A study on conformal Ricci solitons in the framework of $(LCS)_n$-manifolds

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Abstract

The main aim of this paper is to study Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) admitting the conformal Ricci soliton and to characterize when the soliton is shrinking, steady or expanding. Next we establish some results on the $(LCS)_n$-manifold whose metric is a conformal Ricci soliton. Finally some interesting results have been obtained by applying certain curvature conditions on $(LCS)_n$-manifolds admitting conformal Ricci solitons.

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1 Introduction

In 1982, R.S. Hamilton [7] introduced the Ricci soliton as a self similar solution to the Ricci flow equation given by: $\frac{\partial}{\partial t}(g(t)) = -2Ric(g(t))$, where $g(t)$ is an one parameter family of metrics on the manifold.

A Riemannian metric g defined on a smooth manifold M,of dimension n, is said to be a Ricci soliton if for some constant λ, there exists a smooth vector field V on M satisfying the equation

$$Ric + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative in the direction of V and Ric is the Ricci tensor. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by an one-parameter group of diffeomorphisms and scaling [8]. After Hamilton's work many authors have studied Ricci flow and a rigorous literature on this topic can be found in [4, 17].

A.E. Fischer [6] in 2005, introduced conformal Ricci flow equation which is a modified version of the Hamilton's Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented manifold M, of dimension n, are given by

$$\frac{\partial g}{\partial t} + 2(Ric + \frac{g}{n}) = -pg, \tag{1.2}$$

$$r(g) = -1,$$

where p is a non-dynamical (time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence free. That is why sometimes p is also called the conformal pressure.

Recently, in 2015, N.Basu et.al. [3] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

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Definition 1. *A Riemannian metric g on a smooth manifold M, of dimension n, is called a conformal Ricci soliton if there exists a constant λ and a vector field V such that*

$$\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.3}$$

where $S = Ric$ is the Ricci tensor, λ is a constant and p is the conformal pressure.

It can be easily checked that the above soliton equation satisfies the conformal Ricci flow equation(1.2). Later, T. Dutta et.al. [5] studied the conformal Ricci soliton in the framework of Lorentzian α-Sasakian manifolds.

A.A. Shaikh [14] in 2003, introduced the study of Lorentzian concircular structure manifolds (or, briefly, $(LCS)_n$-manifolds) which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [10]. After that, a lot of study has been carried out on $(LCS)_n$-manifolds and on locally ϕ-symmetric $(LCS)_n$-manifolds [16]. Moreover, in 2005, A.A. Shaikh et.al. [15] have shown the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology.

Motivated by the above studies, here we study conformal Ricci soliton in the framework of $(LCS)_n$-manifold. We find conditions to determine the nature of the soliton for different cases. The paper is organised as follows: After introduction, we discuss some preliminary concepts of $(LCS)_n$-manifolds, in section-2. Then in section-3, we study $(LCS)_n$-manifolds admitting the conformal Ricci soliton and we calculate the value of the soliton constant λ and hence we find the condition for the soliton to be shrinking, steady or expanding. After that, we prove that, if a $(LCS)_n$-manifold admits conformal Ricci soliton then it is ξ-projectively flat. In this section we also find conditions for a $(LCS)_n$-manifold admitting conformal Ricci soliton to be ξ-conharmonically flat and ξ-concircularly flat. Finally, in section-4 and section-5 we obtain some interesting results on conformal Ricci soliton on $(LCS)_n$-manifolds satisfying curvature conditions $R(\xi, X) \cdot \tilde{P} = 0$ and $R(\xi, X) \cdot \tilde{M} = 0$; where R is the Riemann curvature tensor, $\tilde{P}$ is the pseudo-projective curvature tensor and $\tilde{M}$ is the M-projective curvature tensor.

2 Brief overview of $(LCS)_n$-manifolds

A smooth connected paracompact Hausdorff n dimensional manifold (M, g) is said to be a Lorentzian manifold if the metric g is Lorentzian metric, i.e; M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM denotes the tangent space of the manifold M at point p and $\mathbb{R}$ is the real line. A non-zero vector $v \in T_pM$ is said to be timelike(respectively; non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (respectively;$\leq 0, = 0, > 0$) [11].

Next, we give the definition of a concircular vector field in a Lorentzian manifold, which is essential for the study of $(LCS)_n$-manifolds.

