Generalisation of a Theorem of Ankeny and Rivlin

Roshan Lal

Department of Mathematics
V.S.K.C. Government Postgraduate College Dakpathar
Dehradun-248125, Uttarakhand, India
Email: rlkeshtwal@gmail.com

Abstract

Let $p(z)$ be a polynomial of degree n. In this paper we prove a result for the bound of maximum modulus of lacunary type of polynomials on a circle of radius greater than unity, the polynomial not vanishing in a disk of prescribed radius. Our result is complement to the results recently proved by Govil and Nwaeze [5] and Mir, Wani and Hussain [6]. Our result also generalizes the earlier proved results.

Subject Classification: [2010]26A15, 26A10
Keywords: Polynomials, zeros, bounds, inequalities, maximum modulus of polynomials.

1 Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n. For simplicity, here and through out, let us define and denote $m = m(p,k) = \min_{|z|=k} |P(z)|$ and $M(p,r) = \max_{|z|=r} |P(z)|$. Concerning the estimate for the maximum modulus of a polynomial on the circle $|z| = R$, $R > 0$, in terms of its degree and the maximum modulus on the unit circle, we know that for every $R \geq 1$,

$$M(p,R) \leq R^n M(p,1). \tag{1.1}$$

The result is best possible for the polynomial having all its zeros at origin.

Inequality (1.1) is a simple deduction from the maximum modulus principle (for reference see [7] or [10]).

For the polynomial of degree n and the case $r \leq 1$, we have the following result due to Varga [12] who attributed it to Zerrantonello.

$$M(p,r) \geq r^n M(p,1). \tag{1.2}$$

Again the result is best possible for the polynomial having all its zeros at origin.

For the class of polynomials having no zeros in $|z| < 1$, the inequalities (1.1) and (1.2) are sharpened by Ankeny and Rivlin [1] and Rivlin [11], by proving following inequality (1.3) and inequality (1.4) respectively

$$M(p,R) \leq \frac{R^n+1}{2} M(p,1). \tag{1.3}$$

$$M(p,r) \geq \left(\frac{1+r}{2}\right)^n M(p,1). \tag{1.4}$$

Aziz and Dawood [3] improved inequality (1.3) under the same hypothesis as

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right) M(p,1) - \left(\frac{R^n-1}{2}\right) m(p,1). \tag{1.5}$$

As a generalization of (1.4), Govil [4] proved if $p(z) \neq 0$ in $|z| < 1$ then for $0 < r \leq R \leq 1$,

$$M(p,r) \geq \left(\frac{1+r}{1+R}\right)^n M(p,R). \tag{1.6}$$

There are several results concerning the refinement and generalizations of above mentioned inequalities (see [5], [6] and [13]).

Recently, Govil and Nwaeze [5] besides proving some other results, also proved the generalization and refinement of inequality (1.6).

Theorem 1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$, be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$. Then for $0 < r < R \leq 1$,*

$$M(P,r) \geq \frac{(1+r)^n}{(1+r)^n + (R+k)^n - (k+r)^n} \left\{ M(P,R) + nm \ln\left(\frac{(R+k)}{(r+k)}\right) \right\}, \tag{1.7}$$

where $m = \min_{|z|=k} |P(z)|$ and $M(p,r) = \max_{|z|=r} |P(z)|$ etc.

More recently, Mir et al. [6] proved the following interesting result and generalized Theorem 1.1 due to Govil and Nwaeze [5] and many other results improving the theorem of T.J. Rivlin [11].

Theorem 2. *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$ be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$. Then for $0 < r < R \leq 1$,*

$$\begin{aligned} M(P,r) \geq {} & \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + (R^\mu + k^\mu)^{n/\mu} - (k^\mu + r^\mu)^{n/\mu}} \\ & \cdot \left\{ M(P,R) + m \ln\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{n/\mu} \right\}, \end{aligned} \tag{1.8}$$

where $m = \min_{|z|=k} |P(z)|$ and $M(p,r) = \max_{|z|=r} |P(z)|$ etc.

