On The Ring Of Hyperbolic Valued Functions

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Abstract

In this article we study the ring structure of all hyperbolic valued functions on an arbitrary topological space. Also we defined the $\mathbb{D}$-continuity of such functions and investigated the ring structure of that class of functions. Finally we study some properties of $\mathbb{D}$-zero sets of those functions. The methods used here are similar to those of L. Gillman and M. Jerison [4].

Keywords: $\mathbb{D}$-continuous, $\mathbb{D}$-pseudocompact space, $\mathbb{D}$-zero sets.

1 Introduction

The set of hyperbolic numbers $\mathbb{D}$ is defined as

$$\mathbb{D} = \{x + y\mathbf{k} : x, y \in \mathbb{R}\}.$$

where $\mathbf{k}$ is an imaginary element such that $\mathbf{k}^2 = 1$ but $\mathbf{k} \notin \mathbb{R}$.

The conjugate of a hyperbolic numbers $\zeta = x + y\mathbf{k}$ is $\overline{\zeta} = x - y\mathbf{k}$. Different algebraic operations (additive, involutive and multiplicative) on $\mathbb{D}$ with respect to the hyperbolic conjugation are.

1. $\overline{\zeta + \eta} = \overline{\zeta} + \overline{\eta}$;
2. $\overline{(\overline{\zeta})} = \zeta$;
3. $\overline{\zeta\eta} = \overline{\zeta}\overline{\eta}$.

Note that,

$$\zeta\overline{\zeta} = x^2 - y^2 \in \mathbb{R}.$$

The modulus of a hyperbolic number $\zeta = x + y\mathbf{k} \in \mathbb{D}$, is defined by

$$|\zeta| = \sqrt{|\zeta\overline{\zeta}|} = \sqrt{|x^2 - y^2|} \in \mathbb{R}.$$

Any hyperbolic number ζ with $\zeta\overline{\zeta} \neq 0$ is said to be invertible (or non-singular), and its inverse is given by

$$\zeta^{-1} = \frac{\overline{\zeta}}{|\zeta|}.$$

If $\zeta \neq 0$ but $\zeta\overline{\zeta} = x^2 - y^2 = 0$ then ζ is a zero divisor. The only zero divisors in $\mathbb{D}$ are those $\zeta\,(\neq 0)$ for which $\zeta\overline{\zeta} = x^2 - y^2 = 0$. We denote the set of zero divisors by $\mathcal{O}_{\mathbb{D}}$. Thus

$$\mathcal{O}_{\mathbb{D}} = \left\{\zeta = x + y\mathbf{k} : \zeta \neq 0, \zeta\overline{\zeta} = x^2 - y^2 = 0\right\}.$$

There are two important zero divisors in $\mathbb{D}$, $\frac{1}{2} + \frac{1}{2}\mathbf{k}$ and its conjugate $\frac{1}{2} - \frac{1}{2}\mathbf{k}$. Set

$$\begin{aligned} \mathbf{e}_1 &= \frac{1+\mathbf{k}}{2}, \\ \mathbf{e}_2 &= \frac{1-\mathbf{k}}{2}. \end{aligned}$$

One can check that

$$\begin{aligned} \mathbf{e}_1 + \mathbf{e}_2 &= 1 \text{ and } \mathbf{e}_1\mathbf{e}_2 = 0 \\ \mathbf{e}_1^2 &= \mathbf{e}_1 \text{ and } \mathbf{e}_2^2 = \mathbf{e}_2. \end{aligned}$$

So these two elements are called mutually orthogonal idempotent elements in $\mathbb{D}$. Thus $\{\mathbf{e}_1, \mathbf{e}_2\}$ forms a basis of $\mathbb{D}$.

The two sets

$$\mathbb{D}_{\mathbf{e}_1} = \mathbf{e}_1 \cdot \mathbb{D} \text{ and } \mathbb{D}_{\mathbf{e}_2} = \mathbf{e}_2 \cdot \mathbb{D}$$

are (principal) ideals in the ring $\mathbb{D}$ and they have the properties:

$$\mathbb{D}_{\mathbf{e}_1} \cap \mathbb{D}_{\mathbf{e}_2} = \{0\}$$

and

$$\mathbb{D} = \mathbb{D}_{\mathbf{e}_1} + \mathbb{D}_{\mathbf{e}_2}. \tag{1.1}$$

Formula (1.1) is called the idempotent decomposition of $\mathbb{D}$. Every hyperbolic number $\zeta = x + y\mathbf{k}$ can be written as

$$\zeta = (x+y)\mathbf{e}_1 + (x-y)\mathbf{e}_2 = \nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2, \nu_1, \nu_2 \in \mathbb{R}. \tag{1.2}$$

Formula (1.2) is called the idempotent representation of a hyperbolic number. It has remarkable features: the algebraic operations of addition, multiplication, taking of inverse, etc. can be realized component-wise.

