On the location of zeros of transcendental entire functions

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Abstract

The aim of this paper is to establish some results focusing on the location of zeros of transcendental entire functions. A few examples with related figures are given here to justify the results obtained.

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1 Introduction, Definitions and Notations.

The study of the location of zeros of polynomials has a long history. The earliest contributors to this area of subject were Gauss, Cauchy and Enstrӧm-Kakeya {cf.[6]} and consequently a lot of papers devoted in this branch can be found in the literature{cf.[1],[4],[5],[7] & [8]}. A function of one complex variable analytic in the finite complex plane $\mathbb{C}$ is called an entire function and whenever it has an essential singularity at point at infinity it will be transcendental. If a function $f(z)$ is entire then it can be represented by an every where convergent power series like

$$f(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

Thus the entire functions form natural generalization of polynomials.

The prime purpose of this paper is to derive zero free region for some transcendental entire functions of finite order under various conditions using the coefficients a_n's. We do not explain the standard theories, notations and definitions of entire functions as those are available in [9] & [10].

The following definitions are well known:

Definition 1. *[9] The order ρ of an entire function $f(z)$ is defined as*

$$\rho = \inf\{k > 0 : M_f(r) < e^{r^k}, r > r_0\}$$

where $M(r, f) := M_f(r) = \max_{|z|=r} |f(z)|$.

Also the type σ of $f(z)$ with $0 < \rho < \infty$ is defined as

$$\sigma = \inf\{\lambda > 0 : M_f(r) < e^{\lambda r^\rho}, r > r_0\}.$$

Definition 1 can be alternatively stated as:

Definition 2. *[9] The order ρ and the type σ of an entire function $f(z)$ is defined as*

$$\rho = \limsup_{r\longrightarrow\infty} \frac{\log^{[2]} M(r, f)}{\log r}$$

and

$$\sigma = \limsup_{r\longrightarrow\infty} \frac{\log M(r, f)}{r^\rho}, 0 < \rho < \infty$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Definition 3. *[3] Let $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ be any entire function. Then the n^{th} Jensen polynomial associated with $f(z)$ is defined by*

$$g_n(z) = \sum_{k=0}^{n} {}^n c_k a_k z^k, n = 0, 1, 2, ...$$

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[3] Let $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ be any entire function and also let $\{g_n(z)\}_{n=0}^{\infty}$ be the sequence of Jensen polynomials associated with $f(z)$. Then $\{g_n(\frac{z}{n})\}_{n=0}^{\infty}$ converges uniformly to $f(z)$ on every compact subset of $\mathbb{C}$.*

Lemma 2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be any entire function. Then there is a polynomial*

$$g_n(z) = a_0 + a_1 z + \sum_{k=2}^{n} (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k$$

associated with $f(z)$ such that $\lim_{n\longrightarrow\infty} g_n(z) = f(z)$ uniformly on every compact subset of $\mathbb{C}$.

Proof. For the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we get in view of Cauchy's inequality {cf.[2]},

$$|a_k| \leq \frac{M_f(R)}{R^k}.$$

Therefore for $\mid z \mid \leq r < R$, it follows that

$$\mid \sum_{k=m+1}^{n}(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k-1}{n})a_k z^k \mid \leq \sum_{k=m+1}^{n}(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k-1}{n}) \mid a_k \mid\mid z \mid^k$$

$$\leq \sum_{k=m+1}^{n} \frac{M_f(R)}{R^k} r^k$$
$$\leq M_f(r) \sum_{k=m+1}^{\infty} (\tfrac{r}{R})^k$$
$$= M_f(R)(\tfrac{r}{R})^{m+1} \frac{1}{1-\frac{r}{R}}$$

and also

$$\mid \sum_{k=m+1}^{\infty} a_k z^k \mid \leq M_f(R)(\frac{r}{R})^{m+1} \frac{1}{1-\frac{r}{R}}.$$

Taking $R = 2r$, we obtain for sufficiently large values of m that

$$\mid \sum_{k=m+1}^{n}(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k-1}{n})a_k z^k \mid < \frac{\epsilon}{3}$$

and

$$\mid \sum_{k=m+1}^{\infty} a_k z^k \mid < \frac{\epsilon}{3}.$$

Furthermore for sufficiently large values of n, we get that

$$\mid \sum_{k=2}^{m}(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k-1}{n})a_k z^k - \sum_{k=2}^{m} a_k z^k \mid < \frac{\epsilon}{3}.$$

