac{-2}{17}, \frac{2}{17})\}$$

And

$$D_2 = \frac{1}{-4}D_1 + (-2,-2)$$
$$= \{(\frac{1}{30}, \frac{1}{30}), (\frac{-1}{30}, \frac{-1}{30}), (\frac{-1}{34}, \frac{1}{34}), (\frac{1}{34}, \frac{-1}{34})\} + (-2,-2)$$
$$= \{(\frac{-61}{30}, \frac{-61}{30}), (\frac{-59}{30}, \frac{-59}{30}), (\frac{-67}{34}, \frac{-69}{34}), (\frac{-69}{34}, \frac{-67}{34})\}.$$

So the generalized scaling set $S^B = D_1 \cup D_2$
$= \{(\frac{2}{15}, \frac{2}{15}), (\frac{-2}{15}, \frac{-2}{15}), (\frac{2}{17}, \frac{-2}{17}), (\frac{-2}{17}, \frac{2}{17}), (\frac{-61}{30}, \frac{-61}{30}), (\frac{-59}{30}, \frac{-59}{30}), (\frac{-67}{34}, \frac{-69}{34}), (\frac{-69}{34}, \frac{-67}{34})\}$.
and it can be seen in Figure 4.

And the corresponding wavelet set can be written as
$W^B = \lambda S^B \backslash S^B$
$= -4 * \{(\frac{2}{15}, \frac{2}{15}), (\frac{-2}{15}, \frac{-2}{15}), (\frac{2}{17}, \frac{-2}{17}), (\frac{-2}{17}, \frac{2}{17}), (\frac{-61}{30}, \frac{-61}{30}), (\frac{-59}{30}, \frac{-59}{30}), (\frac{-67}{34}, \frac{-69}{34}), (\frac{-69}{34}, \frac{-67}{34})\} \backslash$
$\{(\frac{2}{15}, \frac{2}{15}), (\frac{-2}{15}, \frac{-2}{15}), (\frac{2}{17}, \frac{-2}{17}), (\frac{-2}{17}, \frac{2}{17}), (\frac{-61}{30}, \frac{-61}{30}), (\frac{-59}{30}, \frac{-59}{30}), (\frac{-67}{34}, \frac{-69}{34}), (\frac{-69}{34}, \frac{-67}{34})\}$

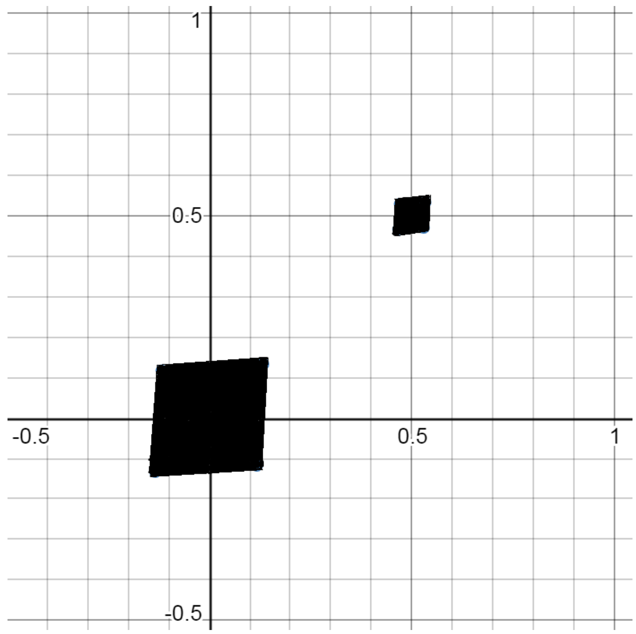

Fig. 2: Scaling set S^B if $D_2 = \frac{1}{\lambda} D_1 + ([\frac{\lambda}{2}], [\frac{\lambda}{2}])$

$W^B = \{(\frac{8}{15}, \frac{8}{15}), (\frac{-8}{15}, \frac{-8}{15}), (\frac{8}{17}, \frac{-8}{17}), (\frac{-8}{17}, \frac{8}{17}), (\frac{122}{15}, \frac{122}{15}), (\frac{-118}{15}, \frac{118}{15}), (\frac{134}{17}, \frac{138}{17}), (\frac{138}{17}, \frac{134}{17})\} \setminus \{(\frac{2}{15}, \frac{2}{15}), (\frac{-2}{15}, \frac{-2}{15}), (\frac{2}{17}, \frac{-2}{17}), (\frac{-2}{17}, \frac{2}{17}), (\frac{-61}{30}, \frac{-61}{30}), (\frac{-59}{30}, \frac{-59}{30}), (\frac{-67}{34}, \frac{-69}{34}), (\frac{-69}{34}, \frac{-67}{34})\}$ and it can be seen in Figure 5.

In a similar way, We can construct so many Simple wavelet sets.

