Notes On Conformal Ricci Soliton In Lorentzian Para Sasakian Manifolds

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Abstract

The object of the present paper is to study Weyl projective curvature tensor, pseudo projective curvature tensor, W_1-curvature tensor, Ricci curvature tensor in Lorentzian para Sasakian manifold admitting conformal Ricci soliton. We study Lorentzian para Sasakian manifold and prove its existence by an example. We prove that Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).P = 0$, $P(\xi, X).S = 0$, $R(\xi, X).\widehat{P} = 0$, $\widehat{P}(\xi, X).S = 0$, $R(\xi, X).W_1 = 0$ are an Einstein Manifolds.

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1 INTRODUCTION

The Ricci flow concept and its proof was introduced by Hamilton [11] in the year 1982. It was degined to answer the Thurston's geometric conjecture, according to it each closed three dimension manifold admits a geometric decomposition. Categorization of all compact manifolds with positive curvature operator in fourth dimension was done by Hamilton [12]. After which, the Ricci flow became one of the powerful tool in the study of Riemannian manifolds, especially in the manifolds having positive curvatures.

The Ricci flow is presented as

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

for a compact Riemannian manifold M with Riemannian metric g. Ricci soliton has come as the limit of the solutions of Ricci flow. The solution for the Ricci flow is known as a Ricci soliton in case it moves only by a one parameter group of diffeomorphism and scaling. Ramesh Sharma [13] begin study of Ricci soliton for compact manifold and later it was studied by Bejan, Crasmareanu [3] analysed Ricci soliton in contact metric manifolds. The Ricci soliton equation is presented as

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar.

A. E. Fischer [1] in the year 2005 established a new concept known as conformal Ricci flow, a variation of the classical Ricci flow equation which has revised the unit volume constraint of that equation to a scalar curvature constraint. After that a conformal geometry has played a prevalent role to constraint the scalar curvature and equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the equation resulting to this is said as conformal Ricci flow equations. The new equations are presented as

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \tag{1.3}$$

and $R(g) = -1$, where p is a scalar non-dynamical field (time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

N. Basu and A. Bhattacharyya [18] in 2015 brought the notion of conformal Ricci soliton and the equation given as follows

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.4}$$

The above equation is the generalization of the Ricci soliton equation and it also assures the conformal Ricci flow equation.

A Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is Levi-Civita connection on the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi symmetric spaces and their generalization. A Riemannian manifold is said to be semi symmetric if its curvature tensor R satisfies $R(X,Y).R = 0$ for all $X, Y \in TM$, where $R(X,Y)$ acts on R as a derivation.

In this paper, we have studied Weyl projective curvature tensor, pseudo projective curvature tensor, W_1-curvature tensor, Ricci curvature tensor in Lorentzian para Sasakian manifold admitting conformal Ricci soliton. We defined Lorentzian para Sasakian manifold and an example of such a manifold. We have studied Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).P = 0$, is Einstein manifold. We have proved Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $P(\xi, X).S = 0$, is an Einstein manifold. We have found that Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).\hat{P} = 0$, is an Einstein manifold. We have studied Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $\hat{P}(\xi, X).S = 0$, is an Einstein manifold. We have found that Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).W_1 = 0$ is an Einstein manifold.

2 PRELIMINARIES

A $(2n+1)$ dimensional differential manifold M is a Lorentzian para Sasakian manifold (LP-Sasakian manifold) if it admits a $(1,1)$ tensor field ϕ, a covariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \tag{2.2}$$

$$\eta(\xi) = -1, \phi\xi = 0, \eta o\phi = 0, \tag{2.3}$$

$$\eta(\phi X) = 0, \ rank(\phi) = 2n, \tag{2.4}$$

Then M admits a Lorentzian metric g, such that

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$

and M is said to admits a Lorentzian almost paracontact structure (ϕ, ξ, η, g). In this case, we have

$$\triangledown_X \xi = \phi X, \tag{2.7}$$

$$(\triangledown_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$$

where $\triangledown$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Also, since the vector field is closed in an LP-Sasakian manifold, we have

$$(\triangledown_X \eta)Y = (\triangledown_Y \eta)X = g(\phi X, Y), \quad \triangledown_\xi \eta = 0$$

for any vector field X and Y.