It is natural to ask that what would be if $R > 1$ in Theorem 1.1 and Theorem 1.2. In an attempt to find bounds for $M(p,r)$ in terms of $M(p,R)$, similar to inequalities (1.7) and (1.8), when we are given $R > 1$, we have been able to find inequalities which are complements to (1.7) and (1.8) and generalizes inequalities (1.3) and (1.5) as well.

2 Main theorem

In this paper, we prove the following interesting result for lacunary type of polynomial not vanishing in $|z| < k$, $k \geq 1$, which generalizes (1.3) and (1.5) and also provides a complement of Theorem 1.2 by Mir et al. [6] in the sense that $R > 1$.

Theorem 3. *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$ be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$. Then for $1 \leq r < R$,*

$$M(p,r \geq)M(p,R) - \frac{n}{\mu}\left(\frac{R^n + k^\mu}{1+k^\mu}\right)\{M(p,1) - m(p,k)\}\ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right). \tag{2.1}$$

If we take $\mu = 1$ in Theorem 2.1, we get the following:

Corollary 1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$, be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$. Then for $1 \leq r < R$,*

$$M(p,r \geq)M(p,R) - n\left(\frac{R^n + k}{1+k}\right)\{M(p,1) - m(p,k)\}\ln\left(\frac{R+k}{r+k}\right). \tag{2.2}$$

If we had use the fact that $\int_r^R t^{\mu-1}\left(\frac{t^n-1}{t^\mu+k^\mu}\right)dt \leq \frac{(r^n-1)}{\mu}\ln\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)$, in inequality (4.3) in the proof of Theorem 2.1, under the same hypothesis, we have the following

Corollary 2. *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$ be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$. Then for $1 \leq r < R$,*

$$\begin{aligned} M(p,r) \geq M(p,R) &- \frac{n}{\mu(1+k^\mu)}\left[(R^n + k^\mu)M(p,1) - (r^n + k^\mu)m(p,k)\right] \\ &\cdot \ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right). \end{aligned} \tag{2.3}$$

If we take $\mu = 1$ in Corollary 2.3, we get the following:

Corollary 3. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$, be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$. Then for $1 \leq r < R$,*

$$M(p,r) \geq M(p,R) - \frac{n}{(1+k)}\left[(R^n + k)M(p,1) - (r^n + k)m(p,k)\right]\ln\left(\frac{R+k}{r+k}\right).$$

Which, obviously is equivalent to

(2.4)
$$M(p,R) \le M(p,r) + \frac{n}{(1+k)}\left[(R^n+k)M(p,1) - (r^n+k)m(p,k)\right]\ln\left(\frac{R+k}{r+k}\right).$$

3 Lemmas

For the proof of the theorem, we need the following lemmas.

Lemma 1. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$ *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \ge 1$, *then*

$$M(p',1) \le \frac{n}{1+k^\mu}\{M(p,1) - m(p,k)\} \tag{3.1}$$

The result is sharp and equality holds for the polynomial $p(z) = (z^\mu + k^\mu)^{n/\mu}$.

The above Lemma 3.1 is due to Pukhta [8].
The next lemma is due to Aziz [2].

Lemma 2. *Let* $p(z)$ *is a polynomial of degree* n, *then for* $R > 1$, *and* $q(z) = z^n\overline{p(1/\bar{z})}$,

$$|p(Rz) - p(z)| + |q(Rz) - q(z)| \le (R^n - 1)M(p,1). \tag{3.2}$$

We also need the following lemma due to Rather [9].

Lemma 3. *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n, *having no zeros in* $|z| < k$, $k \ge 1$. *Then for* $R \ge 1$, *and* $q(z) = z^n\overline{p(1/\bar{z})}$,

$$k^\mu|p(Rz) - p(z)| \le |(q(Rz) - q(z))| - (R^n - 1)\min_{|z|=k}|p(z)|. \tag{3.3}$$

We now prove the following generalization of inequality (1.5).

