On a new class of fractional operator associated with k-uniformly convex functions with negative coefficients

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Abstract

Using the fractional calculus operator $D_z^\lambda f(z)$ (fractional derivatives and functional integral) for functions $f(z)$ which are analytic in the open unit disc U, an interesting fractional operator $O^\lambda f(z)$ for $f(z)$ is defined by Owa (Proc. Int symp on new development of GFTA 2008). Infact this fractional operator is the generalization of some historical operators. A systematic investigation of a new class of k-uniformly convex function, which is defined in terms of this fractional operator is presented. The results obtained here include coefficient bounds, distortion theorem and many other useful properties.

Keywords and Phrases: Analytic, univalent, starlike, convex, k-uniformly convex functions, fractional derivative.

1 Introduction

Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular in the open unit disc $U = \{z : |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the classes of convex function of order α and starlike functions of order α respectively; here and here after in this paper, α and k are real numbers such that $0 \le \alpha < 1$ and $k \ge 0$. A function f of S is said to belong to $K(\alpha)$ if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge \alpha, z \in U$ and is said to belong $S^*(\alpha)$ if $Re\left\{\frac{zf'(z)}{f(z)}\right\} \ge \alpha,\ z \in U$. It is well known that function of these subclasses of S are univalent in U.

The class K and S^* of convex functions and starlike functions respectively are identical by $K \equiv K(0)$ and $S^* \equiv S^*(0)$. Bharti et al. [3] defined $k - S_p(\alpha)$ to be the class of functions f with $0 \le k < \infty$ and $0 \le \alpha < 1$ that satisfy the condition,

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \ge k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha.$$

Bharati et al. [3] also defined $k - UCV(\alpha)$ to be class of functions f with $0 \leq k < \infty$ and $0 \leq \alpha < 1$ that satisfy the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| + \alpha.$$

Owa [8] introduced the operator $O^{\lambda}_{\gamma,z} : s \to S$ defined by

$$O^{\lambda}_{\gamma,z} f(z) = \frac{\Gamma(\gamma + 1 - \lambda)}{\Gamma(\gamma + 1)} z^{1+\lambda-\gamma} O^{\lambda}_{z}(z^{\gamma-1} f(z))$$

where $O^{\lambda}_{z}(f(z))$ is the fractional derivative of f of order λ defined by Owa [8].

Clearly, we have

$$O^{\lambda}_{\gamma,z} f(z) = z + \sum_{n=2}^{\infty} \phi(n, \gamma, \lambda) a_n z^n$$

where, for convenience

$$\phi(n, \gamma, \lambda) = \frac{\Gamma(\gamma + 1 - \lambda)\Gamma(n + \gamma)}{\Gamma(\gamma + 1)\Gamma(n + \gamma - \lambda)}$$

for any real λ and γ.

Remark 1. *From the definition for the fractional operator $O^{\lambda}_{\gamma,z}(f(z))$, we see that*

(1) If $\gamma = 0$ and $\lambda = -1$, then we have Alexander integral operator [1].

(2) If $\gamma = 1$ and $\lambda = -1$, then we have Libera integral operator [6].

(3) If $\gamma = 1$ and $\lambda = 1$, then we have Salagean differential operator [9].

(4) If $\lambda = -1$, then we have Barnardi operator [2].

In view of Remark 1.1, we see that fractional operator $O^{\lambda}_{\gamma,z}(f(z))$ is generalization of some historical operators. therefore by using this fractional operator, we can get many interesting and fruitful results connecting with some operators.

Using this fractional operator, we define and investigate a family $k - S_p(\alpha, \lambda, \gamma)$ consisting of functions $f \in S$, which satisfy

$$Re\left\{\frac{z(O^{\lambda}_{\gamma,z} f(z))'}{O^{\lambda}_{\gamma,z} f(z)}\right\} \geq k\left|\frac{z(O^{\lambda}_{\gamma,z} f(z))'}{O^{\lambda}_{\gamma,z} f(z)} - 1\right| + \alpha \quad (1.1)$$

where $0 \leq k < \infty$ and $0 \leq \alpha < 1$.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0. \quad (1.2)$$

Also let

$$k - TS_p(\alpha, \lambda, \gamma) = S_p(\alpha, \lambda, \gamma) \cap T$$

The object of this paper is to investigate several basic properties of the class $k - TS_p(\alpha, \lambda, \gamma)$.

2 Main Results

The following theorem plays an important role in our systematic study of the class $k - TS_p(\alpha, \lambda, \gamma)$.