Hence for sufficiently large values of n, it follows for $\mid z \mid \leq r$ that

$$\mid f(z) - g_n(z) \mid = \mid \sum_{k=0}^{\infty} a_k z^k - a_0 - a_1 z - \sum_{k=2}^{n}(1-\tfrac{1}{n})...(1-\tfrac{k-1}{n})a_k z^k \mid$$
$$= \mid \sum_{k=2}^{\infty} a_k z^k - \sum_{k=2}^{n}(1-\tfrac{1}{n})...(1-\tfrac{k-1}{n})a_k z^k \mid$$
$$= \mid \sum_{k=2}^{m} a_k z^k + \sum_{k=m+1}^{\infty} a_k z^k - \sum_{k=2}^{m}(1-\tfrac{1}{n})...(1-\tfrac{k-1}{n})a_k z^k - \sum_{k=m+1}^{n}(1-\tfrac{1}{n})...(1-\tfrac{k-1}{n})a_k z^k \mid$$
$$\leq \mid \sum_{k=2}^{m} a_k z^k - \sum_{k=2}^{m}(1-\tfrac{1}{n})...(1-\tfrac{k-1}{n})a_k z^k \mid + \mid \sum_{k=m+1}^{\infty} a_k z^k \mid +$$
$$\mid \sum_{k=m+1}^{n}(1-\tfrac{1}{n})...(1-\tfrac{k-1}{n})a_k z^k \mid$$
$$< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

This completes the proof of the lemma.

□

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let* $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ *be a transcendental entire function with* $f(0) = a_0 \neq 0$ *such that* $\mid a_1 \mid \geq \mid a_2 \mid \geq \ldots$ *. Then* $f(z)$ *does not vanish in*

$$\mid z \mid < \ln(1 + \mid \frac{a_0}{a_1} \mid).$$

Proof. The Jensen Polynomial associated with $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$ is

$$g_n(z) = \sum_{k=0}^{n} {}^n c_k a_k z^k.$$

Therefore,

$$g_n(\frac{z}{n}) = \sum_{k=0}^{n} {}^n c_k a_k (\frac{z}{n})^k$$

$$= a_0 + \sum_{k=1}^{n} {}^n c_k a_k (\frac{z}{n})^k. \tag{3.1}$$

Now,

$$\mid \sum_{k=1}^{n} {}^n c_k a_k (\frac{z}{n})^k \mid \leq \sum_{k=1}^{n} {}^n c_k \mid a_k \mid (\frac{\mid z \mid}{n})^k$$

$$\leq \mid a_1 \mid \textstyle\sum_{k=1}^{n} {}^n c_k (\frac{|z|}{n})^k$$

$$= \mid a_1 \mid \{(1 + \tfrac{|z|}{n})^n - 1\}.$$

Hence from (1), we obtain that

$$\mid g_n(\frac{z}{n}) \mid \geq \mid a_0 \mid - \mid \sum_{k=1}^{n} {}^n c_k a_k (\frac{z}{n})^k \mid$$

$$\geq \mid a_0 \mid - \mid a_1 \mid \{(1 + \tfrac{|z|}{n})^n - 1\} .$$

Now, taking limit as $n \longrightarrow \infty$ on any compact subset of $\mathbb{C}$, it follows that

$$\mid f(z) \mid \geq \mid a_0 \mid - \mid a_1 \mid (e^{|z|} - 1).$$

Hence $\mid f(z) \mid > 0$ if $\mid a_0 \mid - \mid a_1 \mid (e^{|z|} - 1) > 0$

i.e, if $e^{|z|} < (1 + \frac{|a_0|}{|a_1|})$

i.e, if $\mid z \mid < ln(1+\frac{|a_0|}{|a_1|})$.

This proves the theorem.

□

Remark 1. *The following example with related figure ensures the validity of Theorem 3.1.*

Let $f(z) = e^z + 1$.
Then the zeros of $f(z)$ *are at* $z = (2n+1)\pi i, n = 0, \pm 1, \ldots$.
Now the Taylor's series expansion of $f(z)$ *is*

$$f(z) = 2 + z + \frac{z^2}{2!} + \ldots.$$

Here, $a_0 = 2$ *and* $a_1 = 1$.

Hence by Theorem 3.1, $f(z)$ *does not vanish in*

$$\mid z \mid < ln(1+\frac{2}{1}) = ln3 \simeq 1.0986.$$

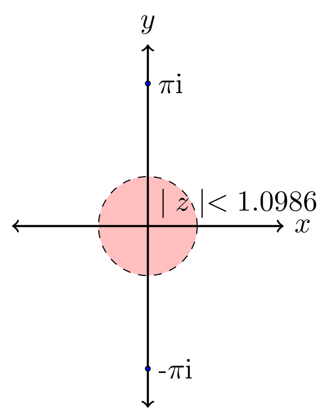

Fig. 1: Zero free region of $f(z) = e^z + 1$

Theorem 2. *Let* $f(z)$ *be an entire function with* $f(0) \neq 0$ *such that for all* z *in* $\mathbb{C}$, $\mid f(z) \mid \leq Me^{\alpha|z|^\beta}$ *where* $M > 0, \alpha > 0$ *and* $0 < \beta < 1$. *Then* $f(z)$ *does not vanish in*

$$\mid z \mid < \ln(1 + \frac{\mid f(0) \mid}{Me^{\frac{1-\beta}{\beta}(\alpha\beta e^{\beta})^{\frac{1}{1-\beta}}}}).$$

Proof. In view of the Taylor's series expansion for $f(z)$,

$$f(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ..., \forall z \in \mathbb{C} \tag{1}$$

where $a_0 = f(0), a_k = \frac{f^k(0)}{k!}$

and by Lemma 2.2 there is a polynomial

$$g_n(z) = a_0 + \sum_{k=1}^{n} (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k$$

associated with $f(z)$ such that $\lim_{n \longrightarrow \infty} g_n(z) = f(z)$ on every compact subset of $\mathbb{C}$.