In a Lorentzian para Sasakian manifold the following relations holds:

$$\eta(\triangledown_X \xi) = 0, \quad \triangledown_\xi \xi = 0$$

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.8}$$

$$R(\xi, X)Y = (\triangledown_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$

$$R(X,Y)\xi = -\eta(X)Y + \eta(Y)X, \tag{2.10}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.11}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.12}$$

$$S(X,Y) = g(QX,Y), \tag{2.13}$$

for all $X, Y \in \chi(M)$, where R is a Riemannian curvature, S is the Ricci tensor and Q is the Ricci operator.

Now from definition of Lie derivative, we have

$$(\pounds_\xi g)(X,Y) = g(\bigtriangledown_\xi X, Y) + g(X, \bigtriangledown_\xi Y),$$

where $\bigtriangledown$ is the Levi- Civita connection associated to g. Using (2.7) in above equation, we have

$$(\pounds_\xi g)(X,Y) = 2g(\phi X, Y). \tag{2.14}$$

Applying conformal Ricci soliton equation (1.4) in (2.14), we obtain

$$S(X,Y) = Ag(X,Y) - g(\phi X, Y), \tag{2.15}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{n})]$, also

$$QX = AX - \phi X, \tag{2.16}$$

$$S(X,\xi) = A\eta(X), \tag{2.17}$$

$$S(\xi,\xi) = A. \tag{2.18}$$

Using these results, we shall prove some important results of Lorentzian para Sasakian manifold in the following sections.

2.1 Example of 3-dimensional Lorentzian para Sasakian manifold

Consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are the standard cocordinates in R^3. The vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), e_3 = e^z \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0,$$

$$g(e_1,e_1) = g(e_2,e_2) = 1, \quad g(e_3,e_3) = -1.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any $X \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi e_1 = -e_1, \phi e_2 = -e_2, \phi e_3 = 0$.

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = -1, \phi^2(X) = X + \eta(X)\xi \quad \& \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1,e_2] = 0, \quad [e_1,e_3] = -e_1, \quad [e_2,e_3] = -e_2.$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z] + g(Z,[X,Y]). \end{aligned}$$

Which is known as Koszul's formula.

From Koszul's formula, we have

$$\nabla_{e_1} e_3 = -e_1,\ \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_1 = -e_3,$$

$$\nabla_{e_2} e_3 = -e_2,\ \nabla_{e_2} e_2 = -e_3,\ \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_1 = 0.$$

From the above result it can be easily seen that the manifold satisfies $\nabla_X \xi = \phi X$ for $\xi = e_3$. Hence the manifold under consideration is Lorentzian para Sasakian manifold.

3 Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).P = 0$

Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The Weyl projective curvature tensor P on M is defined by [9]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{3.1}$$

Now we prove the following theorem:

Theorem 1. *If a Lorentzian para Sasakian manifold admits conformal Ricci soliton and is Weyl projective semi symmetric i.e. $R(\xi, X).P = 0$, then the manifold is an Einstein manifold, where P is Weyl projective curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Putting $Z = \xi$ in (3.1), we have

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{2n}[S(Y,\xi)X - S(X,\xi)Y], \tag{3.2}$$

using (2.2), (2.10), (2.16) and (2.17) in (3.2), we get

$$P(X,Y)\xi = (1 - \frac{A}{2n})[\eta(Y)X - \eta(X)Y], \tag{3.3}$$

Considering

$$E_1 = 1 - \frac{A}{2n},$$

Therefore, equation (3.3) becomes

$$P(X,Y)\xi = E_1[\eta(Y)X - \eta(X)Y], \tag{3.4}$$

and

$$g(P(X,Y)\xi, Z) = E_1[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{3.5}$$

which implies

$$\eta(P(X,Y)Z) = E_1[\eta(X)g(Y,Z) - \eta(Y)g(X,Z))]. \tag{3.6}$$

Now, we consider that the Lorentzian para Sasakian manifold admits conformal Ricci soliton and is Weyl projective semi symmetric i.e. $R(\xi,X).P = 0$ holds in M, which implies

$$\begin{aligned} &R(\xi,X)(P(Y,Z)W) - P(R(\xi,X)Y,Z)W \\ &-P(Y,R(\xi,X)Z)W - P(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{3.7}$$

for all vector field X,Y,Z on M.