Observe that the sets $\mathbb{D}_{\mathbf{e}_1}$ and $\mathbb{D}_{\mathbf{e}_2}$ can be written as

$$\mathbb{D}_{\mathbf{e}_1} = \{s\mathbf{e}_1 : s \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_1;\ \mathbb{D}_{\mathbf{e}_2} = \{t\mathbf{e}_2 : t \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_2.$$

Remark 1. *One should note that*

a) $\zeta \in \mathbb{D}_{\mathbf{e}_1}$ if and only if $\zeta\mathbf{e}_1 = \zeta$;

b) $\zeta \in \mathbb{D}_{\mathbf{e}_2}$ if and only if $\zeta \mathbf{e}_2 = \zeta$.

One can define the set of non-negative hyperbolic numbers as

$$\mathbb{D}^+ = \{\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2 : \nu_1, \nu_2 \geq 0\}.$$

Similarly,

$$\mathbb{D}^- = \{\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2 : \nu_1, \nu_2 \leq 0\}.$$

Observe that

$$\mathbb{D}^+ \cap \mathbb{D}^- = \{0\}.$$

A hyperbolic number ζ is said to be (**strictly**) **positive** if $\zeta \in \mathbb{D}^+ \setminus \{0\}$ and (**strictly**) **negative** if $\zeta \in \mathbb{D}^- \setminus \{0\}$.The set of **non-negative** hyperbolic numbers is also defined as

$$\mathbb{D}^+ = \left\{x + y\mathbf{k} : x^2 - y^2 \geq 0, x \geq 0\right\}.$$

Also we have two more sets:

$$\begin{aligned} \mathbb{D}^+_{\mathbf{e}_1} &= \mathbb{D}_{\mathbf{e}_1} \cap \mathbb{D}^+ \setminus \{0\}, \\ \mathbb{D}^+_{\mathbf{e}_2} &= \mathbb{D}_{\mathbf{e}_2} \cap \mathbb{D}^+ \setminus \{0\}. \end{aligned}$$

On the realization of $\mathbb{D}^+$, M.E. Luna-Elizarraras et.al.[5] defined a partial order relation on $\mathbb{D}$. For two hyperbolic numbers ζ_1, ζ_2 the relation $\preceq_{\mathbb{D}}$ is defined as

$$\zeta_1 \preceq_{\mathbb{D}} \zeta_2 \text{ if and only if } \zeta_2 - \zeta_1 \in \mathbb{D}^+.$$

One can check that this relation is reflexive, transitive and antisymmetric. Therefore $\preceq_{\mathbb{D}}$ is a partial order relation on $\mathbb{D}$. This partial order relation $\preceq_{\mathbb{D}}$ on $\mathbb{D}$ is an extension of the total order relation $\leq$ on $\mathbb{R}$. We say $\zeta_1 \prec_{\mathbb{D}} \zeta_2$ if $\zeta_1 \preceq_{\mathbb{D}} \zeta_2$ but $\zeta_1 \neq \zeta_2$. Also we say $\zeta_2 \succeq_{\mathbb{D}} \zeta_1$ if $\zeta_1 \preceq_{\mathbb{D}} \zeta_2$ and $\zeta_2 \succ_{\mathbb{D}} \zeta_1$ if $\zeta_1 \prec_{\mathbb{D}} \zeta_2$.

For any hyperbolic number $\zeta = \nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2$, the $\mathbb{D}-$**valued** (**or hyperbolic valued**) **modulus** of ζ is defined by

$$|\zeta|_{\mathbb{D}} = |\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2|_{\mathbb{D}} = |\nu_1| \, \mathbf{e}_1 + |\nu_2| \, \mathbf{e}_2 \in \mathbb{D}^+$$

where $|\nu_1|$ and $|\nu_2|$ are the usual modulus of real numbers.

A subset A of $\mathbb{D}$ is said to be $\mathbb{D}-$**bounded** if $\exists \ M \in \mathbb{D}^+$ such that $|\zeta|_{\mathbb{D}} \preceq_{\mathbb{D}} M$ for any $\zeta \in A$.

Set

$$\begin{aligned} A_1 &= \{x \in \mathbb{R} : \exists \ y \in \mathbb{R}, \ x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}, \\ A_2 &= \{y \in \mathbb{R} : \exists \ x \in \mathbb{R}, \ x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}. \end{aligned}$$

It is immediate task to verify that if A is $\mathbb{D}-$bounded then A_1 and A_2 are bounded subset of $\mathbb{R}$.

For a $\mathbb{D}-$bounded subset A of $\mathbb{D}$, the **supremum** and **infimum** of A with respect to the $\mathbb{D}-$valued (or hyperbolic valued) modulus are defined by

$$\sup_{\mathbb{D}} A = \sup A_1 \mathbf{e}_1 + \sup A_2 \mathbf{e}_2$$

and

$$\inf_{\mathbb{D}} A = \inf A_1 \mathbf{e}_1 + \inf A_2 \mathbf{e}_2$$

respectively.