Definition 2. *Let (M, g) be a Lorentzian manifold and P is a vector field in M defined by $g(U, P) = B(U)$, for any vector field U in M. Then the vector field P is said to be a concircular vector field if*

$$(\nabla_U B)(Y) = \alpha[g(U, Y) + \omega(U)B(Y)],$$

where α is a non-zero scalar and ω is closed 1-form and ∇ denotes the covariant differentiation operator of the manifold M with respect to the Lorentzian metric g.

Let (M, g) be a Lorentzian manifold of dimension n and let M admits a unit timelike concircular vector field ξ satisfying $g(\xi, \xi) = -1$. The vector field ξ is called the characteristic vector field of the manifold (M, g). Then ξ being unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X) \quad \textit{and} \quad (\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad \alpha \neq 0. \tag{2.1}$$

Also the non-zero scalar α satisfies the equation

$$(\nabla_X \alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.2}$$

where ρ is a scalar function given by $\rho = -(\xi\alpha)$ and ∇ denotes the covariant differentiation operator of the manifold M with respect to the Lorentzian metric g. Now we consider a $(1, 1)$ tensor field ϕ given by, $\phi X = \frac{1}{\alpha}\nabla_X \xi$. Therefore it is to be noted that the tensor field ϕ also satisfies $\phi X = X + \eta(X)\xi$ and this implies that ϕ is a symmetric $(1, 1)$ tensor field, called the structure tensor of the manifold.

So now, we are in a position to define $(LCS)_n$-manifolds, introduced by A.A. Shaikh [14] to generalize the notion of LP-Sasakian manifolds of Matsumoto [10].

Definition 3. *Let (M, g) be an n-dimensional Lorentzian manifold. Then the manifold (M, g) together with the unit timelike concircular vector field ξ, associated 1-form η an $(1, 1)$ tensor field ϕ and the non-zero scalar function α is said to be a Lorentzian concircular structure manifold $(M, g, \xi, \eta, \phi, \alpha)$(briefly, $(LCS)_n$-manifold) [14].*

It is to be noted that, if we consider the scalar function $\alpha = 1$, then we can obtain the LP-Sasakian structure introduced by Matsumoto [10]. So, in that sense $(LCS)_n$-manifolds are a generalization of LP-Sasakian manifolds. Furthermore, in a $(LCS)_n$-manifold $(n > 2)$, the following relations hold [14, 15, 16]:

$$\phi^2 X = X + \eta(X)\xi \ , \ \eta(\xi) = -1, \tag{2.3}$$
$$\phi(\xi) = 0 \ , \ \eta \circ \phi = 0, \tag{2.4}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.5}$$
$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.6}$$
$$R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \tag{2.7}$$
$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.8}$$
$$S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

for all vector fields X, Y, Z in TM, where TM is tangent bundle of M. Here R is the Riemannian curvature tensor of the manifold M defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and S is the Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is the Ricci operator.

Next, we discuss an illustrative example of an $(LCS)_n$-manifold of dimension $n = 3$ as follows:

Example: Let us consider the manifold $M = \{(u, v, w) \in \mathbb{R}^3 : u \neq 0\}$, where $\{u, v, w\}$ are usual Euclidean coordinates in $\mathbb{R}^3$. Now we choose a set $\{E_i : 1 \leq i \leq 3\}$ of linearly independent vector fields on the manifold M as follows,

$$E_1 = u\frac{\partial}{\partial u}, \quad E_2 = u\frac{\partial}{\partial v}, \quad E_3 = u\frac{\partial}{\partial w}.$$

Define the Lorentzian metric g on M as,

$$g(E_1, E_1) = -1, \quad g(E_2, E_2) = g(E_3, E_3) = 1; \qquad g(E_i, E_j) = 0, \forall i \neq j.$$

Now if we choose $\xi = E_1$ and define a 1-form η on M by, $\eta(X) = g(X, E_1)$, $\forall X \in TM$, where TM is the tangent bundle of M, then it is easy to see that $\eta(\xi) = -1$.
Next let us define a $(1, 1)$ tensor field ϕ on M as,

$$\phi(E_1) = 0, \quad \phi(E_2) = E_3, \quad \phi(E_3) = E_2.$$

Again as g and ϕ are both linear maps, for all $X, Y \in TM$, from the above one can easily check that,

$$\phi^2(X) = X + \eta(X)\xi,$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y).$$