Remark 1. *Above examples suggest that flexibity in the choice of Power on A used to get Scalar potent matrix. Thus, by taking a scalar matrix, we can construct so many scaling sets and corresponding simple wavelet set as to how many times it becomes a scalar matrix. i.e. if B is any scalar matrix such that for some positive integers $m_1, m_2, m_3, ...$, the matrices $B^{m_1}, B^{m_2}, B^{m_3}, ...$, are scalar matrices then we can obtain so many simple wavelets set with matrix dilations $B^{m_1}, B^{m_2}, B^{m_3}, ...$ and all of the corresponding simple wavelet sets have the same shape. But one of the using larger power of B will have a smaller notch.*

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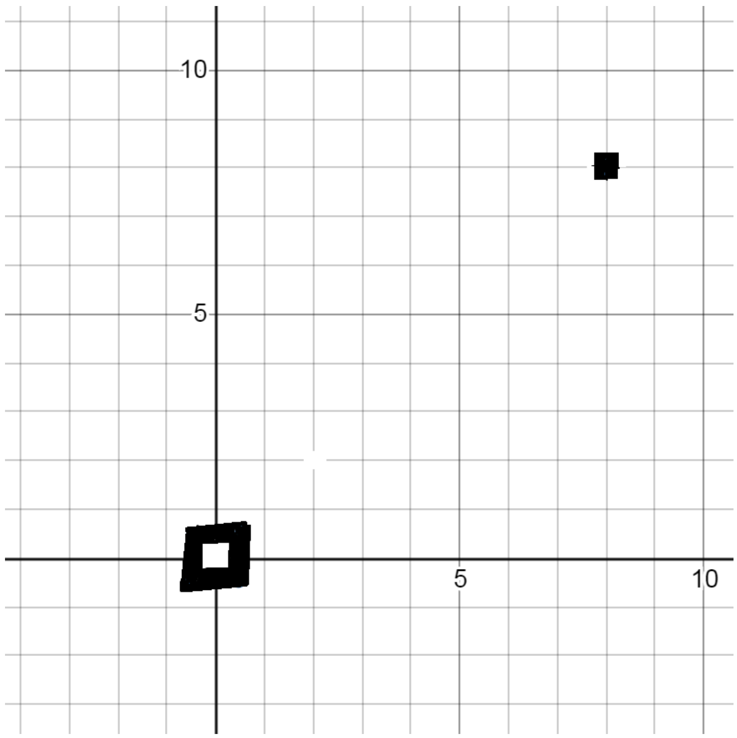

Fig. 3: simple wavelet set W^B if $D_2 = \frac{1}{\lambda}D_1 + ([\frac{\lambda}{2}], [\frac{\lambda}{2}])$

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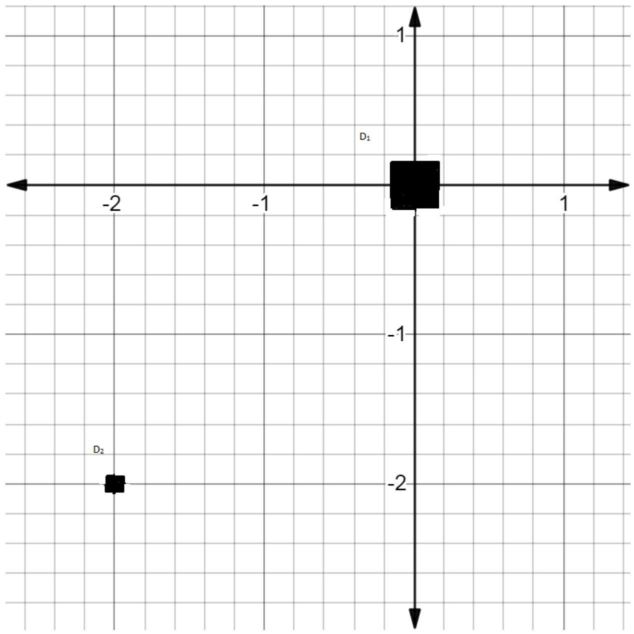

Fig. 4: Scaling set

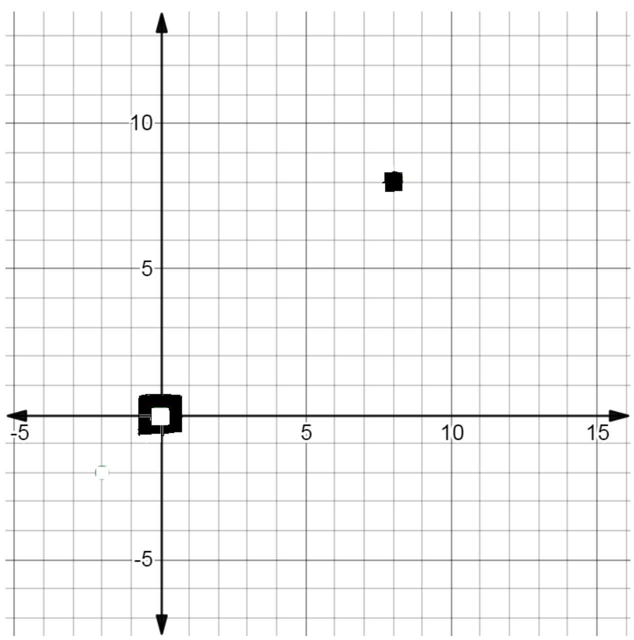

Fig. 5: Simple Wavelet set