Wedge Functions for Degree-n Approximation over Pentagonal Discretization

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Abstract

Wachspress devised the concept of rational wedge basis functions to obtain linear approximation over the 2D domain discretized with help of convex polygons. In the current paper we explore the concept of interpolants for degree-n approximation over a pentagonal element of the domain discretized using pentagons.

Also, the error in approximation has been studied and it has been found that the interpolants for higher degree approximation provide a better approximation.

Subject Classification: MSC[2010] 41A20, 65M60 , 65D05 , 65D17.

Keywords: Adjoint, Pentagonal discretization, Wedge functions, Mathematica Program and Degree-n Approximation.

1 Introduction

In the method proposed by Wachspress, the domain($D_0 \subseteq \mathbb{R}^2$) is subdivided into certain number of elements(polygons), then corresponding to each element a class of wedge functions is defined, which obeys certain rules. This class of functions provides a linear approximation for any function or data over the considered element. In the same way we can compute class of wedge functions corresponding to each element of the domain and an appropriate linear approximation for the given function or data may be obtained over the entire domain.

The wedge functions $W_i(x, y)$ devised by Wachspress[5], for linear approximation over a domain discretized by polygons P_m of order m were computed in such a way that these wedge functions abide the following properties:

(i) There is a node at each vertex of the polygon. For each node there is an associated wedge within each polygon containing the node.

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(ii) $W_i(x, y)$ associated with node i is normalized to unity at i, $(i \in \mathbb{Z}_m)$.

(iii) $W_i(x, y)$ is linear on sides adjacent to i, $(i \in \mathbb{Z}_m)$.

(iv) $W_i(x, y)$ vanishes on sides opposite to node i.

(v) The class of wedge functions associated with P_m form a basis for degree one approximation over it. For the polygon P_m, there must be at least m nodes. For these to suffice, we must have:

$$\sum_{i=1}^{m} W_i(x, y) = 1 \tag{1.1}$$

$$\sum_{i=1}^{m} x_i W_i(x, y) = x \tag{1.2}$$

$$\sum_{i=1}^{m} y_i W_i(x, y) = y \tag{1.3}$$

(vi) Each wedge function and all its derivatives are continuous within the polygon for which the wedge is a basis function.

Wachspress[6] exploited the geometric properties of an element for the computation of the wedge function. The numerator corresponding to the i^{th} node, is the product of linear forms which vanish on all the sides of the element, other than the sides adjacent to the node i. Thus it is a bi-variate polynomial $P^{m-2}(x, y)$ of degree m-2, where m is the number of sides in the considered element. With the help of certain other properties of algebraic geometry, Wachspress successfully established that the adjoint(denominator function) associated with an element is the unique curve $P^{m-3}(x, y) = 0$, passing through the Exterior Intersection Points(ETPs) of the element, where an ETP is the point where the extended sides of the element meet. Thus, the rational wedge function for linear approximation can be regarded as a ratio $\frac{P^{m-2}}{P^{m-3}}$.

In this paper, we have extended the concept of linear approximation to degree-n approximation. To attain degree-n approximation[4], we have introduced some additional nodes in the element. The nodes have been introduced in such a way that the inter-element continuity of the approximation may be preserved and the approximation remains well defined. Corresponding to each node of the element a wedge function is obtained. Inheriting the method of Wachspress we have considered the wedge functions of the form $\frac{P^{m+n-3}}{P^{m-3}}$, where n is the desired degree of approximation. Thus, the adjoint function of Wachspress, for linear approximation is retained in the extension to degree-n approximation.

The interpolants derived in this paper, provide a better approximation than any other such type of interpolants. Also the bi-variate polynomials of degree-n can be exactly generated with the help of these interpolants. We have illustrated the entire process with the help of an example, by taking n=2.

A program in mathematica has been developed which computes all the wedge functions and also the approximation over the element.

2 Construction and Setup

Consider the domain $D \subseteq \mathbb{R}^2$ be discretized with the convex pentagons. Let P_5 be an element of this discretization, having vertices $i(i = 1, 2, 3, 4, 5)$ and the sides s_i joining vertices $i-1$ and i. To get the desired degree-n approximation over the pentagon we introduce the side nodes $\{i_j\}_{j=1}^{n-1}$ on side s_i(such that $i_j \neq i-1$ or i for any j and $i_j \neq i_k$ for $j \neq k$) and interior nodes $\{c_k\}_{k=1}^{\frac{(n-1)(n-2)}{2}}$ (The interior nodes have been chosen in such a way that, there exists a unique curve of degree (n-3) passing through $\frac{n(n-3)}{2}$ number of interior nodes). Through out this work, the i^{th} node(vertex), i_j^{th} node(side node) and c_k^{th} node(interior node) are considered to have Cartesian coordinates (x_i, y_i), (x_{i_j}, y_{i_j}) and (x_{c_k}, y_{c_k}) respectively.