Now, it is well known that the connection ∇ of the metric g is given by the Koszul's formula,

$$\begin{aligned}2g(\nabla_X Y, Z) &= \nabla_X g(Y,Z) + \nabla_Y g(X,Z) - \nabla_Z g(X,Y)\\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),\end{aligned}$$

for all $X, Y, Z \in TM$ and the Lie bracket operation $[X,Y]$ is given by $[X,Y] = \nabla_X Y - \nabla_Y X$. Then one can easily calculate $[E_1, E_2] = E_2$, $[E_2, E_3] = 0$, $[E_1, E_3] = E_3$. Again using the above Koszul's formula and after a straightforward calculation we get,

$$\begin{gathered}\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0,\\ \nabla_{E_2} E_1 = -E_2, \quad \nabla_{E_2} E_2 = -E_1, \quad \nabla_{E_2} E_3 = 0,\\ \nabla_{E_3} E_1 = -E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_1.\end{gathered}$$

Thus from the above we can easily verify that for $\alpha = -1$, the relation $\phi X = \frac{1}{\alpha}\nabla_X \xi$ holds for all $X \in TM$. Hence we can conclude that $(M, g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_n$-manifold of dimension $n = 3$.

3 Conformal Ricci soliton on $(LCS)_n$-manifolds

Let us consider $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Again we know that, for all vector fields X, Y in TM, the 1-form η satisfies the equation

$$(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y). \tag{3.1}$$

Using the equation (2.1) in the above equation (3.1), after a simple calculation, we get

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha[g(X,Y) + \eta(X)\eta(Y)]. \tag{3.2}$$

Now applying the conformal Ricci soliton equation (1.3) in the above equation (3.2) we have

$$S(X,Y) = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]g(X,Y) - \alpha\eta(X)\eta(Y). \tag{3.3}$$

Let us take $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$. Then we can rewrite the above equation (3.3) as

$$S(X,Y) = \sigma g(X,Y) - \alpha\eta(X)\eta(Y). \tag{3.4}$$

which shows that the manifold is an η-Einstein manifold.

Now since the above is true for all vector fields X and Y, using the relation $S(X,Y) = g(QX,Y)$ in the above equation (3.4) we have

$$QX = \sigma X - \alpha\eta(X)\xi. \tag{3.5}$$

Again taking $Y = \xi$ in the equation (3.4) we get

$$S(X, \xi) = (\sigma + \alpha)\eta(X). \tag{3.6}$$

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold (M, g). Then putting $X = Y = e_i$ in the equation (3.4) and summing over $1 \leq i \leq n$, we have $r(g) = n\sigma + \alpha$. But we know that for conformal Ricci flow, $r(g) = -1$, which leads us to get $\sigma = -(\frac{\alpha+1}{n})$. Again we have $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, using this in the previous result we get

$$\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha. \tag{3.7}$$

So, from the above discussions, using equations (3.4) and (3.7), we can state the following theorem

Theorem 1. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton. Then*

a) The manifold becomes an η-Einstein manifold.

b) The value of the soliton scalar λ is equal to $\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha$.

c) The soliton is shrinking, steady or expanding according as the conformal pressure $p < 2(\frac{1-n}{n})\alpha$, $p = 2(\frac{1-n}{n})\alpha$ or $p > 2(\frac{1-n}{n})\alpha$.

Next, we discuss about the projective curvature tensor which plays an important role in the study of differential geometry. The projective curvature has an one-to-one correspondence between each coordinate neighbourhood of an n-dimensional Riemannian manifold and a domain of Euclidean space such that there is an one-to-one correspondence between geodesics of the Riemannian manifold with the straight lines in the Euclidean space. The projective curvature tensor in an n-dimensional Riemannian manifold (M, g) is defined by [19]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(QY, Z)X - g(QX, Z)Y], \tag{3.8}$$

for any vector fields $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold M, R is the Riemannian curvature tensor of M and Q is the Ricci operator.
The manifold (M, g) is called ξ-projectively flat if $P(X, Y)\xi = 0$ for any vector fields $X, Y \in \chi(M)$ and ξ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z = \xi$ in (3.8) we get

$$P(X, Y)\xi = R(X, Y)\xi - \frac{1}{(n-1)}[S(Y, \xi)X - S(X, \xi)Y].$$

Using (2.6) and (3.6) the above equation becomes

$$P(X, Y)\xi = [(\alpha^2 - \rho) - \frac{\sigma + \alpha}{(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{3.9}$$