Lemma 4. *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree* n, *having no zeros in* $|z| < k$, $k \ge 1$. *Then for* $R > 1$,

$$M(p,R) \le \left(\frac{R^n + k^\mu}{1+k^\mu}\right)M(p,1) - \frac{(R^n - 1)}{1+k^\mu}\min_{|z|=k}|p(z)|. \tag{3.4}$$

If we put $k = 1$ *in Lemma 3.4, we get inequality* (1.5). *For* $\mu = 1$, *under the same hypothesis of Lemma 3.4, we get*

$$M(p,R) \le \left(\frac{R^n + k}{1+k}\right)M(p,1) - \left(\frac{R^n - 1}{1+k}\right)\min_{|z|=k}|p(z)|. \tag{3.5}$$

This is clearly generalization of (1.5).

Proof of Lemma 3.4. Since $p(z)$ does not vanish in $|z| < k$, $k \geq 1$ and $|p(z)| \geq m = \min_{|z|=k} |p(z)|$, therefore by Lemma 3.3, we have

$$k^{\mu}|p(Rz) - p(z)| \leq |(q(Rz) - q(z))| - (R^n - 1) \min_{|z|=k} |p(z)|. \tag{3.6}$$

Adding both the sides of (3.6) the term $|p(Rz) - p(z)|$, we have

$$(1 + k^{\mu}|p(Rz) - p(z)|) \leq |p(Rz) - p(z)| + |q(Rz) - q(z)| - (R^n - 1) \min_{|z=k|} |p(z)|. \tag{3.7}$$

Inequality (3.7), in conjunction with Lemma 3.2, gives

$$(1 + k^{\mu})|p(Rz) - p(z)| \leq (R^n - 1)M(p, 1) - (R^n - 1) \min_{|z=k|} |p(z)|.$$

Equivalently, we have

$$|p(Rz) - p(z)| \leq \frac{(R^n - 1)}{(1 + k^{\mu})}\{M(p, 1) - m(p, k)\}. \tag{3.8}$$

The inequality (3.8), is obviously equivalent to

$$M(p, R) \leq \left(\frac{R^n + k^{\mu}}{1 + k^{\mu}}\right) M(p, 1) - \frac{(R^n - 1)}{1 + k^{\mu}} \min_{|z|=k} |p(z)|.$$

Thus the desired lemma is proved. □

4 Proof of the Main Theorem

Proof of Theorem 2.1. Let $0 < t \leq k$. Since $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, the polynomial $F(z) = p(tz)$ does not vanish in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$, therefore applying Lemma 3.1 to $F(z)$, we have

$$\max_{|z|=1} |F'(z)| \leq \frac{n}{1 + \left(\frac{k}{t}\right)^{\mu}} \left\{ \max_{|z|=1} |F(z)| - \min_{|z|=\frac{k}{t}} |F(z)| \right\},$$

which is equivalent to

$$\max_{|z|=t} |p'(z)| \leq \frac{n\, t^{\mu-1}}{t^{\mu} + k^{\mu}} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{4.1}$$

We have, now for $0 \leq \theta < 2\pi$ and $1 \leq r < R$,

$$|p(Re^{i\theta}) - p(re^{i\theta})| \leq \int_r^R |p'(te^{i\theta})| dt.$$

Which, on using (4.1) gives

$$|p(Re^{i\theta}) - p(re^{i\theta})| \le \int_r^R \frac{n\, t^{\mu-1}}{t^\mu + k^\mu}\Big\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \Big\} dt. \tag{4.2}$$

