A study on conformal Ricci solitons in the framework of 
\((LCS)_n\)-manifolds

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Abstract

The main aim of this paper is to study Lorentzian concircular structure manifolds (briefly, \((LCS)_n\)-manifolds) admitting the conformal Ricci soliton and to characterize when the soliton is shrinking, steady or expanding. Next we establish some results on the \((LCS)_n\)-manifold whose metric is a conformal Ricci soliton. Finally some interesting results have been obtained by applying certain curvature conditions on \((LCS)_n\)-manifolds admitting conformal Ricci solitons.

Subject Classification: 53C15, 53C25, 53D10.

Keywords: Ricci soliton, Conformal Ricci soliton, \((LCS)_n\)-manifold, pseudo-projective curvature tensor, concircular curvature tensor.

1 Introduction

In 1982, R.S. Hamilton [7] introduced the Ricci soliton as a self similar solution to the Ricci flow equation given by: \[
\frac{dg}{dt}(g(t)) = -2Ric(g(t)),
\]
where \(g(t)\) is an one parameter family of metrics on the manifold.

A Riemannian metric \(g\) defined on a smooth manifold \(M\), of dimension \(n\), is said to be a Ricci soliton if for some constant \(\lambda\), there exists a smooth vector field \(V\) on \(M\) satisfying the equation

\[
Ric + \frac{1}{2}L_V g = \lambda g,
\]

where \(L_V\) denotes the Lie derivative in the direction of \(V\) and \(Ric\) is the Ricci tensor. The Ricci soliton is called shrinking if \(\lambda > 0\), steady if \(\lambda = 0\) and expanding if \(\lambda < 0\). Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by an one-parameter group of diffeomorphisms and scaling [8]. After Hamilton’s work many authors have studied Ricci flow and a rigorous literature on this topic can be found in [4, 17].

A.E. Fischer [6] in 2005, introduced conformal Ricci flow equation which is a modified version of the Hamilton’s Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci
flow equations on a smooth closed connected oriented manifold $M$, of dimension $n$, are given by

$$
\frac{\partial g}{\partial t} + 2(Ric + \frac{g}{n}) = -pg,
$$

where $p$ is a non-dynamical (time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence free. That is why sometimes $p$ is also called the conformal pressure.

Recently, in 2015, N. Basu et al. [3] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

**Definition 1.** A Riemannian metric $g$ on a smooth manifold $M$ of dimension $n$, is called a conformal Ricci soliton if there exists a constant $\lambda$ and a vector field $V$ such that

$$
\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g,
$$

where $S = Ric$ is the Ricci tensor, $\lambda$ is a constant and $p$ is the conformal pressure.

It can be easily checked that the above soliton equation satisfies the conformal Ricci flow equation (1.2). Later, T. Dutta et al. [5] studied the conformal Ricci soliton in the framework of Lorentzian $\alpha$-Sasakian manifolds.

A.A. Shaikh [14] in 2003, introduced the study of Lorentzian concircular structure manifolds (or, briefly, $(LCS)_n$-manifolds) which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [10]. After that, a lot of study has been carried out on $(LCS)_n$-manifolds and on locally $\psi$-symmetric $(LCS)_n$-manifolds [16]. Moreover, in 2005, A.A. Shaikh et al. [15] have shown the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology.

Motivated by the above studies, here we study conformal Ricci soliton in the framework of $(LCS)_n$-manifold. We find conditions to determine the nature of the soliton for different cases. The paper is organised as follows: After introduction, we discuss some preliminary concepts of $(LCS)_n$-manifolds, in section-2. Then in section-3, we study $(LCS)_n$-manifolds admitting the conformal Ricci soliton and we calculate the value of the soliton constant $\lambda$ and hence we find the condition for the soliton to be shrinking, steady or expanding. After that, we prove that, if a $(LCS)_n$-manifold admits conformal Ricci soliton then it is $\xi$-projectively flat. In this section we also find conditions for a $(LCS)_n$-manifold admitting conformal Ricci soliton to be $\xi$-conharmonically flat and $\xi$-concircularity flat. Finally, in section-4 and section-5 we obtain some interesting results on conformal Ricci soliton on $(LCS)_n$-manifolds satisfying curvature conditions $R(\xi, X) \cdot \tilde{P} = 0$ and $R(\xi, X) \cdot \tilde{M} = 0$; where $R$ is the Riemann curvature tensor, $\tilde{P}$ is the pseudo-projective curvature tensor and $\tilde{M}$ is the $M$-projective curvature tensor.
2 Brief overview of \((LCS)_n\)-manifolds

A smooth connected paracompact Hausdorff \(n\) dimensional manifold \((M, g)\) is said to be a Lorentzian manifold if the metric \(g\) is Lorentzian metric, i.e; \(M\) admits a smooth symmetric tensor field \(g\) of type \((0,2)\) such that for each point \(p \in M\), the tensor \(g_p : T_pM \times T_pM \to \mathbb{R}\) is a non-degenerate inner product of signature \((-,+,...,+\)) where \(T_pM\) denotes the tangent space of the manifold \(M\) at point \(p\) and \(\mathbb{R}\) is the real line. A non-zero vector \(v \in T_pM\) is said to be timelike (respectively; non-spacelike, null, spacelike) if it satisfies \(g_p(v,v) < 0\) (respectively; \(\leq 0\), \(= 0\), \(> 0\))\[11\].