The details of the hyperbolic numbers are discussed in [1], [2] and [3].

2 Main Results

Consider the partially ordered relation $\succeq_{\mathbb{D}}$ defined on $\mathbb{D}$. Now $\mathbb{D}$ is a partially ordered ring as

$$a \succeq_{\mathbb{D}} b \Rightarrow a + x \succeq_{\mathbb{D}} b + x \ \forall \ x$$

and

$$a \succeq_{\mathbb{D}} 0, b \succeq_{\mathbb{D}} 0 \Rightarrow ab \succeq_{\mathbb{D}} 0.$$

In the partially ordered ring $\mathbb{D}$, we define

$$a \vee_{\mathbb{D}} b = \sup_{\mathbb{D}} \{a, b\}$$

i.e., an element $c \in \mathbb{D}$ such that $c \succeq_{\mathbb{D}} a$ and $c \succeq_{\mathbb{D}} b$; furthermore for all $x \in \mathbb{D}$ such that $x \succeq_{\mathbb{D}} a$ and $x \succeq_{\mathbb{D}} b$ then $x \succeq_{\mathbb{D}} c$.

Likewise we can define

$$a \wedge_{\mathbb{D}} b = \inf_{\mathbb{D}} \{a, b\}.$$

Since $\inf_{\mathbb{D}} \{a, b\} = -\sup_{\mathbb{D}} \{-a, -b\}$, $a \wedge_{\mathbb{D}} b = -(-a \vee_{\mathbb{D}} -b)$. So $\mathbb{D}$ is a lattice-ordered ring since both $a \vee_{\mathbb{D}} b$ and $a \wedge_{\mathbb{D}} b$ exist for each a and b. In the lattice ordered ring $\mathbb{D}$, $\forall a \in \mathbb{D}$ we define $|a|_{\mathbb{D}} = a \vee_{\mathbb{D}} -a$ and consequently $|a|_{\mathbb{D}} \succeq_{\mathbb{D}} 0$.

Let $\mathbb{D}^X$ be the collection of all hyperbolic valued function from a topological space X into $\mathbb{D}$ i.e., $f : X \to \mathbb{D}$ be defined by $f(x) = f_1(x)\mathbf{e}_1 + f_2(x)\mathbf{e}_2$ where $f_1, f_2 \in \mathbb{R}^X$, the set of all functions from X into $\mathbb{R}$. So for every $f \in \mathbb{D}^X$ we get a natural idempotent decomposition of f as $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ for some $f_1,\ f_2 \in \mathbb{R}^X$ and consequently we have

$$\begin{aligned} \mathbb{D}^X &= \mathbb{R}^X\mathbf{e}_1 + \mathbb{R}^X\mathbf{e}_2 \\ &= \mathbb{D}^X_{\mathbf{e}_1} + \mathbb{D}^X_{\mathbf{e}_2} \end{aligned}$$

where

$$\mathbb{D}^X_{\mathbf{e}_1} = \left\{ f\mathbf{e}_1 : f \in \mathbb{R}^X \right\}, \mathbb{D}^X_{\mathbf{e}_2} = \left\{ g\mathbf{e}_2 : g \in \mathbb{R}^X \right\}.$$

Now $\forall f, g \in \mathbb{D}^X$, define $f + g,\ f.g$ by

$$(f+g)(x) = f(x) + g(x)$$

and

$$(fg)(x) = f(x)g(x)$$

$\forall \ x \in X$.

Obviously $(f+g), (fg) \in \mathbb{D}^X$. Also $\forall f \in \mathbb{D}^X$, define $-f$ by

$$(-f)(x) = -f(x)$$

$\forall \ x \in X$.

Moreover $\forall \ s\mathbf{e}_1 + t\mathbf{e}_2 \in \mathbb{D}$, define $\mathbf{r_{s,t}} : X \to \mathbb{D}$ by $\mathbf{r_{s,t}}(x) = \mathbf{s}(x)\mathbf{e}_1 + \mathbf{t}(x)\mathbf{e}_2$, $\forall \ x \in X$ where $\mathbf{s} : X \to \mathbb{R}$ be defined by $\mathbf{s}(x) = s$ and $\mathbf{t} : X \to \mathbb{R}$ be defined by $\mathbf{t}(x) = t$, $\forall \ x \in X$. In particular if $s = t = r$ (say), we write $\mathbf{r_{s,t}} = \mathbf{r}$. Then $\mathbf{0}$ and $\mathbf{1}$ are the zero and identity elements respectively in $\mathbb{D}^X$.

Further if $g = f_1\mathbf{e}_1$ and $h = f_2\mathbf{e}_2$ then $gh = \mathbf{0}$. So g and h are divisors of zero in $\mathbb{D}^X$.

One can easily verify that $\mathbb{D}^X$ is a commutative ring with identity having divisors of zeros with respect to the operations defined above.