Again combining equations (2.9) and (3.6) we have

$$[(\alpha^2-\rho)(n-1)-\sigma-\alpha]\eta(X)=0, \tag{3.10}$$

which essentially gives us

$$[(\alpha^2-\rho)(n-1)]=(\sigma+\alpha). \tag{3.11}$$

Now in view of (3.11), the equation (3.9) yields us $P(X,Y)\xi=0$ for any vector fields $X,Y\in\chi(M)$. Thus we have the following

Theorem 2. *If $(M,g,\xi,\eta,\phi,\alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-projectively flat, ξ being the characteristic vector field of the manifold.*

A transformation of a Riemannian manifold of dimension n, which transforms every geodesic circle of the manifold M into a geodesic circle, is called a concircular transformation [18]. Here a geodesic circle is a curve in M whose first curvature is constant and second curvature (that is, torsion) is identically equal to zero. The concircular curvature tensor in a Riemannian manifold (M,g) of dimension n is defined by [13, 18]

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y], \tag{3.12}$$

for any vector fields $X,Y,Z\in\chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold M and r is the scalar curvature of M.
The manifold (M,g) is called ξ-concircularly flat if $C(X,Y)\xi=0$ for any vector fields $X,Y\in\chi(M)$ and ξ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z=\xi$ in (3.12) we get

$$C(X,Y)\xi=R(X,Y)\xi-\frac{r}{n(n-1)}[\eta(Y)X-\eta(X)Y].$$

Using (2.6) the above equation becomes

$$C(X,Y)\xi=[(\alpha^2-\rho)-\frac{r}{n(n-1)}][\eta(Y)X-\eta(X)Y]. \tag{3.13}$$

Again in view of equation (3.11), the above equation (3.13) becomes

$$C(X,Y)\xi=[\frac{(\sigma+\alpha)}{(n-1)}-\frac{r}{n(n-1)}][\eta(Y)X-\eta(X)Y]. \tag{3.14}$$

Now in view of equation (3.14), we can say that $C(X,Y)\xi=0$ iff $r=n(\sigma+\alpha)$. Again using the fact that for conformal Ricci flow $r=-1$ and using $\sigma=[(\lambda-\alpha)-(\frac{p}{2}+\frac{1}{n})]$ we eventually get $C(X,Y)\xi=0$ iff $\lambda=\frac{p}{2}$. This leads to the following theorem

Theorem 3. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-concircularly flat iff $\lambda = \frac{p}{2}$, ξ being the characteristic vector field of the manifold and p is the conformal pressure.*

The conharmonic curvature tensor plays an important role in the study of manifolds. The conharmonic curvature tensor of an n-dimensional Riemannian manifold (M, g) is defined as [9]

(3.15)

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y]$$

for any vector fields $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold M, R is the Riemannian curvature tensor of M, S is the Ricci tensor and Q is the Ricci operator.

The manifold (M, g) is called ξ-conharmonically flat if $H(X,Y)\xi = 0$ for any vector fields $X, Y \in \chi(M)$ and ξ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension n, putting $Z = \xi$ in (3.15) we have

$$H(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)}[\eta(Y)QX - \eta(X)QY + S(Y,\xi)X - S(X,\xi)Y].$$

Using (2.6), (3.5) and (3.6) the above equation yields

$$H(X,Y)\xi = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{3.16}$$

Again in view of equation (3.11), the above equation (3.16) becomes

$$H(X,Y)\xi = [\frac{(-n\sigma - \alpha)}{(n-1)(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{3.17}$$

Thus from the above (3.17) we can conclude that $H(X,Y)\xi = 0$ iff $n\sigma = -\alpha$. Moreover, using the value $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$ and after few steps of calculations we have $H(X,Y)\xi = 0$ iff $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$. Thus we can state the following:

Theorem 4. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-conharmonically flat iff $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$, ξ being the characteristic vector field of the manifold and p is the conformal pressure.*

Next, let us consider a conformal Ricci soliton (g, V, λ) on an n-dimensional $(LCS)_n$-manifold M as

$$\mathcal{L}_V g(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y), \tag{3.18}$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the Lorentzian metric g in the direction of the vector field V. This vector field V is also called the potential vector field.