Next, we give the definition of a concircular vector field in a Lorentzian manifold, which is essential for the study of \((LCS)_n\)-manifolds.

**Definition 2.** Let \((M, g)\) be a Lorentzian manifold and \(P\) is a vector field in \(M\) defined by \(g(U, P) = B(U)\), for any vector field \(U\) in \(M\). Then the vector field \(P\) is said to be a concircular vector field if
\[
(\nabla_U B)(Y) = \alpha [g(U,Y) + \omega(U)B(Y)],
\]
where \(\alpha\) is a non-zero scalar and \(\omega\) is closed 1-form and \(\nabla\) denotes the covariant differentiation operator of the manifold \(M\) with respect to the Lorentzian metric \(g\).

Let \((M, g)\) be a Lorentzian manifold of dimension \(n\) and let \(M\) admits a unit timelike concircular vector field \(\xi\) satisfying \(g(\xi, \xi) = -1\). The vector field \(\xi\) is called the characteristic vector field of the manifold \((M, g)\). Then \(\xi\) being unit concircular vector field, there exists a non-zero 1-form \(\eta\) such that
\[
g(X, \xi) = \eta(X) \quad \text{and} \quad (\nabla_X \eta)(Y) = \alpha [g(X,Y) + \eta(X)\eta(Y)], \quad \alpha \neq 0.
\]

Also the non-zero scalar \(\alpha\) satisfies the equation
\[
(\nabla_X \alpha) = (X \alpha) = d\alpha(X) = \rho \eta(X),
\]
where \(\rho\) is a scalar function given by \(\rho = -\alpha\) and \(\nabla\) denotes the covariant differentiation operator of the manifold \(M\) with respect to the Lorentzian metric \(g\). Now we consider a \((1,1)\) tensor field \(\phi\) given by \(\phi X = \frac{1}{\alpha} \nabla_X \xi\). Therefore it is to be noted that the tensor field \(\phi\) also satisfies \(\phi X = X + \eta(X)\xi\) and this implies that \(\phi\) is a symmetric \((1,1)\) tensor field, called the structure tensor of the manifold.

So now, we are in a position to define \((LCS)_n\)-manifolds, introduced by A.A. Shaikh [14] to generalize the notion of LP-Sasakian manifolds of Matsumoto [10].

**Definition 3.** Let \((M, g)\) be an \(n\)-dimensional Lorentzian manifold. Then the manifold \((M, g)\) together with the unit timelike concircular vector field \(\xi\), associated 1-form \(\eta\) an \((1,1)\) tensor field \(\phi\) and the non-zero scalar function \(\alpha\) is said to be a Lorentzian concircular structure manifold \((M, g, \xi, \eta, \phi, \alpha)\)(briefly, \((LCS)_n\)-manifold) [14].
It is to be noted that, if we consider the scalar function $\alpha = 1$, then we can obtain the LP-Sasakian structure introduced by Matsumoto [10]. So, in that sense $(LCS)_n$-manifolds are a generalization of LP-Sasakian manifolds. Furthermore, in a $(LCS)_n$-manifold ($n > 2$), the following relations hold [14, 15, 16]:

\begin{align}
\phi^2 X &= X + \eta(X)\xi, \quad \eta(\xi) = -1, \\
\phi(\xi) &= 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
R(\phi Y, \phi X) &\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \\
S(X, \xi) &= (n - 1)(\alpha^2 - \rho)\eta(X),
\end{align}

for all vector fields $X,Y,Z$ in $TM$, where $TM$ is tangent bundle of $M$. Here $R$ is the Riemannian curvature tensor of the manifold $M$ defined by

\[ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \]

and $S$ is the Ricci tensor defined by $S(X,Y) = g(QX,Y)$, where $Q$ is the Ricci operator.

Next, we discuss an illustrative example of an $(LCS)_n$-manifold of dimension $n = 3$ as follows:

**Example:** Let us consider the manifold $M = \{(u,v,w) \in \mathbb{R}^3 : u \neq 0\}$, where \{u,v,w\} are usual Euclidean coordinates in $\mathbb{R}^3$. Now we choose a set \{\(E_i : 1 \leq i \leq 3\}\} of linearly independent vector fields on the manifold $M$ as follows,

\[ E_1 = u \frac{\partial}{\partial u}, \quad E_2 = u \frac{\partial}{\partial v}, \quad E_3 = u \frac{\partial}{\partial w}. \]

Define the Lorentzian metric $g$ on $M$ as,

\[ g(E_1, E_1) = -1, \quad g(E_2, E_2) = g(E_3, E_3) = 1; \quad g(E_i, E_j) = 0, \forall i \neq j. \]

Now if we choose $\xi = E_1$ and define a 1-form $\eta$ on $M$ by, $\eta(X) = g(X, E_1)$, $\forall X \in TM$, where $TM$ is the tangent bundle of $M$, then it is easy to see that $\eta(\xi) = -1$.