We define the partial order on $\mathbb{D}^X$ by

$$f \leq g \text{ (or } g \geq f) \text{ if } f(x) \preceq_{\mathbb{D}} g(x) \ \forall x \in X.$$

Note that

$$f \geq \mathbf{0} \text{ iff } f(x) \in \mathbb{D}^+$$

and

$$f \leq \mathbf{0} \text{ iff } f(x) \in \mathbb{D}^-.$$

Also $f \geq \mathbf{0}$ iff $f(x) \in \mathbb{D}^+$ iff $\exists\, f_1,\, f_2 \in \mathbb{R}^X$ with $f_1(x),\, f_2(x) \geq 0\ \forall\, x \in X$ such that $f(x) = f_1(x)\,\mathbf{e}_1 + f_2(x)\,\mathbf{e}_2$ iff $\exists\, k_1,\, k_2 \in \mathbb{R}^X$ such that $f_1 = k_1^2$ and $f_2 = k_2^2$ iff $\exists\, k \in \mathbb{D}^X$ such that $k(x) = k_1(x)\,\mathbf{e}_1 + k_2(x)\,\mathbf{e}_2$ and $f = k^2$.

One can easily check that $\left(\mathbb{D}^X, \leq\right)$ is a partially ordered ring such that $\forall\, f, g, h \in \mathbb{D}^X$

(i) $f \leq g \Rightarrow f + h \leq g + h.$

(ii) $f \leq g, 0 \leq h \Rightarrow fh \leq gh.$

Now for $f, g \in \mathbb{D}^X$, let $k(x) = f(x) \vee_{\mathbb{D}} g(x)$. Then $k \geq f$, $k \geq g$ and for all h such that $h \geq f$, $h \geq g$ then $h \geq k$. So $f \vee g = k$ exists in $\mathbb{D}^X$. Dually, we can define $(f \wedge g)(x) = f(x) \wedge_{\mathbb{D}} g(x)$. Thus $\mathbb{D}^X$ is a lattice-ordered ring.

Next we define $|f| = f \vee -f$ such that

$$|f|(x) = |f(x)|_{\mathbb{D}}.$$

Let $C(X)$ denote the set of all real valued continuous functions on the topological space X. A function $f : X \to \mathbb{D}$ defined by $f(x) = f_1(x)\,\mathbf{e}_1 + f_2(x)\,\mathbf{e}_2$ is said to be a $\mathbb{D}-$**continuous** function if and only if $f_1, f_2 \in C(X)$. In other words $f : X \to \mathbb{D}$ is $\mathbb{D}-$**continuous** at $x \in X$ if for every $\epsilon \in \mathbb{D}^+$ there exists an open set U containing x in X such that

$$|f(x) - f(a)|_{\mathbb{D}} \prec_{\mathbb{D}} \epsilon$$

whenever $x \in U \subseteq X$. If f is $\mathbb{D}-$**continuous** at every $x \in X$, then f is said to be a $\mathbb{D}-$**continuous** function on X.

In that case we denote the set of all such fuctions by $C_{\mathbb{D}}(X)$.

Clearly, $f \in C_{\mathbb{D}}(X) \Rightarrow |f| \in C_{\mathbb{D}}(X)$. Now we have $\forall\, f, g \in C_{\mathbb{D}}(X)$ and $\forall\, x \in X$, $(f \vee g)(x) = f(x) \vee_{\mathbb{D}} g(x) = \frac{\mathbf{1}}{\mathbf{2}}\left[f(x) + g(x) + |f(x) - g(x)|_{\mathbb{D}}\right] = \left(\frac{\mathbf{1}}{\mathbf{2}}\left[f + g + |f - g|\right]\right)(x).$

Thus $f \vee g = \frac{\mathbf{1}}{\mathbf{2}}\left[f + g + |f - g|\right] \in C_{\mathbb{D}}(X)$. Therefore $C_{\mathbb{D}}(X)$ is a sublattice of $\mathbb{D}^X$.

A function $f \in C_{\mathbb{D}}(X)$ is said to be bounded if $|f(x)|_{\mathbb{D}} \preceq_{\mathbb{D}} M$ for some $M \in \mathbb{D}^+$. Set $C^*_{\mathbb{D}}(X)$ consists of all such functions. One can check that $C^*_{\mathbb{D}}(X)$ is a subring and sublattice of $C_{\mathbb{D}}(X)$ with respect to the algebraic and order operations discussed above.

Obviously $C^*_{\mathbb{D}}(X) \subset C_{\mathbb{D}}(X) \subset \mathbb{D}^X$. Also note that if $f, g \in C_{\mathbb{D}}(X)$, then $f + g$, $f.g$, $-f$ and $|f|$ all belong to $C_{\mathbb{D}}(X)$.

Obviously $C_{\mathbb{D}}(X)$ is a subring of $\mathbb{D}^X$. All the facts, listed avove, hold when $C_{\mathbb{D}}(X)$ is replaced by $C^*_{\mathbb{D}}(X)$. Hence $C^*_{\mathbb{D}}(X)$ is also a subring of $C_{\mathbb{D}}(X)$. Both of

$C_{\mathbb{D}}(X)$ and $C^*_{\mathbb{D}}(X)$ are partially ordered ring with respect to the partial ordering of $\mathbb{D}^X$.