Now assume that the vector field V be pointwise collinear with the characteristic vector field ξ, that is, $V = b\xi$, where b is a smooth function on the manifold M. Then for any vector fields $X, Y \in \chi(M$, the equation (3.18) implies

$$\mathcal{L}_{b\xi} g(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y). \tag{3.19}$$

Again from the property of the Lie derivative of the Levi-Civita connection we know that $\mathcal{L}_Z g(X,Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X)$. Applying this formula in the above equation (3.19) and then using $\phi X = \frac{1}{\alpha}\nabla_X \xi$ we get

(3.20)

$$b\alpha g(\phi X,Y) + (Xb)\eta(Y) + b\alpha g(\phi Y, X) + (Yb)\eta(X) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y).$$

Putting $Y = \xi$ in (3.20) and using the equations (2.4) we obtain

$$2S(X,\xi) - (Xb) + (\xi b)\eta(X) = [2\lambda - (p + \frac{2}{n})]\eta(X). \tag{3.21}$$

Using equation (3.6) in the above(3.21) and then putting the value $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$ gives us

$$(Xb) = (\xi b)\eta X. \tag{3.22}$$

Again putting $X = \xi$ in the equation (3.21) we have

$$S(\xi,\xi) - (\xi b) + [\lambda - (\frac{p}{2} + \frac{1}{n})] = 0. \tag{3.23}$$

Now, in view of equation (3.6) and $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, the above equation (3.23) yields $(\xi b) = 0$. Furthermore, using $(\xi b) = 0$ in equation (3.22) we can conclude that (Xb)=0, for any vector field $X \in \chi(M)$. And this implies that the function b is constant and hence V is a constant multiple of ξ. Therefore we have the following theorem

Theorem 5. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton (g, V, λ), V being the potential vector field of the manifold. If the potential vector field V is pointwise collinear with the characteristic vector field ξ, i.e; if $V = b\xi$, then b is constant, i.e; V becomes constant multiple of ξ.*

Next, we study an important curvature property called ξ-Ricci semi symmetry.

Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Then we say that the manifold M is ξ-Ricci semi symmetric if $R(\xi, X) \cdot S = 0$ in M, where ξ is the characteristic vector field, R is the Riemannian curvature tensor, S is the Ricci tensor.

Let us start with the known formula that for any vector fields X, Y, Z on M,

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z). \tag{3.24}$$

Now, using (2.7) the above equation (3.24) yields

$$R(\xi, X)\cdot S = (\alpha^2-\rho)[g(X,Y)S(\xi,Z)-\eta(Y)S(X,Z)+S(Y,\xi)g(X,Z)-\eta(Z)S(Y,X)].$$

Using (2.9) in the above equation and after few steps we get

$$R(\xi, X)\cdot S = \alpha(\alpha^2-\rho)[g(X,Y)\eta(Z)+g(X,Z)\eta(Y)+2\eta(X)\eta(Y)\eta(Z)]. \tag{3.25}$$

Now note that $(\alpha^2-\rho)=0$ implies $\lambda = \frac{p}{2}+\frac{1}{n}$, which is the trivial case. Thus for non-triviality we assume $(\alpha^2-\rho)\neq 0$. Again as α is a non-zero scalar, from (3.25) we can state the following:

Theorem 6. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton, then the manifold becomes ξ-Ricci semi symmetric, i.e; $R(\xi, X)\cdot S = 0$ iff the Lorentzian metric g satisfies the relation*

$$g(X,Y)\eta(Z)+g(X,Z)\eta(Y)+2\eta(X)\eta(Y)\eta(Z)=0$$

for any vector fields X, Y, Z on M, ξ being the characteristic vector field, R is the Riemannian curvature tensor and S is the Ricci tensor.

4 Conformal Ricci soliton on $(LCS)_n$-manifolds satisfying certain curvature conditions

First let (M, g) be an n-dimensional $(LCS)_n$-manifold. Then from equation (3.15) the conharmonic curvature tensor on M is given by

$$H(X,Y)Z = R(X,Y)Z-\frac{1}{n-2}[g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y]. \tag{4.1}$$

Interchanging Z and X and the putting $Z=\xi$, we can rewrite the above equation (4.1) as

$$H(\xi,X)Y = R(\xi,X)Y-\frac{1}{n-2}[S(X,Y)\xi-S(\xi,Y)X+g(X,Y)Q\xi-g(\xi,Y)QX].$$

Using (2.7), (3.4), (3.5) and (3.6) in the above we get

$$H(\xi,X)Y = [(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{(n-2)}][g(X,Y)\xi-\eta(Y)X]. \tag{4.2}$$