Next let us define a $(1,1)$ tensor field $\phi$ on $M$ as,

\[ \phi(E_1) = 0, \quad \phi(E_2) = E_3, \quad \phi(E_3) = E_2. \]

Again as $g$ and $\phi$ are both linear maps, for all $X, Y \in TM$, from the above one can easily check that,

\[ \phi^2(X) = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y). \]
Now, it is well known that the connection $\nabla$ of the metric $g$ is given by the Koszul’s formula,

$$2g(\nabla_X Y, Z) = \nabla_X g(Y, Z) + \nabla_Y g(X, Z) - \nabla_Z g(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

for all $X, Y, Z \in TM$ and the Lie bracket operation $[X, Y]$ is given by $[X, Y] = \nabla_X Y - \nabla_Y X$. Then one can easily calculate $[E_1, E_2] = E_2$, $[E_2, E_3] = 0$, $[E_1, E_3] = E_3$. Again using the above Koszul’s formula and after a straightforward calculation we get,

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0,$$

$$\nabla_{E_2} E_1 = -E_2, \quad \nabla_{E_2} E_2 = -E_1, \quad \nabla_{E_2} E_3 = 0,$$

$$\nabla_{E_3} E_1 = -E_3, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_1.$$

Thus from the above we can easily verify that for $\alpha = -1$, the relation $\phi X = \frac{1}{\alpha} \nabla X \xi$ holds for all $X \in TM$. Hence we can conclude that $(M, g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_n$-manifold of dimension $n = 3$.

### 3 Conformal Ricci soliton on $(LCS)_n$-manifolds

Let us consider $(M, g, \xi, \eta, \phi, \alpha)$ be an $n$-dimensional $(LCS)_n$-manifold. Again we know that, for all vector fields $X, Y$ in $TM$, the 1-form $\eta$ satisfies the equation

$$(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y).$$

Using the equation (2.1) in the above equation (3.1), after a simple calculation, we get

$$(\mathcal{L}_\xi g)(X, Y) = 2\alpha[g(X, Y) + \eta(X)\eta(Y)].$$

Now applying the conformal Ricci soliton equation (1.3) in the above equation (3.2) we have

$$(3.3) \quad S(X, Y) = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]g(X, Y) - \alpha\eta(X)\eta(Y).$$

Let us take $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$. Then we can rewrite the above equation (3.3) as

$$(3.4) \quad S(X, Y) = \sigma g(X, Y) - \alpha\eta(X)\eta(Y).$$

which shows that the manifold is an $\eta$-Einstein manifold.

Now since the above is true for all vector fields $X$ and $Y$, using the relation $S(X, Y) = g(QX, Y)$ in the above equation (3.4) we have

$$(3.5) \quad QX = \sigma X - \alpha\eta(X)\xi.$$
Again taking $Y = \xi$ in the equation (3.4) we get
\begin{equation}
(3.6) \quad S(X, \xi) = (\sigma + \alpha)\eta(X).
\end{equation}

Let us consider an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ of the manifold $(M, g)$. Then putting $X = Y = e_i$ in the equation (3.4) and summing over $1 \leq i \leq n$, we have $r(g) = n\sigma + \alpha$. But we know that for conformal Ricci flow, $r(g) = -1$, which leads us to get $\sigma = -\left(\frac{\alpha + 1}{n}\right)$. Again we have $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, using this in the previous result we get
\begin{equation}
(3.7) \quad \lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha.
\end{equation}

So, from the above discussions, using equations (3.4) and (3.7), we can state the following theorem

**Theorem 1.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $n$-dimensional $(LCS)_n$-manifold admitting a conformal Ricci soliton. Then

a) The manifold becomes an $\eta$-Einstein manifold.

b) The value of the soliton scalar $\lambda$ is equal to $\lambda = \frac{p}{2} + (1 - \frac{1}{n})\alpha$.

c) The soliton is shrinking, steady or expanding according as the conformal pressure $p < 2(\frac{1-n}{n})\alpha$, $p = 2(\frac{1-n}{n})\alpha$ or $p > 2(\frac{1-n}{n})\alpha$.

Next, we discuss about the projective curvature tensor which plays an important role in the study of differential geometry. The projective curvature has an one-to-one correspondence between each coordinate neighbourhood of an $n$-dimensional Riemannian manifold and a domain of Euclidean space such that there is an one-to-one correspondence between geodesics of the Riemannian manifold with the straight lines in the Euclidean space. The projective curvature tensor in an $n$-dimensional Riemannian manifold $(M, g)$ is defined by [19]
\begin{equation}
(3.8) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(QY, Z)X - g(QX, Z)Y],
\end{equation}
for any vector fields $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of the manifold $M$, $R$ is the Riemannian curvature tensor of $M$ and $Q$ is the Ricci operator.