Further note that $\forall\ f,g \in C_{\mathbb{D}}(X))$ [or $f,g \in C^*_{\mathbb{D}}(X)$] $f \vee g \in C_{\mathbb{D}}(X)$ [or $f \vee g \in C^*_{\mathbb{D}}(X)$].

Thus both $C_{\mathbb{D}}(X)$ and $C^*_{\mathbb{D}}(X)$ are lattice ordered ring.

Let $f \in C_{\mathbb{D}}(X)$, X being a space, such that $f \geq 0$.

Let us define $\forall x \in X$, $k(x)$ be the non-negative square root of $f(x)$. Hence $f(x) = k(x).k(x) = (k.k)(x) = k^2(x)$.

Thus $f = k^2$ and $k \in C_{\mathbb{D}}(X)$.

Note that the word non-negative in the above sentences can be replaced by the word non-positive.

Suppose X is a discrete topological space. Choose $a \in X$ and $\epsilon \in \mathbb{D}^+$ be arbitrary.

Then $N_a = \{a\}$.

Therefore $\forall x \in N_a$,

$$\begin{aligned} |f(x) - f(a)|_{\mathbb{D}} &= |\{f_1(x)\mathbf{e}_1 + f_2(x)\mathbf{e}_2\} - \{f_1(a)\mathbf{e}_1 + f_2(a)\mathbf{e}_2\}|_{\mathbb{D}} \\ &= |\{f_1(a)\mathbf{e}_1 + f_2(a)\mathbf{e}_2\} - \{f_1(a)\mathbf{e}_1 + f_2(a)\mathbf{e}_2\}|_{\mathbb{D}} \\ &= 0\mathbf{e}_1 + 0\mathbf{e}_2 \\ &\prec_{\mathbb{D}} \epsilon \end{aligned}$$

Number of open sets increases in topology then chances of $\mathbb{D}-$continuity of f increases.

Theorem 1. *Let X, Y be spaces and $t : C_{\mathbb{D}}(X) \to C_{\mathbb{D}}(Y)$ be a homomorphism. Then $t(f \vee g) = tf \vee tg \quad \forall f, g \in C_{\mathbb{D}}(X)$.*

Proof. First let us prove that t is order preserving. For this it is sufficient to show that $\forall\ f \in C_{\mathbb{D}}(X)$, $f \geq 0 \Rightarrow tf \geq 0$

Since $f \geq 0, \exists\ k \in C_{\mathbb{D}}(X)$ s.t $f = k^2$.

Therefore $tf = t(k^2) = tk.tk = (tk)^2$

Hence $tf \geq 0$. So t preserves order.

Now note that $\forall\ f \in C_{\mathbb{D}}(X)$, $|f| \in C_{\mathbb{D}}(X)$ and

$(t|f|)^2 = (t|f|)(t|f|) = t(|f|^2) = t(f^2) = tf.tf = (tf)^2 = |tf|^2$.

Hence $t|f| = |tf|$.

Let, $f,\ g \in C_{\mathbb{D}}(X)$.

Then $(f \vee g) + (f \vee g) = f + g + |f - g|$.

Therefore $t(f \vee g) + t(f \vee g) = tf + tg + t|f - g| = tf + tg + |t(f - g)| = tf + tg + |tf - tg| = (tf \vee tg) + (tf \vee tg)$.

Since all the functions involved in the above equality are hyperbolic valued function, it follows that $t(f \vee g) = tf \vee tg$. □

Theorem 2. *Let $t : C_{\mathbb{D}}(X) \to C_{\mathbb{D}}(Y)$ be a homomorphism. Then $\forall f \in C^*_{\mathbb{D}}(X)$, $tf \in C^*_{\mathbb{D}}(Y)$.*

Proof. Let $f \in C^*_{\mathbb{D}}(X)$. Then $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ where $f_1,\ f_2 \in C^*(X)$, the set of all bounded functions in $C(X)$.Thus $\exists\ n_1, n_2 \in \mathbb{N}$ s.t $|f_1| \leq \mathbf{n}_1$, $|f_2| \leq \mathbf{n}_2$.

Choose $n = \max\{n_1, n_2\}$. Then $|f| = |f_1\mathbf{e}_1 + f_2\mathbf{e}_2| = |f_1|\,\mathbf{e}_1 + |f_2|\,\mathbf{e}_2 \leq \mathbf{n}\mathbf{e}_1 + \mathbf{n}\mathbf{e}_2 = \mathbf{r}_{\mathbf{n},\mathbf{n}} = \mathbf{n}$.