Using (2.9) in (3.7) and putting $W = \xi$, we get

$$\begin{aligned} &g(X,P(Y,Z)\xi)\xi - \eta(P(Y,Z)\xi)X - g(X,Y)P(\xi,Z)\xi \\ &+\eta(Y)P(X,Z)\xi - g(X,Z)P(Y,\xi)\xi + \eta(Z)P(Y,X)\xi \\ &-g(X,\xi)P(Y,Z)\xi + \eta(\xi)P(Y,Z)X = 0, \end{aligned} \tag{3.8}$$

which implies

$$\begin{aligned} &g(X,P(Y,Z)\xi)\xi - g(X,Y)P(\xi,Z)\xi \\ &+\eta(Y)P(X,Z)\xi - g(X,Z)P(Y,\xi)\xi + \eta(Z)P(Y,X)\xi \\ &-g(X,\xi)P(Y,Z)\xi - P(Y,Z)X = 0. \end{aligned} \tag{3.9}$$

Taking inner product with ξ in (3.9) and using (2.3), we get

$$\begin{aligned} &-g(X,P(Y,Z)\xi) - g(X,Y)\eta(P(\xi,Z)\xi) \\ &+\eta(Y)\eta(P(X,Z)\xi) - g(X,Z)\eta(P(Y,\xi)\xi) + \eta(Z)\eta(P(Y,X)\xi) \\ &-g(X,\xi)\eta(P(Y,Z)\xi) - \eta(P(Y,Z)X) = 0, \end{aligned} \tag{3.10}$$

using (3.4) in (3.10), we have

$$-E_1[\eta(Z)g(X,Y) - \eta(Y)g(X,Z)] - \eta(P(Y,Z)X) = 0 \tag{3.11}$$

putting $Z = \xi$ in (3.11) and using (2.3) we get

$$E_1[g(X,Y) + \eta(Y)\eta(X)] - \eta(P(Y,\xi)X) = 0. \tag{3.12}$$

Now from (3.1), we get

$$\eta(P(Y,\xi)X) = g(X,Y) + E_1\eta(Y)\eta(X) - \frac{1}{2n}S(X,Y). \tag{3.13}$$

After putting (3.13) in (3.12), we get

$$E_1[g(X,Y) + \eta(Y)\eta(X)] - g(X,Y) - E_1\eta(Y)\eta(X) - \frac{1}{2n}S(X,Y) = 0.$$

Simplifying above equation, we get

$$S(X,Y) = Ag(X,Y).$$

So, from above we conclude that manifold is an Einstein manifold. □

4 Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $P(\xi, X).S = 0$

Theorem 2. *If a Lorentzian para Sasakian manifold M admits conformal Ricci soliton and the manifold is Weyl projective Ricci symmetric i.e., $P(\xi, X).S = 0$ then the manifold is an Einstein manifold, P is Weyl projective curvature tensor and S is a Ricci tensor.*

Proof. Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). From (3.1), we can write

$$P(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n}[S(X, Y)\xi - S(\xi, Y)X], \tag{4.1}$$

using (2.9), (2.16) and (2.17) in (4.1), we have

$$P(\xi, X)Y = [g(X, Y)\xi - \eta(Y)X] - \frac{1}{2n}[S(X, Y)\xi - A\eta(Y)X], \tag{4.2}$$

and similarly, we have

$$P(\xi, X)Z = [g(X, Z)\xi - \eta(Z)X] - \frac{1}{2n}[S(X, Z)\xi - A\eta(Z)X]. \tag{4.3}$$

Now, we consider that the tensor derivative of S by $P(\xi, X)$ is zero i.e., $P(\xi, X).S = 0$. Then the Lorentzian para sasakian manifold M admitting conformal Ricci soliton is Weyl projective Ricci symmetric. It gives

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0, \tag{4.4}$$

using (4.2) and (4.3) in (4.4), we get

$$\begin{aligned} &S([g(X, Y)\xi - \eta(Y)X] - \frac{1}{2n}[S(X, Y)\xi \\ &-A\eta(Y)X], Z) + S(Y, [g(X, Z)\xi - \eta(Z)X] \\ &-\frac{1}{2n}[S(X, Z)\xi - A\eta(Z)X]) = 0. \end{aligned} \tag{4.5}$$