The manifold $(M, g)$ is called $\xi$-projectively flat if $P(X, Y)\xi = 0$ for any vector fields $X, Y \in \chi(M)$ and $\xi$ is the characteristic vector field of the manifold. Now for an $(LCS)_n$-manifold of dimension $n$, putting $Z = \xi$ in (3.8) we get
\begin{equation}
P(X, Y)\xi = R(X, Y)\xi - \frac{1}{(n-1)}[S(Y, \xi)X - S(X, \xi)Y].
\end{equation}

Using (2.6) and (3.6) the above equation becomes
\begin{equation}
(3.9) \quad P(X, Y)\xi = [(\alpha^2 - \rho) - \frac{\sigma + \alpha}{(n-1)}] [\eta(Y)X - \eta(X)Y].
\end{equation}
Again combining equations (2.9) and (3.6) we have

\[(\alpha^2 - \rho)(n - 1) - \sigma - \alpha] \eta(X) = 0,\]

which essentially gives us

\[(\alpha^2 - \rho)(n - 1) = (\sigma + \alpha).\]

Now in view of (3.11), the equation (3.9) yields us \(P(X,Y)\xi = 0\) for any vector fields \(X,Y \in \chi(M)\). Thus we have the following

**Theorem 2.** If \((M,g,\xi,\eta,\phi,\alpha)\) is an \(n\)-dimensional \((LCS)_n\)-manifold admitting a conformal Ricci soliton, then the manifold becomes \(\xi\)-projectively flat, \(\xi\) being the characteristic vector field of the manifold.

A transformation of a Riemannian manifold of dimension \(n\), which transforms every geodesic circle of the manifold \(M\) into a geodesic circle, is called a concircular transformation [18]. Here a geodesic circle is a curve in \(M\) whose first curvature is constant and second curvature (that is, torsion) is identically equal to zero. The concircular curvature tensor in a Riemannian manifold \((M,g)\) of dimension \(n\) is defined by [13, 18]

\[
C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],
\]

for any vector fields \(X,Y,Z \in \chi(M)\), \(\chi(M)\) being the Lie algebra of vector fields of the manifold \(M\) and \(r\) is the scalar curvature of \(M\).

The manifold \((M,g)\) is called \(\xi\)-concircularly flat if \(C(X,Y)\xi = 0\) for any vector fields \(X,Y \in \chi(M)\) and \(\xi\) is the characteristic vector field of the manifold. Now for an \((LCS)_n\)-manifold of dimension \(n\), putting \(Z = \xi\) in (3.12) we get

\[
C(X,Y)\xi = R(X,Y)\xi - \frac{r}{n(n-1)}[\eta(Y)X - \eta(X)Y].
\]

Using (2.6) the above equation becomes

\[
C(X,Y)\xi = [(\alpha^2 - \rho) - \frac{r}{n(n-1)}][\eta(Y)X - \eta(X)Y].
\]

Again in view of equation (3.11), the above equation (3.13) becomes

\[
C(X,Y)\xi = \left(\frac{\sigma + \alpha}{n-1} - \frac{r}{n(n-1)}\right)[\eta(Y)X - \eta(X)Y].
\]

Now in view of equation (3.14), we can say that \(C(X,Y)\xi = 0\) iff \(r = n(\sigma + \alpha)\). Again using the fact that for conformal Ricci flow \(r = -1\) and using \(\sigma = [(\lambda - \alpha) - \left(\frac{p}{n} + \frac{1}{n}\right)]\) we eventually get \(C(X,Y)\xi = 0\) iff \(\lambda = \frac{p}{2}\). This leads to the following theorem
**Theorem 3.** If \((M, g, \xi, \eta, \phi, \alpha)\) is an \(n\)-dimensional \((LCS)\)_\(n\)-manifold admitting a conformal Ricci soliton, then the manifold becomes \(\xi\)-concircularly flat iff \(\lambda = \frac{p}{2}\), \(\xi\) being the characteristic vector field of the manifold and \(p\) is the conformal pressure.

The conharmonic curvature tensor plays an important role in the study of manifolds. The conharmonic curvature tensor of an \(n\)-dimensional Riemannian manifold \((M, g)\) is defined as [9]

\[
H(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y]
\]  
(3.15)

for any vector fields \(X, Y, Z \in \chi(M)\), \(\chi(M)\) being the Lie algebra of vector fields of the manifold \(M\), \(R\) is the Riemannian curvature tensor of \(M\), \(S\) is the Ricci tensor and \(Q\) is the Ricci operator.

The manifold \((M, g)\) is called \(\xi\)-conharmonically flat if \(H(X, Y)\xi = 0\) for any vector fields \(X, Y \in \chi(M)\) and \(\xi\) is the characteristic vector field of the manifold. Now for an \((LCS)\)_\(n\)-manifold of dimension \(n\), putting \(Z = \xi\) in (3.15) we have

\[
H(X, Y)\xi = R(X, Y)\xi - \frac{1}{n-2}[\eta(Y)QX - \eta(X)QY + S(Y, \xi)X - S(X, \xi)Y].
\]

Using (2.6), (3.5) and (3.6) the above equation yields

\[
H(X, Y)\xi = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n - 2)}][\eta(Y)X - \eta(Y)Y].
\]

Again in view of equation (3.11), the above equation (3.16) becomes

\[
H(X, Y)\xi = \left[\frac{(-n\sigma - \alpha)}{(n - 1)(n - 2)}\right][\eta(Y)X - \eta(Y)Y].
\]

(3.17)

Thus from the above (3.17) we can conclude that \(H(X, Y)\xi = 0\) iff \(n\sigma = -\alpha\). Moreover, using the value \(\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]\) and after few steps of calculations we have \(H(X, Y)\xi = 0\) iff \(\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha\). Thus we can state the following:

**Theorem 4.** If \((M, g, \xi, \eta, \phi, \alpha)\) is an \(n\)-dimensional \((LCS)\)_\(n\)-manifold admitting a conformal Ricci soliton, then the manifold becomes \(\xi\)-conharmonically flat iff \(\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n})\alpha\), \(\xi\) being the characteristic vector field of the manifold and \(p\) is the conformal pressure.