Also from (4.2) we can write

$$\eta(H(\xi,X)Y) = -[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{(n-2)}][g(X,Y)+\eta(X)\eta(Y)]. \tag{4.3}$$

Now we assume that $H(\xi, X) \cdot S = 0$ holds. Then we have

$$S(H(\xi, X)Y, Z) + S(Y, H(\xi, X)Z) = 0. \tag{4.4}$$

In view of (3.4) the above (4.4) yields

$$\sigma[g(H(\xi, X)Y, Z)+g(Y, H(\xi, X)Z)]-\alpha[\eta(H(\xi, X)Z)\eta(Y)+\eta(H(\xi, X)Y)\eta(Z)] = 0.$$

Using (4.2) and (4.3) in the above equation we get

$$\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{4.5}$$

Finally taking $Z = \xi$ in equation (4.5) and then using (2.5) we arrive at

$$\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n-2)}]g(\phi X, \phi Y) = 0. \tag{4.6}$$

Since α is non-zero and $g(\phi X, \phi Y) \neq 0$ always; then $[(\alpha^2 - \rho) - \frac{(2\sigma+\alpha)}{(n-2)}] = 0$ i.e; $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$. Therefore we can state the following theorem:

Theorem 7. *If $(M, g, \xi, \eta, \phi, \alpha)$ is an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton, and satisfies the condition $H(\xi, X) \cdot S = 0$ i.e; the manifold is ξ-Ricci conharomnically symmetric. Then the soliton constant is given by $\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha$; where H is the conharmonic curvature tensor and S is the Ricci tensor of the manifold and ξ is the characteristic vector field.*

Next we study another important curvature tensor called $\tilde{M}$-projective curvature tensor. The $\tilde{M}$-projective curvature tensor on an $(LCS)_n$-manifold is defined by [1]

$$\tilde{M}(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \tag{4.7}$$

Taking inner product with respect to the vector field ξ, the above (4.6) yields

$$\eta(\tilde{M}(X, Y)Z) = \eta(R(X, Y)Z) - \frac{1}{2(n-1)}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y) + g(Y, Z)\eta(QX) - g(X, Z)\eta(QY)]. \tag{4.8}$$

Using (2.8), (3.4) and (3.5) in the above equation we get

$$\eta(\tilde{M}(X, Y)Z) = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{2(n-1)}][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{4.9}$$

Now we assume the condition that $R(\xi, X) \cdot \tilde{M} = 0$. Then we have

$$R(\xi, X)\tilde{M}(Y, Z)W - \tilde{M}(R(\xi, X)Y, Z)W - \tilde{M}(Y, R(\xi, X)Z)W - \tilde{M}(Y, Z)R(\xi, X)W = 0. \tag{4.10}$$

Using (2.7) in (4.9) and then taking an inner product with respect to ξ we get

$$(4.11)\quad -g(X,\tilde{M}(Y,Z)W)-\eta(X)\eta(\tilde{M}(Y,Z)W)-g(X,Y)\eta(\tilde{M}(\xi,Z)W) +\eta(Y)\eta(\tilde{M}(X,Z)W)-g(X,Z)\eta(\tilde{M}(Y,\xi)W)+\eta(Z)\eta(\tilde{M}(Y,X)W) -g(X,W)\eta(\tilde{M}(Y,Z)\xi)+\eta(W)\eta(\tilde{M}(Y,Z)X)=0.$$

Then in view of (4.8) the above (4.10) becomes

(4.12)

$$[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{2(n-1)}][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,\tilde{M}(Y,Z)W)=0.$$

From (4.6) and (4.11) we get

(4.13)

$$[(\alpha^2-\rho)-\frac{(2\sigma+\alpha)}{2(n-1)}][g(Y,W)g(X,Z)-g(X,Y)g(Z,W)]+g(X,R(Y,Z)W) -\frac{1}{2(n-1)}[S(Z,W)g(X,Y)-S(Y,W)g(X,Z)+g(Z,W)S(Y,X)-g(Y,W)S(Z,X)]=0.$$

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold (M,g). Then putting $X = Y = e_i$ in the equation (4.12) and summing over $1 \leq i \leq n$, we get

$$(4.14)\qquad 2nS(Z,W)=[2(n-1)^2(\alpha^2-\rho)-(n-1)(2\sigma+\alpha)-r]g(Z,W).$$