Putting $Z = \xi$, we get

$$S(X, Y) = -Ag(X, Y),$$

which implies

$$S(X, Y) = Bg(X, Y),$$

where $B = -A$, we conclude that the manifold becomes an Einstein manifold. □

5 Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).\hat{P} = 0$

Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The pseudo projective curvature tensor $\widehat{P}$ on M is defined by [9]

$$\begin{aligned}\hat{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{1}{2n+1}[\frac{a}{2n} + b][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \quad (5.1)$$

Now we prove the following theorem:

Theorem 3. *If a Lorentzian para Sasakian manifold admits conformal Ricci soliton and is pseudo projective semi symmetric i.e.,* $R(\xi, X).\hat{P} = 0$*, then the manifold is an Eistein manifold where* $\hat{P}$ *is pseudo projective curvature tensor and* $R(\xi, X)$ *is derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ).Putting $Z = \xi$ in (5.1), we have

$$\begin{aligned}\hat{P}(X,Y)\xi &= aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] \\ &+ \frac{1}{2n+1}[\frac{a}{2n} + b][g(Y,\xi)X - g(X,\xi)Y], \end{aligned} \quad (5.2)$$

using (2.2), (2.10), (2.17) in (5.2), we get

$$\hat{P}(X,Y)\xi = [a + bA + \frac{1}{2n+1}(\frac{a}{2n} + b)][\eta(Y)X - \eta(X)Y], \quad (5.3)$$

considering

$$\rho = [a + bA + \frac{1}{2n+1}(\frac{a}{2n} + b)],$$

therefore, (5.3) becomes

$$\hat{P}(X,Y)\xi = \rho[\eta(Y)X - \eta(X)Y], \quad (5.4)$$

and

$$g(\hat{P}(X,Y)\xi, Z) = \rho[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \quad (5.5)$$

which implies

$$\eta(\hat{P}(X,Y)Z) = \rho[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]. \quad (5.6)$$

Now, we consider that Lorentzian para Sasakian manifold admits conformal Ricci soliton and is pseudo projective semi symmetric i.e., $R(\xi, X).\widehat{P} = 0$ holds in M, which implies

$$\begin{aligned}&R(\xi,X)(\hat{P}(Y,Z)W) - \hat{P}(R(\xi,X)Y,Z)W \\ &-\hat{P}(Y,R(\xi,X)Z)W - \hat{P}(Y,Z)R(\xi,X)W = 0, \end{aligned} \quad (5.7)$$

for all vector field X, Y, Z on M.

Using (2.9) in (5.7) and putting $W = \xi$, we get

$$\begin{aligned} & g(X, \hat{P}(Y,Z)\xi)\xi - \eta(\hat{P}(Y,Z)\xi)X - \hat{P}(g(X,Y)\xi \\ & -\eta(Y)X, Z)\xi - \hat{P}(Y, g(X,Z)\xi - \eta(Z)X)\xi \\ & -\hat{P}(Y,Z)(g(X,\xi)\xi + \eta(\xi)X) = 0, \end{aligned} \tag{5.8}$$

which implies

$$\begin{aligned} & g(X, \hat{P}(Y,Z)\xi)\xi - g(X,Y)\hat{P}(\xi,Z)\xi \\ & +\eta(Y)\hat{P}(X,Z)\xi - g(X,Z)\hat{P}(Y,\xi)\xi + \eta(Z)\hat{P}(Y,X)\xi \\ & -\eta(X)\hat{P}(Y,Z)\xi - \hat{P}(Y,Z)X = 0, \end{aligned} \tag{5.9}$$

taking inner product with ξ in (5.9) and using (2.3), we get

$$\begin{aligned} & g(X, \hat{P}(Y,Z)\xi)\xi - g(X,Y)\eta(\hat{P}(\xi,Z)\xi) \\ & +\eta(Y)\eta(\hat{P}(X,Z)\xi) - g(X,Z)\eta(\hat{P}(Y,\xi)\xi) + \eta(Z)\eta(\hat{P}(Y,X)\xi) \\ & -\eta(X)\eta(\hat{P}(Y,Z)\xi) - \eta(\hat{P}(Y,Z)X) = 0. \end{aligned} \tag{5.10}$$