Next, let us consider a conformal Ricci soliton \((g, V, \lambda)\) on an \(n\)-dimensional \((LCS)\)_\(n\)-manifold \(M\) as

\[
\mathcal{L}_V g(X, Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{n})]g(X, Y),
\]

(3.18)

where \(\mathcal{L}_V g\) denotes the Lie derivative of the Lorentzian metric \(g\) in the direction of the vector field \(V\). This vector field \(V\) is also called the potential vector field.

Now assume that the vector field \(V\) be pointwise collinear with the characteristic
vector field $\xi$, that is, $V = b\xi$, where $b$ is a smooth function on the manifold $M$. Then for any vector fields $X, Y \in \chi(M)$, the equation (3.18) implies

$$\mathcal{L}_{b\xi} g(X, Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{n})] g(X, Y).$$

Again from the property of the Lie derivative of the Levi-Civita connection we know that $\mathcal{L}_{Z} g(X, Y) = g(\nabla_{X} Z, Y) + g(\nabla_{Y} Z, X)$. Applying this formula in the above equation (3.19) and then using $\phi X = \frac{1}{\alpha} \nabla_{X} \xi$ we get

$$b\alpha g(\phi X, Y) + (Xb)\eta(Y) + (\xi b)\eta(X) + 2S(X, Y) = [2\lambda - (p + \frac{2}{n})] g(X, Y).$$

Putting $Y = \xi$ in (3.20) and using the equations (2.4) we obtain

$$2S(X, \xi) - (Xb)\eta(\xi) + (\xi b)\eta(X) = [2\lambda - (p + \frac{2}{n})] \eta(X).$$

Using equation (3.6) in the above (3.21) and then putting the value $\sigma = [(\lambda - \alpha) - (\frac{\lambda}{2} + \frac{1}{n})]$ gives us

$$(Xb) = (\xi b)\eta X.$$

Again putting $X = \xi$ in the equation (3.21) we have

$$S(\xi, \xi) - (\xi b) + [\lambda - (\frac{p}{2} + \frac{1}{n})] = 0.$$

Now, in view of equation (3.6) and $\sigma = [(\lambda - \alpha) - (\frac{p}{2} + \frac{1}{n})]$, the above equation (3.23) yields $(\xi b) = 0$. Furthermore, using $(\xi b) = 0$ in equation (3.22) we can conclude that $(Xb) = 0$, for any vector field $X \in \chi(M)$. And this implies that the function $b$ is constant and hence $V$ is a constant multiple of $\xi$. Therefore we have the following theorem

**Theorem 5.** Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $n$-dimensional $(LCS)_{n}$-manifold which admits a conformal Ricci soliton $(g, V, \lambda)$, $V$ being the potential vector field of the manifold. If the potential vector field $V$ is pointwise collinear with the characteristic vector field $\xi$, i.e; if $V = b\xi$, then $b$ is constant, i.e; $V$ becomes constant multiple of $\xi$.

Next, we study an important curvature property called $\xi$-Ricci semi symmetry. Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $n$-dimensional $(LCS)_{n}$-manifold. Then we say that the manifold $M$ is $\xi$-Ricci semi symmetric if $R(\xi, X) \cdot S = 0$ in $M$, where $\xi$ is the characteristic vector field, $R$ is the Riemannian curvature tensor, $S$ is the Ricci tensor.

Let us start with the known formula that for any vector fields $X, Y, Z$ on $M$,

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z).$$

Now, using (2.7) the above equation (3.24) yields

\[ R(\xi, X) \cdot S = (\alpha^2 - \rho)[g(X, Y)S(\xi, Z) - \eta(Y)S(X, Z) + S(Y, X)g(X, Z) - \eta(Z)S(Y, X)]. \]

Using (2.9) in the above equation and after few steps we get

\[ R(\xi, X) \cdot S = \alpha(\alpha^2 - \rho)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)]. \]

Now note that \((\alpha^2 - \rho) = 0\) implies \(\lambda = \frac{p}{2} + \frac{1}{n}\), which is the trivial case. Thus for non-triviality we assume \((\alpha^2 - \rho) \neq 0\). Again as \(\alpha\) is a non-zero scalar, from (3.25) we can state the following:

**Theorem 6.** If \((M, g, \xi, \eta, \phi, \alpha)\) is an \(n\)-dimensional \((LCS)_n\)-manifold admitting a conformal Ricci soliton, then the manifold becomes \(\xi\)-Ricci semi symmetric, i.e; \(R(\xi, X) \cdot S = 0\) iff the Lorentzian metric \(g\) satisfies the relation

\[ g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z) = 0 \]

for any vector fields \(X, Y, Z\) on \(M\), \(\xi\) being the characteristic vector field, \(R\) is the Riemannian curvature tensor and \(S\) is the Ricci tensor.