Since t is a homomorphism, $t\mathbf{1} = t(\mathbf{1}\cdot\mathbf{1}) = (t\mathbf{1})(t\mathbf{1})$.
Therefore $\forall\, x \in X$, $t\mathbf{1}(x) = 0, 1, \mathbf{r_{1,0}}\,(= \mathbf{e}_1)$ or $\mathbf{r_{0,1}}\,(= \mathbf{e}_2)$.
Hence $t\mathbf{1}(x) \leq \mathbf{1}(x)$.
Thus $|tf| = t|f| \leq t\mathbf{n} = (t\mathbf{1} + t\mathbf{1} + ...n \text{ times}) \leq \mathbf{1}+\ \mathbf{1} + ...n \text{ times} = \mathbf{n}$.
Thus $tf \in C^*_{\mathbb{D}}(Y)$. □

Definition 1 ($\mathbb{D}-$Pseudocompact)**.** *A topological space X is said to be $\mathbb{D}-$pseudocompact if $C_{\mathbb{D}}(X) = C^*_{\mathbb{D}}(X)$.*

Corollary 1. *If Y is not $\mathbb{D}-$pseudocompact then for every space X, $C_{\mathbb{D}}(Y)$ can not be a homomorphic image of $C^*_{\mathbb{D}}(X)$*

Proof. If possible let there be a space X and a homomorphism $t : C^*_{\mathbb{D}}(X) \to C_{\mathbb{D}}(Y)$ such that $tC^*_{\mathbb{D}}(X) = C_{\mathbb{D}}(Y)$.
Then by the above result $C_{\mathbb{D}}(Y) = tC^*_{\mathbb{D}}(X) \subset C^*_{\mathbb{D}}(Y) \subset C_{\mathbb{D}}(Y)$.
Which implies that $C_{\mathbb{D}}(Y) = C^*_{\mathbb{D}}(Y)$.
So Y is $\mathbb{D}-$pseudocompact. □

Corollary 2. *For a space X , $C_{\mathbb{D}}(X)$ and $C^*_{\mathbb{D}}(X)$ be isomorphic iff $C_{\mathbb{D}}(X) = C^*_{\mathbb{D}}(X)$.*

Proof. Let $C_{\mathbb{D}}(X)$ and $C^*_{\mathbb{D}}(X)$ be isomorphic.
So there exists an isomorphism t of $C^*_{\mathbb{D}}(X)$ onto $C_{\mathbb{D}}(X)$.
Then $C_{\mathbb{D}}(X) = t\,(C^*_{\mathbb{D}}(X)) \subset C^*_{\mathbb{D}}(X)$.
Hence $C_{\mathbb{D}}(X) \subset C^*_{\mathbb{D}}(X)$ and $C^*_{\mathbb{D}}(X) \subset C_{\mathbb{D}}(X)$.
Therefore $C^*_{\mathbb{D}}(X) = C_{\mathbb{D}}(X)$. □

Remark 2. *Let t be an isomorphism of $C_{\mathbb{D}}(X)$ onto $C_{\mathbb{D}}(Y)$, then $t\,(C^*_{\mathbb{D}}(X)) = C^*_{\mathbb{D}}(Y)$.*

Theorem 3. *Let t be a homomorphism of $C_{\mathbb{D}}(X)$ into $C_{\mathbb{D}}(Y)$ such that $C^*_{\mathbb{D}}(Y) \subset t\,(C_{\mathbb{D}}(X))$ then $t\,(C^*_{\mathbb{D}}(X)) = C^*_{\mathbb{D}}(Y)$.*

Proof. We have already proved earlier that $t\,(C^*_{\mathbb{D}}(X)) \subset C^*_{\mathbb{D}}(Y)$.
Choose an $h \in C_{\mathbb{D}}(X)$ such that $th = \mathbf{1}$, since $\mathbf{1} \in C^*_{\mathbb{D}}(Y)$ and $C^*_{\mathbb{D}}(Y) \subset t\,(C_{\mathbb{D}}(X))$.
Thus $\mathbf{1} = th = t(h.\mathbf{1}) = th.t\mathbf{1} = \mathbf{1}.t\mathbf{1} = t\mathbf{1}$.
Therefore $t\mathbf{n} = \mathbf{n}\ \forall n \in \mathbb{N}$.
Let $f \in C^*_{\mathbb{D}}(Y)$. Then $\exists\ n \in \mathbb{N}$ such that $|f| \leq \mathbf{n}$.
Choose $g \in C_{\mathbb{D}}(X)$ such that $tg = f$.
Put $k = (-\mathbf{n} \vee g) \wedge \mathbf{n}$.
Obviously $-n \leq k(x) = (-n \vee_{\mathbb{D}} g\,(x)) \wedge_{\mathbb{D}} n \leq n$.
Thus $k \in C^*_{\mathbb{D}}(X)$.
Now note that $tk = (-t\mathbf{n} \vee tg) \wedge t\mathbf{n} = (-\mathbf{n} \vee f) \wedge \mathbf{n} = f$, since $|f| \leq \mathbf{n}$.
Therefore $t\,(C^*_{\mathbb{D}}(X)) = C^*_{\mathbb{D}}(Y)$. □

Let X be a topological space and $f \in C_{\mathbb{D}}(X)$. Then the set $\{x \in X : f(x) = 0\}$ is called the $\mathbb{D}-$zero-set of f in X. This set is denoted by $\mathbf{Z}_X^{\mathbb{D}}(f)$ or simply $\mathbf{Z}^{\mathbb{D}}(f)$.