Again putting $Z = W = \xi$ in above and using equation (3.6) we get

$$(4.15)\qquad 2(n-1)^2(\alpha^2-\rho)-(5n-2)[\lambda-(\frac{p}{2}+\frac{1}{n})]+2n\alpha=0.$$

Now using (3.11) in the above equation (4.14) and after a simple calculation we arrive at

$$(4.16)\qquad \lambda=(\frac{p}{2}+\frac{1}{n})-2\alpha.$$

Thus we have the following theorem

Theorem 8. *Let $(M,g,\xi,\eta,\phi,\alpha)$ be an n-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton and the manifold is ξ-$\tilde{M}$-projectively semi symmetric i.e; it satisfies the condition $R(\xi,X)\cdot\tilde{M}=0$; ξ being the characteristic vector field, $\tilde{M}$ is the M-projective curvature tensor of the manifold. Then the soliton is shrinking, steady or expanding according as $p>(4\alpha-\frac{2}{n})$, $p=(4\alpha-\frac{2}{n})$ or $p<(4\alpha-\frac{2}{n})$*

Next we prove an interesting result on $(LCS)_n$-manifold admitting a conformal Ricci soliton and satisfying the condition $R(\xi,X)\cdot\tilde{P}=0$, where $\tilde{P}$ denotes the well-known Pseudo-projective curvature tensor. But before that let us recall some well-known results that will be used later in this section:

Theorem 9. *[11] If $S : g(x, y, z) = c$ is a surface in $\mathbb{R}^3$ then the gradient vector field ∇g (connected only at a point of S) is a non-vanishing normal vector field on the entire surface S.*

S.R. Ashoka et.al. in their paper [1] have given the higher dimensional version of the above theorem as follows:

Corollary 1. *[1] If $S : g(x, y, z) = c$ is a surface (abstract surface or manifold) in $\mathbb{R}^n$ then the gradient vector field ∇g (connected only at points of S) is a non-vanishing normal vector field on the entire surface (abstract surface or manifold) S.*

Then the above mentioned authors in [1] also gave the following remark from the above corollary as:

Remark 1. *[1] Taking a real valued scalar function α associated with an $(LCS)_n$-manifold with $M = \mathbb{R}^3$ and $g = \alpha$ in the above corollary we have, $\nabla\alpha$ as a non-vanishing normal vector field on $S \subset M$ and directional derivative of α with respect to ξ, $\xi\alpha = \xi$, $\nabla\alpha = |\xi||\nabla\alpha|cos(\hat{\xi}, \nabla\alpha)$*
1) If ξ is tangent to S then $\xi\alpha = 0$.
2) If ξ is tangent to M but not to S then $\xi\alpha \neq 0$.
3) If the angle between ξ and $\nabla\alpha$ is acute then $0 < cos(\hat{\xi}, \nabla\alpha) < 1$, then $\xi\alpha = k|\nabla\alpha|$, $0 < k < 1$ and $\xi\alpha > 0$.
4) If the angle between ξ and $\nabla\alpha$ is obtuse then $-1 < cos(\hat{\xi}, \nabla\alpha) < 0$, then $\xi\alpha = k|\nabla\alpha|$, $-1 < k < 0$ and $\xi\alpha < 0$.

Now we see the dependance of the conformal Ricci soliton on $\xi\alpha$ for $(LCS)_n$-manifolds satisfying $R(\xi, X) \cdot \tilde{P} = 0$. The Pseudo projective curvature tensor $\tilde{P}$ is defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y,Z)X - g(X,Z)Y], \tag{4.17}$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (4.16) we get

$$\tilde{P}(X,Y)\xi = aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] - \frac{r}{n}(\frac{a}{n-1} + b)[\eta(Y)X - \eta(X)Y]. \tag{4.18}$$

Using (2.6) and (3.6) the above equation (4.17) yields

$$\tilde{P}(X,Y)\xi = [a(\alpha^2 - \rho) + b(\sigma + \alpha) - \frac{r}{n}(\frac{a}{n-1} + b)][\eta(Y)X - \eta(X)Y], \tag{4.19}$$

where σ is as described in the previous section. Again from (4.16) we can write

$$\eta(\tilde{P}(X,Y)Z) = a\eta(R(X,Y)Z) + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].$$