Now, in view of Cauchy's inequality , we get from (1) that

$$| a_k | \leq \frac{M_f(r)}{r^k}, k = 1, 2, ...$$

$$\text{i.e, } | a_k | < \tfrac{M e^{\alpha r^\beta}}{r^k}.$$

Now, we see that right hand side of the above inequality is minimum for $r = (\frac{k}{\alpha\beta})^{\frac{1}{\beta}}$. Thus,

$$| a_k | < M(\frac{\alpha\beta e}{k})^{\frac{k}{\beta}} \text{ for } k = 1, 2, ...$$

Therefore,

$$| \sum_{k=1}^{n} (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k | \leq \sum_{1}^{n} {}^n c_k k! \mid a_k \mid (\frac{| z |}{n})^k$$

$$< \textstyle\sum_1^n {}^n c_k k^k M(\frac{\alpha\beta e}{k})^{\frac{k}{\beta}} (\frac{|z|}{n})^k. \tag{2}$$

As $k^k(\frac{\alpha\beta e}{k})^{\frac{k}{\beta}}$ is maximum for $k = (\alpha\beta e^\beta)^{\frac{1}{1-\beta}}$ and the maximum value of it is $e^{\frac{1-\beta}{\beta}(\alpha\beta e^\beta)^{\frac{1}{1-\beta}}}$, it follows from (2) that

$$| \sum_{k=1}^{n} (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k | < M e^{\frac{1-\beta}{\beta}(\alpha\beta e^\beta)^{\frac{1}{1-\beta}}} \sum_{k=1}^{n} {}^n c_k (\frac{| z |}{n})^k$$

$$\text{i.e, } | \textstyle\sum_{k=1}^n (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k | < M e^{\frac{1-\beta}{\beta}(\alpha\beta e^\beta)^{\frac{1}{1-\beta}}} \{(1 + \frac{|z|}{n})^n - 1\}.$$

Therefore

$$\begin{aligned} | g_n(z) | &= | f(0) + \textstyle\sum_{k=1}^n (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k | \\ &\geq | f(0) | - | \textstyle\sum_{k=1}^n (1 - \frac{1}{n})...(1 - \frac{k-1}{n}) a_k z^k | \end{aligned}$$

$$>| f(0) | -Me^{\frac{1-\beta}{\beta}(\alpha\beta e^{\beta})^{\frac{1}{1-\beta}}}\{(1+\frac{|z|}{n})^n - 1\}.$$

Now taking limit as $n \longrightarrow \infty$ on any compact subset of $\mathbb{C}$, we get that

$$| f(z) |>| f(0) | -Me^{\frac{1-\beta}{\beta}(\alpha\beta e^{\beta})^{\frac{1}{1-\beta}}}(e^{|z|} - 1).$$

Hence $| f(z) |> 0$ if $| f(0) | -Me^{\frac{1-\beta}{\beta}(\alpha\beta e^{\beta})^{\frac{1}{1-\beta}}}(e^{|z|} - 1) > 0$

i.e, $| f(z) |> 0$ if $| z |< \ln(1 + \frac{|f(0)|}{Me^{\frac{1-\beta}{\beta}(\alpha\beta e^{\beta})^{\frac{1}{1-\beta}}}})$.

Thus the theorem is established.

□

Remark 2. *The following example with related figure justifies the validity of Theorem 3.2.*

Let $f(z) = \cos\sqrt{z}$.

Then the zeros of $f(z)$ *are at* $z = \{(2n+1)\frac{\pi}{2}\}^2, n = 0, \pm1, \pm2...$

Clearly $| f(z) |\leq e^{|z|^{\frac{1}{2}}}$.

Here, $M = 1, \alpha = 1$ *and* $\beta = \frac{1}{2}$.

Hence by Theorem 3.4, $f(z)$ *does not vanish in*

$$| z |< ln(1 + \frac{1}{e^{\frac{e}{4}}}) \simeq 0.41.$$

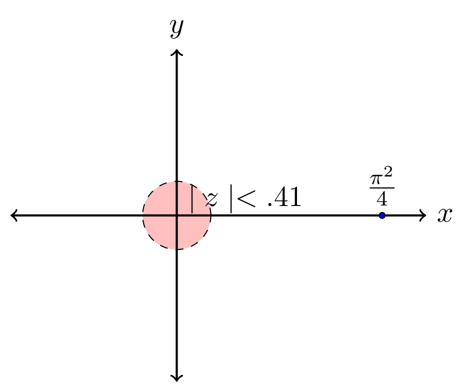

Fig. 2: Zero free region of $f(z) = \cos\sqrt{z}$

Future prospect. In the line of the works as carried out in the paper one may think of proving analogous results for transcendental entire functions of infinite order.

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