Note that for $f \in C(X)$, the zero-set of f i.e., $\mathbf{Z}_X(f)$ or $\mathbf{Z}(f) = \{x \in X : f(x) = 0\}$ is a closed set in X. Thus for $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$, $\mathbf{Z}_X^{\mathbb{D}}(f) = \mathbf{Z}_X(f_1) \cap \mathbf{Z}_X(f_2)$, which is also a closed set in X.

Also any $\mathbb{D}-$zero-set of some function $f \in C_{\mathbb{D}}(X)$ is called a $\mathbb{D}-$zero-set in X. This $\mathbf{Z}^{\mathbb{D}}$ is a mapping from $C_{\mathbb{D}}(X)$ onto the set of all $\mathbb{D}-$zero-sets in X.

Let $f \in C_{\mathbb{D}}(X)$ then $\frac{f}{1+f^2} \in C_{\mathbb{D}}^*(X)$. Now

$$\begin{aligned}
x &\in \mathbf{Z}^{\mathbb{D}}(f) \Leftrightarrow f(x) = 0 \\
&\Leftrightarrow \frac{f(x)}{1+f^2(x)} = 0 \\
&\Leftrightarrow \frac{f_1(x)}{1+f_1^2(x)}\mathbf{e}_1 + \frac{f_2(x)}{1+f_2^2(x)}\mathbf{e}_2 = 0 \\
&\Leftrightarrow \frac{f_1(x)}{1+f_1^2(x)} = 0, \frac{f_2(x)}{1+f_2^2(x)} = 0 \\
&\Leftrightarrow x \in \mathbf{Z}\left(\frac{f_1}{1+f_1^2}\right), x \in \mathbf{Z}\left(\frac{f_2}{1+f_2^2}\right) \\
&\Leftrightarrow x \in \mathbf{Z}^{\mathbb{D}}\left(\frac{f}{1+f^2}\right)
\end{aligned}$$

Hence $\mathbf{Z}^{\mathbb{D}}(f) = \mathbf{Z}^{\mathbb{D}}\left(\frac{f}{1+f^2}\right)$.

Also note that $\forall n \in \mathbb{N}$, $\mathbf{Z}^{\mathbb{D}}(f) = \mathbf{Z}^{\mathbb{D}}(|f|) = \mathbf{Z}^{\mathbb{D}}(f^n)$.

Theorem 4. *Let X be a topological space and $f, g \in C_{\mathbb{D}}(X)$, then*
i) $\mathbf{Z}^{\mathbb{D}}(f) \cup \mathbf{Z}^{\mathbb{D}}(g) = \mathbf{Z}^{\mathbb{D}}(fg)$, provided f and g are not zero divisors.
ii) $\mathbf{Z}^{\mathbb{D}}(f) \cap \mathbf{Z}^{\mathbb{D}}(g) = \mathbf{Z}^{\mathbb{D}}(f^2+g^2) = \mathbf{Z}^{\mathbb{D}}(|f|+|g|)$.

Proof. Let $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$, $g = g_1\mathbf{e}_1 + g_2\mathbf{e}_2 \in C_{\mathbb{D}}(X)$.

i) Suppose that f and g are not zero divisors.
Let $x \in \mathbf{Z}^{\mathbb{D}}(f) \cup \mathbf{Z}^{\mathbb{D}}(g)$
$\Longleftrightarrow f(x) = 0$ or $g(x) = 0$
$\Leftrightarrow f_1(x) = 0,\ f_2(x) = 0$ or $g_1(x) = 0, g_2(x) = 0$
$\Longleftrightarrow (f_1g_1)(x) = 0$ and $(f_2g_2)(x) = 0$
$\Longleftrightarrow x \in \mathbf{Z}(f_1g_1)$ and $x \in \mathbf{Z}(f_2g_2)$
$\Leftrightarrow x \in \mathbf{Z}^{\mathbb{D}}(fg)$.

ii) Now $x \in \mathbf{Z}^{\mathbb{D}}(f) \cap \mathbf{Z}^{\mathbb{D}}(g)$
$\Leftrightarrow x \in \mathbf{Z}(f_1) \cap \mathbf{Z}(f_2) \cap \mathbf{Z}(g_1) \cap \mathbf{Z}(g_2)$
$\Leftrightarrow x \in \mathbf{Z}(f_1^2) \cap \mathbf{Z}(f_2^2) \cap \mathbf{Z}(g_1^2) \cap \mathbf{Z}(g_2^2)$
$\Leftrightarrow f_1^2(x) = 0,\ f_2^2(x) = 0,\ g_1^2(x) = 0,\ g_2^2(x) = 0$
$\Leftrightarrow f^2(x) = 0,\ g^2(x) = 0$
$\Leftrightarrow (f^2+g^2)(x) = 0$
$\Leftrightarrow x \in \mathbf{Z}^{\mathbb{D}}(f^2+g^2)$.