### 4 Conformal Ricci soliton on \((LCS)_n\)-manifolds satisfying certain curvature conditions

First let \((M, g)\) be an \(n\)-dimensional \((LCS)_n\)-manifold. Then from equation (3.15) the conharmonic curvature tensor on \(M\) is given by

\[ H(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y]. \]

Interchanging \(Z\) and \(X\) and the putting \(Z = \xi\), we can rewrite the above equation (4.1) as

\[ H(\xi, X)Y = R(\xi, X)Y - \frac{1}{n-2}[S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX]. \]

Using (2.7), (3.4), (3.5) and (3.6) in the above we get

\[ H(\xi, X)Y = [(\alpha^2 - \rho) - (2\sigma + \alpha)/(n-2)]g(X, Y)\xi - \eta(Y)X]. \]

Also from (4.2) we can write

\[ \eta(H(\xi, X)Y) = -[(\alpha^2 - \rho) - (2\sigma + \alpha)/(n-2)]g(X, Y) + \eta(X)\eta(Y)]. \]
Now we assume that \( H(\xi, X) \cdot S = 0 \) holds. Then we have
\[
S(H(\xi, X)Y, Z) + S(Y, H(\xi, X)Z) = 0.
\]
In view of (3.4) the above (4.4) yields
\[
\sigma g[H(\xi, X)Y, Z] + g(Y, H(\xi, X)Z) = \alpha \eta[H(\xi, X)Z] \eta(Y) + \eta[H(\xi, X)Y] \eta(Z).
\]
Using (4.2) and (4.3) in the above equation we get
\[
\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n - 2)}] [g(X, Y) \eta(Z) + g(X, Z) \eta(Y) + 2\eta(X) \eta(Y) \eta(Z)] = 0.
\]
Finally taking \( Z = \xi \) in equation (4.5) and then using (2.5) we arrive at
\[
\alpha[(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n - 2)}] \eta(\phi X, \phi Y) = 0.
\]
Since \( \alpha \) is non-zero and \( \eta(\phi X, \phi Y) \neq 0 \) always; then \( [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{(n - 2)}] = 0 \) i.e; 
\[ \lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n}) \alpha. \] Therefore we can state the following theorem:

**Theorem 7.** If \((M, g, \xi, \eta, \phi, \alpha)\) is an \(n\)-dimensional \((LCS)_n\)-manifold which admits a conformal Ricci soliton, and satisfies the condition \( H(\xi, X) \cdot S = 0 \) i.e; the manifold is \(\xi\)-Ricci conharmonically symmetric. Then the soliton constant is given by \(\lambda = \frac{p}{2} + \frac{1}{n} + (1 - \frac{1}{n}) \alpha\); where \(H\) is the conharmonic curvature tensor and \(S\) is the Ricci tensor of the manifold and \(\xi\) is the characteristic vector field.

Next we study another important curvature tensor called \(\tilde{M}\)-projective curvature tensor. The \(\tilde{M}\)-projective curvature tensor on an \((LCS)_n\)-manifold is defined by \[1\]
\[
\tilde{M}(X, Y)Z = R(X, Y)Z - \frac{1}{2(n - 1)} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].
\]
Taking inner product with respect to the vector field \(\xi\), the above (4.6) yields
\[
\eta(\tilde{M}(X, Y)Z) = \eta(R(X, Y)Z) - \frac{1}{2(n - 1)} [S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]
+ g(Y, Z)\eta(QX) - g(X, Z)\eta(QY)]
\]
Using (2.8), (3.4) and (3.5) in the above equation we get
\[
\eta(\tilde{M}(X, Y)Z) = [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{2(n - 1)}] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\]
Now we assume the condition that \(R(\xi, X) \cdot \tilde{M} = 0\). Then we have
\[
R(\xi, X)\tilde{M}(Y, Z)W - \tilde{M}(R(\xi, X)Y, Z)W
- \tilde{M}(Y, R(\xi, X)Z)W - \tilde{M}(Y, Z)R(\xi, X)W = 0.
\]
Using (2.7) in (4.9) and then taking an inner product with respect to ξ we get

\begin{equation}
(4.11) \quad -g(X, \hat{M}(Y, Z) W) - \eta(X) \eta(\hat{M}(Y, Z) W) - g(X, Y) \eta(\hat{M}(\xi, Z) W) \\
+ \eta(Y) \eta(\hat{M}(X, Z) W) - g(X, Z) \eta(\hat{M}(Y, \xi) W) + \eta(\xi) \eta(\hat{M}(X, Z) W) \\
- g(X, W) \eta(\hat{M}(Y, Z) \xi) + \eta(W) \eta(\hat{M}(Y, Z) X) = 0.
\end{equation}

Then in view of (4.8) the above (4.10) becomes

\begin{equation}
(4.12) \quad [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{2(n-1)}] [g(Y, W) g(X, Z) - g(X, Y) g(Z, W)] + g(X, \hat{M}(Y, Z) W) = 0.
\end{equation}

From (4.6) and (4.11) we get

\begin{equation}
(4.13) \quad [(\alpha^2 - \rho) - \frac{(2\sigma + \alpha)}{2(n-1)}] [g(Y, W) g(X, Z) - g(X, Y) g(Z, W)] + g(X, R(Y, Z) W) \\
- \frac{1}{2(n-1)} [S(Z, W) g(Y, X) - S(Y, W) g(X, Z) + g(Z, W) S(Y, X) - g(Y, W) S(Z, X)] = 0.
\end{equation}