Now applying inequality (3.4) of Lemma 3.4 to inequality (4.2), we get

$$\begin{aligned}
&|p(Re^{i\theta}) - p(re^{i\theta})| \\
&\le \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu}\left[\left\{\left(\frac{t^n + k^\mu}{1+k^\mu}\right) M(p,1) - \frac{t^n - 1}{1+k^\mu} m(p,k)\right\} - m(p,k)\right] dt \\
&\le \int_r^R \frac{n}{1+k^\mu} t^{\mu-1}\left(\frac{t^n + k^\mu}{t^\mu + k^\mu}\right) M(p,1) dt \\
&\quad - n\left\{\int_r^R \frac{1}{1+k^\mu}\frac{t^{\mu-1}(t^n-1)}{t^\mu + k^\mu} dt + \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} dt\right\} m(p,k).
\end{aligned} \tag{4.3}$$

R.H.S. of the above inequality (4.3), in compact form, can be written as

$$\begin{aligned}
&\le \frac{n}{1+k^\mu}\{M(p,1) - m(p,k)\} \int_r^R t^{\mu-1}\left(\frac{t^n + k^\mu}{t^\mu + k^\mu}\right) dt \\
&\le \frac{n}{1+k^\mu}\{M(p,1) - m(p,k)\}(R^n + k^\mu) \int_r^R \left(\frac{t^{\mu-1}}{t^\mu + k^\mu}\right) dt \\
&\le \frac{n}{\mu}\frac{(R^n + k^\mu)}{1+k^\mu}\{M(p,1) - m(p,k)\} \ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right).
\end{aligned}$$

The above inequality equivalent to

$$M(p,R) - M(p,r) \le \frac{n}{\mu}\frac{(R^n + k^\mu)}{1+k^\mu}\{M(p,1) - m(p,k)\} \ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right).$$

Or equivalently, we have

$$M(p,r \ge) M(p,R) - \frac{n}{\mu}\frac{(R^n + k^\mu)}{1+k^\mu}\{M(p,1) - m(p,k)\} \ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right).$$

Thus finally we have proved the desired result. □

ACKNOWLEDGEMENTS. The author is thankful to the anonymous referee for valuable suggestions.

References

[1] N.C. Ankeny and T.J. Rivlin, On a Theorem of S. Bernstein, *Pacific J. Maths.*, **5**(1955), 249–252.

[2] A. Aziz, Inequalities for the derivative of a polynomial, *Proc. Amer. Math. Soc.,* **89** (1983), 259–266.

[3] A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **54** (1988), 306–313.

[4] N. K. Govil, On the maximum modulus of polynomials, *J. Math. Anal. Appl.*, 112 (1985), 253–258.

[5] N. K. Govil and E. R. Nwaeze, Some sharpening and generalizations of a result of T. J. Rivlin, *Anal. Theory Appl.*, 33 (3) (2017), 219–228.

[6] A. Mir, A. Wani and I. Hussain, A note on a theorem of T. J. Rivlin, *Anal. Theory Appl.*, 34 (4) (2018), 293–296

[7] G. Pölya and G. Szegö, *Problems and Theorems in Analysis*, Vol. 1, Springler-Verlag, Berlin, 1972.

[8] M.S. Pukhta, Extremal problems for polynomials and on location of zeros of polynomials, Ph.D. thesis submitted to the Jamia Millia Islamia, 1995.

[9] N. A. Rather, Extremal properties and location of zeros of polynomials, Ph.D. Thesis submitted to the University of Kashmir, 1998.

[10] M. Riesz, Uber enin Satz der Herrn Serge Bernstein, *Acta Math.*, **40** (1918), 337–347.

[11] T. J. Rivlin, On the maximum modulus of polynomials, *Amer. Math. Monthly*, **67** (1960), 251–253.

[12] R. S. Varga, A comparison of the successive over relaxation method and semi-iterative methods using Chebychev polynomials, *J. Soc. Indust. Appl. Math.*, **5** (1957), 44.

[13] A. Zireh, E. Khojastehnejhad and R. S. Musawi, Some results concerning growth of polynomials, *Anal. Theory Appl.*, **29** (2013), 37–46.