On simplifying (5.10), we have

$$g(X, \hat{P}(Y,Z)\xi) + \eta(\hat{P}(Y,Z)X)) = 0. \tag{5.11}$$

putting $Z = \xi$ in (3.11) and using (2.2) and (2.3), we get

$$-\rho g(X,Y) - \rho\eta(X)\eta(Y) + \eta(\widehat{P}(Y,\xi)X) = 0. \tag{5.12}$$

Now from (5.1), we get

$$\begin{aligned} \eta(\hat{P}(Y,\xi)X) \quad = \quad & [a + bA + \frac{1}{2n+1}(\frac{a}{2n} + b)]\eta(Y)\eta(X) \\ & +[a + \frac{1}{2n+1}(\frac{a}{2n} + b)]g(Y,X) + bS(Y,X). \end{aligned} \tag{5.13}$$

After putting (5.13) in (5.12), the equation reduce

$$\begin{aligned} & -\rho g(X,Y) - \rho\eta(X)\eta(Y) + [a + bA + \frac{1}{2n+1}(\frac{a}{2n} + b)]\eta(Y)\eta(X) \\ & +[a + \frac{1}{2n+1}(\frac{a}{2n} + b)]g(Y,X) + bS(Y,X) = 0 \end{aligned}$$

which implies

$$S(X,Y) = Ag(X,Y). \tag{5.14}$$

So from (5.14), we conclude that the manifold becomes an Einstein manifold. □

6 Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $\hat{P}(\xi, X).S = 0$

Theorem 4. *If a Lorentzian para Sasakian manifold M admits conformal Ricci soliton and $\hat{P}(\xi, X).S = 0$ holds i.e., the manifold is Ricci pseudo projectively symmetric, then the manifold is an Einstein manifold, where $\hat{P}$ is pseudo projective curvature tensor and S is Ricci tensor.*

Proof. Let M be a $(2n + 1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Now the equation (5.1) can be written as

$$\begin{aligned} \hat{P}(\xi, X)Y &= [a + \frac{1}{2n+1}(\frac{a}{2n} + b)]g(X, Y)\xi \\ &\quad +[-a - Ab - \frac{1}{2n+1}(\frac{a}{2n} + b)]\eta(Y)X] + bS(X, Y)\xi, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \hat{P}(\xi, X)Z &= [a + \frac{1}{2n+1}(\frac{a}{2n} + b)]g(X, Z)\xi \\ &\quad +[-a - Ab - \frac{1}{2n+1}(\frac{a}{2n} + b)]\eta(Z)X] + bS(X, Z)\xi. \end{aligned} \tag{6.2}$$

Now, we assume that the manifold is Ricci pseudo projectively symmetric i.e., $\hat{P}(\xi, X).S = 0$ holds in M, which gives

$$S(\hat{P}(\xi, X)Y, Z) + S(Y, \hat{P}(\xi, X)Z) = 0, \tag{6.3}$$

using (6.1), (6.2) in (6.3), we have

$$\begin{aligned} &S([a + \frac{1}{2n+1}(\frac{a}{2n} + b)]g(X, Y)\xi \\ &+[-a - Ab - \frac{1}{2n+1}(\frac{a}{2n} + b)]\eta(Y)X] \\ &+bS(X, Y)\xi, Z) + S(Y, [a + \frac{1}{2n+1}(\frac{a}{2n} + b)]g(X, Z)\xi \\ &+[-a - Ab - \frac{1}{2n+1}(\frac{a}{2n} + b)]\eta(Z)X] + bS(X, Z)\xi) = 0, \end{aligned} \tag{6.4}$$

putting $Z = \xi$ in above equation and simplifying, we get

$$S(X, Y) = (\frac{aA + \frac{A}{2n+1}[\frac{a}{2n} + b]}{-a - 2Ab - \frac{1}{2n+1}[\frac{a}{2n} + b]})g(X, Y), \tag{6.5}$$

let $\alpha = (\frac{aA + \frac{A}{2n+1}[\frac{a}{2n}+b]}{-a-2Ab-\frac{1}{2n+1}[\frac{a}{2n}+b]})$ then above equation becomes

$$S(X, Y) = \alpha g(X, Y).$$

Which proves that the manifold is an Einstein manifold. □

7 Lorentzian para Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).W_1 = 0$

Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The W_1- curvature tensor on M is defined by [9]

$$W_1(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{7.1}$$

Now we prove the following theorem:

Theorem 5. *If a Lorentzian para Sasakian manifold admits conformal Ricci soliton and is W_1 semi symmetric i.e., $R(\xi, X).W_1 = 0$, then the manifold is an Einstein manifold, where W_1 is W_1- curvature tensor and $R(\xi, X)$ is derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let M be a $(2n+1)$ dimensional Lorentzian para Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). Putting $Z = \xi$ in (7.1), we have

$$W_1(X,Y)\xi = R(X,Y)\xi + \frac{1}{2n}[S(Y,\xi)X - S(X,\xi)Y], \tag{7.2}$$

using (2.2), (2.10), (2.16) and (2.17) in (7.2), we get

$$W_1(X,Y)\xi = (1 + \frac{A}{2n})[\eta(Y)X - \eta(X)Y], \tag{7.3}$$

considering

$$f_1 = 1 + \frac{A}{2n},$$

therefore, equation (7.3) becomes

$$W_1(X,Y)\xi = f_1[\eta(Y)X - \eta(X)Y], \tag{7.4}$$

and

$$g(W_1(X,Y)\xi, Z) = f_1[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{7.5}$$

which implies

$$\eta(W_1(X,Y)Z) = f_1[\eta(X)g(Y,Z) - \eta(Y)g(X,Z))]. \tag{7.6}$$

Now, we consider that the Lorentzian para Sasakian manifold admits conformal Ricci soliton and is W_1 semi symmetric i.e., $R(\xi, X).W_1 = 0$ holds in M, which implies

$$\begin{aligned} &R(\xi,X)(W_1(Y,Z)W) - W_1(R(\xi,X)Y,Z)W \\ &-W_1(Y,R(\xi,X)Z)W - W_1(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{7.7}$$

for all vector field X, Y, Z on M.

Using (2.9) in (7.7) and putting $W = \xi$, we get

$$\begin{aligned} &g(X, W_1(Y,Z)\xi)\xi - \eta(W_1(Y,Z)\xi)X - g(X,Y)W_1(\xi,Z)\xi \\ &+\eta(Y)W_1(X,Z)\xi - g(X,Z)W_1(Y,\xi)\xi + \eta(Z)W_1(Y,X)\xi \\ &-g(X,\xi)W_1(Y,Z)\xi + \eta(\xi)W_1(Y,Z)X = 0, \end{aligned} \tag{7.8}$$

which implies

$$\begin{aligned} &g(X, W_1(Y,Z)\xi)\xi - g(X,Y)W_1(\xi,Z)\xi \\ &+\eta(Y)W_1(X,Z)\xi - g(X,Z)W_1(Y,\xi)\xi + \eta(Z)W_1(Y,X)\xi \\ &-g(X,\xi)W_1(Y,Z)\xi - W_1(Y,Z)X = 0. \end{aligned} \tag{7.9}$$

Taking inner product with ξ in (7.9) and using (2.3), we get

$$\begin{aligned} &-g(X, W_1(Y,Z)\xi) - g(X,Y)\eta(W_1(\xi,Z)\xi) \\ &+\eta(Y)\eta(W_1(X,Z)\xi) - g(X,Z)\eta(W_1(Y,\xi)\xi) + \eta(Z)\eta(W_1(Y,X)\xi) \\ &-g(X,\xi)\eta(W_1(Y,Z)\xi) - \eta(W_1(Y,Z)X) = 0, \end{aligned} \tag{7.10}$$

on simplyfing equation (7.10), we have

$$g(X, W_1(Y,Z)\xi) + \eta(W_1(Y,Z)X) = 0. \tag{7.11}$$

Putting $Z = \xi$ in (7.11) and using (2.3), we get

$$f_1[-g(X,Y) - \eta(Y)\eta(X)] + \eta(W_1(Y,\xi)X) = 0, \tag{7.12}$$

now from (7.1), we get

$$\eta(W_1(Y,\xi)X) = g(X,Y) + f_1\eta(Y)\eta(X) + \frac{1}{2n}S(X,Y). \tag{7.13}$$

After putting (7.13) in (7.12), we get

$$f_1[-g(X,Y) - \eta(Y)\eta(X)] + g(X,Y) + f_1\eta(Y)\eta(X) + \frac{1}{2n}S(X,Y) = 0,$$

on simplifying above equation, we get

$$S(X,Y) = Ag(X,Y).$$

Which proves that the manifold is an Einstein manifold. □

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