Let us consider an orthonormal basis \{e_i : 1 \leq i \leq n\} of the manifold \((M, g)\). Then putting \(X = Y = e_i\) in the equation (4.12) and summing over \(1 \leq i \leq n\), we get

\begin{equation}
(4.14) \quad 2n S(Z, W) = [2(n-1)^2(\alpha^2 - \rho) - (n-1)(2\sigma + \alpha) - \rho] g(Z, W).
\end{equation}

Again putting \(Z = W = \xi\) in above and using equation (3.6) we get

\begin{equation}
(4.15) \quad 2(n-1)^2(\alpha^2 - \rho) - (5n - 2)[\lambda - (\frac{p}{2} + \frac{1}{n})] + 2n\alpha = 0.
\end{equation}

Now using (3.11) in the above equation (4.14) and after a simple calculation we arrive at

\begin{equation}
(4.16) \quad \lambda = (\frac{p}{2} + \frac{1}{n}) - 2\alpha.
\end{equation}

Thus we have the following theorem

**Theorem 8.** Let \((M, g, \xi, \eta, \phi, \alpha)\) be an \(n\)-dimensional \((LCS)_n\)-manifold admitting a conformal Ricci soliton and the manifold is \(\xi, \hat{M}\)-projectively semi symmetric i.e; it satisfies the condition \(R(\xi, X) \cdot \hat{M} = 0\); \(\xi\) being the characteristic vector field, \(\hat{M}\) is the \(\hat{M}\)-projective curvature tensor of the manifold. Then the soliton is shrinking, steady or expanding according as \(p > (4\alpha - \frac{2}{n})\), \(p = (4\alpha - \frac{2}{n})\) or \(p < (4\alpha - \frac{2}{n})\).

Next we prove an interesting result on \((LCS)_n\)-manifold admitting a conformal Ricci soliton and satisfying the condition \(R(\xi, X) \cdot \hat{P} = 0\), where \(\hat{P}\) denotes the well-known Pseudo-projective curvature tensor. But before that let us recall some well-known results that will be used later in this section:
Theorem 9. \[11\] If \( S : g(x, y, z) = c \) is a surface in \( \mathbb{R}^3 \) then the gradient vector field \( \nabla g \) (connected only at a point of \( S \)) is a non-vanishing normal vector field on the entire surface \( S \).

S.R. Ashoka et.al. in their paper \[1\] have given the higher dimensional version of the above theorem as follows:

Corollary 1. \[1\] If \( S : g(x, y, z) = c \) is a surface (abstract surface or manifold) in \( \mathbb{R}^n \) then the gradient vector field \( \nabla g \) (connected only at points of \( S \)) is a non-vanishing normal vector field on the entire surface (abstract surface or manifold) \( S \).

Then the above mentioned authors in \[1\] also gave the following remark from the above corollary as:

Remark 1. \[1\] Taking a real valued scalar function \( \alpha \) associated with an \((LCS)_{n^*}\) manifold with \( M = \mathbb{R}^3 \) and \( g = \alpha \) in the above corollary we have, \( \nabla \alpha \) as a non-vanishing normal vector field on \( S \subset M \) and directional derivative of \( \alpha \) with respect to \( \xi, \xi\alpha = \xi, \nabla \alpha = |\xi| |\nabla \alpha| \cos(\xi, \nabla \alpha) \)

1) If \( \xi \) is tangent to \( S \) then \( \xi\alpha = 0 \).
2) If \( \xi \) is tangent to \( M \) but not to \( S \) then \( \xi\alpha \neq 0 \).
3) If the angle between \( \xi \) and \( \nabla \alpha \) is acute then \( 0 < \cos(\hat{\xi}, \nabla \alpha) < 1 \), then \( \xi\alpha = \frac{k|\nabla \alpha|}{n} \), \( 0 < k < 1 \) and \( \xi\alpha > 0 \).
4) If the angle between \( \xi \) and \( \nabla \alpha \) is obtuse then \( -1 < \cos(\hat{\xi}, \nabla \alpha) < 0 \), then \( \xi\alpha = \frac{k|\nabla \alpha|}{n} \), \( -1 < k < 0 \) and \( \xi\alpha < 0 \).

Now we see the dependance of the conformal Ricci soliton on \( \xi\alpha \) for \((LCS)_{n^*}\) manifolds satisfying \( R(\xi, X) \cdot \hat{P} = 0 \). The Pseudo projective curvature tensor \( \hat{P} \) is defined by

\[
\hat{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y]
- \frac{r}{n} \left( \frac{a}{n - 1} + b \right) [g(Y, Z)X - g(X, Z)Y],
\]

where \( a, b \neq 0 \) are constants. Taking \( Z = \xi \) in (4.16) we get

\[
\hat{P}(X, Y)\xi = aR(X, Y)\xi + b[S(Y, \xi)X - S(X, \xi)Y]
- \frac{r}{n} \left( \frac{a}{n - 1} + b \right) [\eta(Y)X - \eta(X)Y].
\]