Using (2.8) and (3.4) the above equation becomes

$$\eta(\tilde{P}(X,Y)Z) = [a(\alpha^2-\rho)+b\sigma-\frac{r}{n}(\frac{a}{n-1}+b)][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]. \tag{4.20}$$

Now we assume the condition that $R(\xi,X)\cdot\tilde{P}=0$. Then we have

$$\begin{aligned} R(\xi,X)\tilde{P}(U,V)W &- \tilde{P}(R(\xi,X)U,V)W \\ &- \tilde{P}(U,R(\xi,X)V)W - \tilde{P}(U,V)R(\xi,X)W = 0, \end{aligned} \tag{4.21}$$

for any vector fields X,U,V,W on M. Using (2.7) in the above equation and then taking an inner product with respect to ξ we get

$$\begin{aligned} &- g(X,\tilde{P}(U,V)W) - \eta(X)\eta(\tilde{P}(U,V)W) - g(X,U)\eta(\tilde{P}(\xi,V)W) \\ &\quad + \eta(U)\eta(\tilde{P}(X,V)W) - g(X,U)\eta(\tilde{P}(U,\xi)W) + \eta(V)\eta(\tilde{P}(U,X)W) \\ &\qquad - g(X,W)\eta(\tilde{P}(U,V)\xi) + \eta(W)\eta(\tilde{P}(U,V)X) = 0. \end{aligned}$$

Then using (4.18) and (4.19) the above equation becomes

$$\begin{aligned} [a(\alpha^2-\rho)+b\sigma-\frac{r}{n}(\frac{a}{n-1}+b)][g(X,V)g(U,W)-g(X,U)g(V,W)] \\ + g(X,\tilde{P}(U,V)W) = 0. \end{aligned} \tag{4.22}$$

Now in view of (4.16) and then using (3.4) in the equation (4.21) we get

$$\begin{aligned} ag(X,R(U,V)W) &- b\alpha[\eta(V)\eta(W)g(X,U)-\eta(U)\eta(W)g(X,V)] \\ &+ a(\alpha^2-\rho)[g(X,V)g(U,W)-g(X,U)g(V,W)] = 0. \end{aligned} \tag{4.23}$$

Let us consider an orthonormal basis $\{e_i : 1\leq i\leq n\}$ of the manifold (M,g). Then putting $X=U=e_i$ in the equation (4.22) and summing over $1\leq i\leq n$, we get

$$aS(V,W) - b(n-1)\alpha\eta(V)\eta(W) - a(n-1)(\alpha^2-\rho)g(V,W) = 0. \tag{4.24}$$

Again setting $V=W=\xi$ in (4.23) and after a few steps of simple calculations we get

$$\lambda = (n-1)[(\alpha^2-\rho)-\frac{b}{a}\alpha] + (\frac{p}{2}+\frac{1}{n}). \tag{4.25}$$

Therefore in view of the above equation (4.24) and Remark-4.1 we can state the following :

Theorem 10. *Let $(M,g,\xi,\eta,\phi,\alpha)$ be an n-dimensional $(LCS)_n$-manifold which admits a conformal Ricci soliton and the manifold is ξ-pseudo-projectively semi symmetric i.e; if it satisfies the condition $R(\xi,X)\cdot\tilde{P}=0$; ξ being the characteristic vector field, $\tilde{P}$ is the pseudo-projective curvature tensor of the manifold and α is a positive function; then*

1) If ξ is orthogonal to $\nabla\alpha$; the soliton is expanding if $\alpha > \frac{b}{a}$, $p > -\frac{2}{n}$; steady if $\alpha = \frac{b}{a}$, $p = -\frac{2}{n}$ and shrinking if $\alpha < \frac{b}{a}$, $p < -\frac{2}{n}$.

2)If the angle between ξ and $\nabla\alpha$ is acute; the soliton is expanding if $\alpha^2 + k|\nabla\alpha| > \frac{b}{a}\alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 + k|\nabla\alpha| = \frac{b}{a}\alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 + k|\nabla\alpha| < \frac{b}{a}\alpha$, $p < -\frac{2}{n}$.

3) If the angle between ξ and $\nabla\alpha$ is obtuse; the soliton is expanding if $\alpha^2 > k|\nabla\alpha| + \frac{b}{a}\alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 = k|\nabla\alpha| + \frac{b}{a}\alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 < k|\nabla\alpha| + \frac{b}{a}\alpha$, $p < -\frac{2}{n}$.

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