Theorem 1. *A function f defined by (1.2) is in $k - TS_p(\alpha, \lambda, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k) - (\alpha+k)]\phi(n,\gamma,\lambda)a_n \leq 1-\alpha. \tag{2.1}$$

Proof. We assume that the inequality (2.1) holds true and let $|z| = 1$. It suffices to show that

$$Re\left\{\frac{z(O^{\lambda}_{\gamma,z}f(z))'}{O^{\lambda}_{\gamma,z}f(z)}\right\} \geq k\left|\frac{z(O^{\lambda}_{\gamma,z}f(z))'}{O^{\lambda}_{\gamma,z}f(z)} - 1\right| + \alpha$$

we have,

$$k\left\{\frac{z(O^{\lambda}_{\gamma,z}f(z))'}{O^{\lambda}_{\gamma,z}f(z)} - 1\right\} - Re\left\{\frac{z(O^{\lambda}_{\gamma,z}f(z))'}{O^{\lambda}_{\gamma,z}f(z)} - \alpha\right\}$$

$$\leq (1+k)\left|\frac{z(O^{\lambda}_{\gamma,z}f(z))'}{O^{\lambda}_{\gamma,z}f(z)} - 1\right| \leq \frac{(1+k)\sum\limits_{n=2}^{\infty}(n-1)\phi(n,\gamma,\lambda)a_n}{1-\sum\limits_{n=2}^{\infty}(n-1)\phi(n,\gamma,\lambda)a_n}.$$

□

The last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[n(1+k) - (\alpha+k)]\phi(n,\gamma,\lambda)a_n \leq 1-\alpha,$$

which is true by hypothesis. Hence we have

$$f \in k - TS_p(\alpha, \lambda, \gamma).$$

To prove the converse, we assume that $f(z)$ is defined by (1.2) and in the class $k - TS_p(\alpha, \lambda, \gamma)$, so that the condition (1.1) readily yields

$$\frac{1-\sum\limits_{n=2}^{\infty} n\phi(n,\gamma,\lambda)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi(n,\gamma,\lambda)a_n z^{n-1}} - \alpha \geq k\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)\phi(n,\gamma,\lambda)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi(n,\gamma,\lambda)a_n z^{n-1}}\right|$$

Letting $z \to 1$ along the real axis, we have,

$$\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\phi(n,\gamma,\lambda)a_n}{1-\sum\limits_{n=2}^{\infty}\phi(n,\gamma,\lambda)a_n} \geq 0$$

upon clearing the denominator, we obtain required assertion (2.1). Finally, we note that the assertion (2.1) of Theorem 2.1 is sharp, the extremal function being

$$f(z) = z - \frac{(1-\alpha)}{[n(1+k)-(\alpha+k)]\phi(n,\gamma,\lambda)} z^n.$$

3 Growth Theorems

Theorem 2. *Let the function $f(z)$ defined by (1.2) be in the class $k-TS_p(n,\gamma,\lambda)$. Then*

$$|f(z)| \geq |z| - \frac{(1-\alpha)(1-\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}|z|^n$$

and

$$|f(z)| \leq |z| + \frac{(1-\alpha)(1-\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}|z|^n$$

with equality for $f(z) = z - \dfrac{(1-\alpha)(1-\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}z^2$, $(z = \pm r)$.

Proof. Since $f(z) \in k - TS_p(\alpha,\lambda,\gamma)$, in view of Theorem 2.1, we have

$$\frac{(1+\gamma)(2+k-\alpha)}{(1+\gamma-\lambda)}\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty}[n(1+k)-(\alpha+k)]\phi(n,\gamma,\lambda)a_n \leq (1-\alpha)$$

Thus immediately yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1-\alpha)(1+\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}$$

From (1.2) and above result, we easily have

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n|z|^n \leq |z| + \frac{(1-\alpha)(1+\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}|z|^2$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n|z|^n \geq |z| - \frac{(1-\alpha)(1+\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}|z|^2$$

Theorem 3.2 If $f \in k - TS_p(\alpha,\lambda,\gamma)$, then

$$1 - \frac{2(1-\alpha)(1+\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}|z| \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)(1+\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}|z|$$

with equality for

$$f(z) = z - \frac{(1-\alpha)(1+\gamma-\lambda)}{(1+\gamma)(2+k-\alpha)}z^2, \qquad (z = \pm r)$$

□

4 Closed-to-convexity, starlikeness and convexity

A function f in T is said to be closed-to-convex of order β in U if

$$\mathrm{Re}\{f'(z)\} > \beta$$

for some $\beta(0 \leq \beta < 1)$ and for all $z \in U$. If $f(z) \in T$ satisfies the inequality

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$$

For some $\beta(0 \leq \beta < 1)$ and for all $z \in U$, the $f(z)$ is to be starlike of order β in U. On the other hand if $f(z) \in T$ satisfies the inequality

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta$$

for some $\beta(0 \leq \beta < 1)$ and for all $z \in U$, the $f(z)$ is said to be convex of order β in U. It follows at once that $f(z) \in T$ is convex of order β if and only if $zf'(z)$ is starlike of order β in U (for details, see [5]).