Using (2.6) and (3.6) the above equation (4.17) yields

\[
\hat{P}(X, Y)\xi = [a(\alpha^2 - \rho) + b(\sigma + \alpha) - \frac{r}{n} \left( \frac{a}{n - 1} + b \right)] [\eta(Y)X - \eta(X)Y],
\]

where \( \sigma \) is as described in the previous section. Again from (4.16) we can write

\[
\eta(\hat{P}(X, Y)Z) = a\eta(R(X, Y)Z) + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]
- \frac{r}{n} \left( \frac{a}{n - 1} + b \right) [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\]
Using (2.8) and (3.4) the above equation becomes

\[
\eta(\ddot{P}(X,Y)Z) = [a(\alpha^2 - \rho) + b\sigma - \frac{r}{n} \left( \frac{a}{n} \right) + b)] [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\]

Now we assume the condition that \( R(\xi, X) \cdot \ddot{P} = 0 \). Then we have

\[
R(\xi, X)\ddot{P}(U,V)W - \ddot{P}(R(\xi, X)U,V)W
- \ddot{P}(U,R(\xi,X)V)W - \ddot{P}(U,V)R(\xi,X)W = 0,
\]

for any vector fields \( X, U, V, W \) on \( M \). Using (2.7) in the above equation and then taking an inner product with respect to \( \xi \) we get

\[
- g(X, \ddot{P}(U,V)W) - \eta(X)\eta(\ddot{P}(U,V)W) - g(X,U)\eta(\ddot{P}(\xi,V)W)
+ \eta(U)\eta(\ddot{P}(U,X)W) + \eta(V)\eta(\ddot{P}(U,X)W)
- g(X,W)\eta(\ddot{P}(U,V)\xi) + \eta(W)\eta(\ddot{P}(U,V)X) = 0.
\]

Then using (4.18) and (4.19) the above equation becomes

\[
[a(\alpha^2 - \rho) + b\sigma - \frac{r}{n} \left( \frac{a}{n} \right) + b)] [g(X,V)g(U,W) - g(X,U)g(V,W)]
+ g(X, \ddot{P}(U,V)W) = 0.
\]

Now in view of (4.16) and then using (3.4) in the equation (4.21) we get

\[
ag(X,R(U,V)W) - ba[\eta(V)\eta(W)g(X,U) - \eta(U)\eta(W)g(X,V)]
+ a(\alpha^2 - \rho)[g(X,V)g(U,W) - g(X,U)g(V,W)] = 0.
\]

Let us consider an orthonormal basis \{\( e_i \) : \( 1 \leq i \leq n \)\} of the manifold \((M, g)\). Then putting \( X = U = e_i \) in the equation (4.22) and summing over \( 1 \leq i \leq n \), we get

\[
aS(V,W) - b(n - 1)\alpha\eta(V)\eta(W) - a(n - 1)(\alpha^2 - \rho)g(V,W) = 0.
\]

Again setting \( V = W = \xi \) in (4.23) and after a few steps of simple calculations we get

\[
\lambda = (n - 1)[(\alpha^2 - \rho) - \frac{b}{a}\alpha] + (\frac{p}{2} + \frac{1}{n}).
\]

Therefore in view of the above equation (4.24) and Remark-4.1 we can state the following:

**Theorem 10.** Let \((M, g, \xi, \eta, \phi, \alpha)\) be an \( n \)-dimensional \((LCS)_n\)-manifold which admits a conformal Ricci soliton and the manifold is \( \xi \)-pseudo-projectively semi symmetric i.e; if it satisfies the condition \( R(\xi, X) \cdot \ddot{P} = 0; \) \( \xi \) being the characteristic vector field, \( \ddot{P} \) is the pseudo-projective curvature tensor of the manifold and \( \alpha \) is a positive function; then

1) If $\xi$ is orthogonal to $\nabla \alpha$; the soliton is expanding if $\alpha > \frac{b}{a}$, $p > -\frac{2}{n}$; steady if $\alpha = \frac{b}{a}$, $p = -\frac{2}{n}$ and shrinking if $\alpha < \frac{b}{a}$, $p < -\frac{2}{n}$.

2) If the angle between $\xi$ and $\nabla \alpha$ is acute; the soliton is expanding if $\alpha^2 + k|\nabla \alpha| > \frac{b}{\alpha} \alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 + k|\nabla \alpha| = \frac{b}{\alpha} \alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 + k|\nabla \alpha| < \frac{b}{\alpha} \alpha$, $p < -\frac{2}{n}$.

3) If the angle between $\xi$ and $\nabla \alpha$ is obtuse; the soliton is expanding if $\alpha^2 > k|\nabla \alpha| + \frac{b}{\alpha} \alpha$, $p > -\frac{2}{n}$; steady if $\alpha^2 = k|\nabla \alpha| + \frac{b}{\alpha} \alpha$, $p = -\frac{2}{n}$ and shrinking if $\alpha^2 < k|\nabla \alpha| + \frac{b}{\alpha} \alpha$, $p < -\frac{2}{n}$.

Acknowledgement: The first author D. Ganguly is thankful to the National Board for Higher Mathematics (NBHM), India, for their financial support (Ref No: 0203/11/2017/RD-II/10440) to carry on this research work.

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