Again, $x \in \mathbf{Z}^{\mathbb{D}}(f) \cap \mathbf{Z}^{\mathbb{D}}(g)$
$\Leftrightarrow f(x) = 0$ and $g(x) = 0$
$\Leftrightarrow |f(x)|_{\mathbb{D}} = 0$ and $|g(x)|_{\mathbb{D}} = 0$
$\Leftrightarrow |f|(x) = 0$ and $|g|(x) = 0$
$\Leftrightarrow (|f| + |g|)(x) = 0$
$x \in \mathbf{Z}^{\mathbb{D}}(|f| + |g|)$. □

Remark 3. *The above result shows that union and intersection of two* $\mathbb{D}-$*zero sets in* X *are again* $\mathbb{D}-$*zero sets in* X*. Every* $\mathbb{D}-$*zero set must be a closed set. Every* $\mathbb{D}-$*zero set is a* G_δ *set for all* $f \in C_{\mathbb{D}}(X)$ *i.e.,*

$$\mathbf{Z}^{\mathbb{D}}(f) = \bigcap_{n=1}^{\infty} \left\{ x \in X : |f(x)|_{\mathbb{D}} \prec_{\mathbb{D}} \frac{1}{n}\mathbf{e}_1 + \frac{1}{n}\mathbf{e}_2 \right\} \text{ since } \cap \left(-\frac{1}{n}, \frac{1}{n} \right) = 0.$$

Remark 4. $\mathbf{Z}^{\mathbb{D}}$ *may be thought to be mapping of* $C_{\mathbb{D}}(X)$ *into the power set of* X *i.e.,* $\mathcal{P}(X)$*. Thus* $\forall C'_{\mathbb{D}} \subset C_{\mathbb{D}}(X)$, $\left\{ \mathbf{Z}^{\mathbb{D}}(f) : f \in C'_{\mathbb{D}} \right\}$ *could be denoted by* $\mathbf{Z}^{\mathbb{D}}(C'_{\mathbb{D}})$*. In view to this notation the set of all* $\mathbb{D}-$*zero sets in* X *should be denoted by* $\mathbf{Z}^{\mathbb{D}}(C_{\mathbb{D}}(X))$ *or* $\mathbf{Z}^{\mathbb{D}}(C^*_{\mathbb{D}}(X))$*. However henceforth we shall use the symbol* $\mathbf{Z}^{\mathbb{D}}(X)$ *to denote the set of all* $\mathbb{D}-$*zero sets in the space* X*. We have already seen that* $\mathbf{Z}^{\mathbb{D}}(X)$ *is closed under finite unions and intersections.*

Theorem 5. *For every space* X, $\mathbf{Z}^{\mathbb{D}}(X)$ *is closed under countable intersection.*

Proof. Let $Z_1, Z_2, Z_3, ...$be elements of $\mathbf{Z}^{\mathbb{D}}(X)$. Choose $f_1, f_2, f_3, ...$in $C_{\mathbb{D}}(X)$ such that $Z_k = \mathbf{Z}^{\mathbb{D}}(f_k), \forall k \in 1, 2, 3....$

Set

$$g_n = \frac{(f_n)^2}{\mathbf{2}^n \left(1 + (f_n)^2\right)}, \forall n \in \mathbb{N}.$$

obviously, $Z_n = \mathbf{Z}^{\mathbb{D}}(f_n) = \mathbf{Z}^{\mathbb{D}}(g_n)$ and $g_n \in C_{\mathbb{D}}(X)$ for all $n \in \mathbb{N}$.
Also $0 \leq |g_n| \leq \frac{\mathbf{1}}{\mathbf{2}^n}$ for all $n \in \mathbb{N}$.
Note that $\sum_{n=1}^{\infty} g_n$ converges uniformly on X.
Set $g = \sum_{n=1}^{\infty} g_n$, then $g \in C_{\mathbb{D}}(X)$.
Now note that for all $x \in X$

$$\begin{aligned} x &\in \mathbf{Z}^{\mathbb{D}}(g) \Leftrightarrow g(x) = 0 \\ &\Leftrightarrow g_n(x) = 0, \forall n \in \mathbb{N} \\ &\Leftrightarrow x \in Z_n, \forall n \in \mathbb{N} \\ &\Leftrightarrow x \in Z_1 \cap Z_2 \cap Z_3 \cap ... . \end{aligned}$$

Hence $Z_1 \cap Z_2 \cap Z_3 \cap ... = \mathbf{Z}^{\mathbb{D}}(g) \in \mathbf{Z}^{\mathbb{D}}(X)$. □

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