We now prove the following theorem.

Theorem 3. *If $f(z) \in k - TS_p(\alpha, \lambda, \gamma)$. Then by virtue of (3.4.1), the function $f(z)$ is close-to-convex of order β, $r = r(\alpha, \lambda, \gamma, \beta)$in $|z| < r_1$ where*

$$\left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leq 1 - \beta, \quad (0 \leq \beta < 1, \;\; z \in U) \tag{4.1}$$

In view of (2.1), the assertion is true if

$$\frac{n|z|^{n-1}}{1-\beta} \leq \frac{[n(1+k) - (\alpha+k)]\Gamma(\gamma+1-\lambda)\Gamma(n+\gamma)}{(1-\alpha)\Gamma(\gamma+1)\Gamma(n+\gamma-\lambda)} \tag{4.2}$$

upon solving (4.2) for $|z|$, we get

$$|z| = r_1 \leq \left[\frac{(1-\beta)[n(1+k) - (\alpha+k)]\Gamma(\gamma+1-\lambda)\Gamma(n+\gamma)}{n(1-\alpha)\Gamma(\gamma+1)\Gamma(n+\gamma-\lambda)}\right]^{1/n-1}$$

Theorem 4. *If $f(z) \in k - TS_p(\alpha, \lambda, \gamma)$. Then $f(z)$ is close-to-convex of order β in $|z| < r_1(\alpha, \lambda, \gamma, \beta)$ where*

$$r_1(\alpha, \lambda, \gamma, \beta) = \inf_n \left[\frac{(1-\beta)\{n(1+k) - (\alpha+k)\}\phi(n, \gamma, \lambda)}{(n-\beta)(1-\alpha)}\right]^{1/n-1} \tag{4.3}$$

$$(n \geq 2; \; n \in N)$$

where

$$\phi(n, \gamma, \lambda)$$

is given in the operator $O^{\lambda}_{\gamma,z}$ introduced by Owa.

Proof. Let $f(z) \in k - TS_p(\alpha,\lambda,\gamma)$. Then by virtue of (4.1) the function $f(z)$ is close-to-convex of order β in U, provided that

$$\left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \le 1-\beta(0 \le \beta < 1), \quad z \in U.$$

In view of (2.1), the assertion is true if

$$\frac{n|z|^{n-1}}{(1-\beta)} \le \frac{[n(1+k)-(\alpha+k)]\phi(n,\gamma,\lambda)}{(1-\alpha)}.$$

Upon solving it for $|z|$, we get (4.3). □

Theorem 5. *If $f(z) \in k - TS_p(\alpha,\lambda,\gamma)$. Then $f(z)$ is starlike of order β in $|z| < r_2(\alpha,\lambda,\gamma,\beta)$ where*

$$r_2(\alpha,\lambda,\gamma,\beta) = \inf_n \left[\frac{(1-\beta)\{n(1+k)-(\alpha+k)\}\phi(n,\gamma,\lambda)}{(n-\beta)(1-\alpha)}\right]^{1/n-1}. \tag{4.4}$$

Proof. Under the hypothesis of Theorem 4.2, $f(z)$ is starlike of order β in U, provided that

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n|z|^{n-1}} \le 1-\beta(0 \le \beta < 1), \quad z \in U$$

In view of (2.1), above assertion is true if

$$\frac{(n-\beta)}{(1-\beta)} \le \frac{[n(1+k)-(\alpha+k)]\phi(n,\gamma,\lambda)}{(1-\alpha)}$$

$$|z| \le \left[\frac{(1-\beta)\{n(1+k)-(\alpha+k)\}\phi(n,\gamma,\lambda)}{(n-\beta)(1-\alpha)}\right]^{1/n-1}$$

which obviously leads to (4.4). □

Theorem 6. *If $f(z) \in k - TS_p(\alpha,\lambda,\gamma)$, then $f(z)$ is convex of order β in $|z| < r_3(\alpha,\lambda,\gamma,\beta)$ where*

$$r_3 = r_3(\alpha,\lambda,\gamma,\beta) = \inf_n \left[\frac{(1-\beta)\{n(1+k)-(\alpha+k)\}\phi(n,\gamma,\lambda)}{n(n-\beta)(1-\alpha)}\right]^{1/n-1}. \tag{4.5}$$

Proof. Under the hypothesis of Theorem 4.3, $f(z)$ is convex of order β in U, provided that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}} \leq (1-\beta), \quad (0 \leq \beta < 1), \quad z \in U$$

By means of (2.1), it is easily seen that above inequality holds true if

$$\frac{n(n-\beta)}{1-\beta}|z|^{n-1} \leq \frac{\{n(1+k)-(\alpha+k)\}\phi(n,\gamma,\lambda)\}}{(1-\alpha)}, \quad \{n \geq 2,\ n \in N\}$$

which yields (4.5). □

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