To achieve degree n-approximation over the convex polygon, we consider the wedge functions corresponding to the node(vertex, side node and interior node) of the form

$$W_i^n = \frac{N_i^n}{D_0} \tag{2.1}$$

where D_0 is the same denominator, which was considered by Wachspress to attain linear approximation and the numerator N_i^n corresponding to the i^{th} node is defined in view of the following properties:

1. There is one node at each vertex of the pentagon, (n-1) side nodes on each side and $\frac{(n-1)(n-2)}{2}$ interior nodes(see Fig.2). For each node there is an associated wedge within each pentagon containing the node.
2. Wedge $W_i^n(x, y)$ associated with node i is normalized to unity at i.
3. Wedge $W_i^n(x, y)$ is of degree n on sides adjacent to i.
4. Wedge $W_i^n(x, y)$ vanishes on all nodes j($\neq i$) lying on the boundary of P_5.
5. The wedges associated with P_5 form a basis for degree n approximation over it. For the pentagon P_5, there must be at least mn boundary and $\frac{(n-1)(n-2)}{2}$ interior nodes. For these to suffice, we must have:

$$\sum_{k=1}^{M} x_k^i y_k^j W_k^n = x^i y^j \qquad 0 \leq i + j \leq n \tag{2.2}$$

where $M = 5n + \frac{(n-1)(n-2)}{2}$

6. Each wedge function and all its derivatives are continuous within the pentagon for which the wedge is a basis function.

3 Computation of the numerators for the wedge functions

Let l_i be the linear form of the edges s_i(i=1,...,5)[6] defined as:

$$l_i = (y_i - y_{i-1})x - (x_i - x_{i-1})y - x_{i-1}(y_i - y_{i-1}) + y_{i-1}(x_i - x_{i-1})(i \in \mathbb{Z}_5) \tag{3.1}$$

and similarly l_{i_j}(j=1,2,...,n-1)be the linear forms joining nodes $(i+1)_{n-j}$ and $(i)_j$. Also, γ_{c_k} $(k = 1, \cdots, \frac{(n-1)(n-2)}{2})$be the unique curve of degree (n-3) passing through the $\frac{n(n-3)}{2}$ interior nodes $\{c_\nu\}_{\nu \neq k}$. Then, the numerator for the wedge functions for degree n approximation over P_5, are defined using properties mentioned in section 2 as:

$$N_i^n = k_i^n \Big(\prod_{j \neq i, i+1}^{5} l_j \Big) \Big(\prod_{j=1}^{n-1} l_{i_j} \Big) \tag{3.2}$$

For i_j, $(j = 1, \cdots, (n-1))$ and $i = 1, 2, ..., m$

$$N_{i_j}^n = k_{i_j}^n \Big(\prod_{\nu \neq i}^{5} l_\nu \Big) \Big(\prod_{\nu \neq j}^{n-1} l_{i_\nu} \Big) \tag{3.3}$$

For c_k, $(k = 1, \cdots, \frac{(n-1)(n-2)}{2})$

$$N_{c_k}^n = k_{c_k}^n \Big(\prod_{\nu=1}^{5} l_\nu \Big) \gamma_{c_k} \tag{3.4}$$

The following figure depicts the above scheme:

4 Computation of the Adjoint

The denominator(Adjoint) for the rational wedge functions is computed in view of the wedge properties in such a way that the wedge functions, defined in equation(2.1) attain degree-n on the sides adjacent to the corresponding node and also it satisfies the other properties mentioned in section 2, so that approximation of any bivariate polynomial up-to degree n can be achieved.

To find the adjoint for degree-n approximation, we inherit the method proposed by Wachspress[6], that the adjoint for the wedge functions for linear approximation is the unique curve passing through the exterior intersection points of the considered element. To assure that the same adjoint works for the wedge functions for degree-n approximation we quote the following lemma:

Lemma 1. *[6] Let the subscripts on P and Q be the degrees of these polynomials. Let P_n, Q_m, and L_1 have s distinct triple points. Then*

$$\frac{P_n(x,y)}{Q_m(x,y)} = \frac{P^1_{n-s}}{Q^1_{m-s}} mod L_1 \tag{4.1}$$

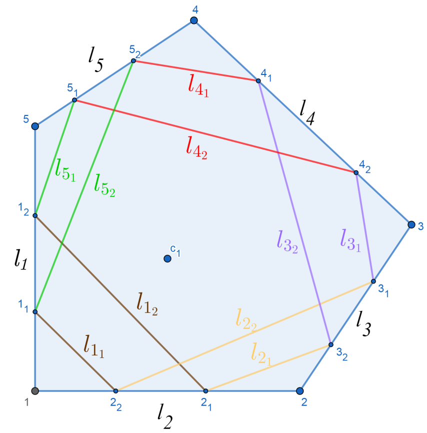

Fig. 1: Degree 3 approximation over polygon of order 5

($\frac{P^1_{n-s}}{Q^1_{m-s}} modL_1$ means $\frac{P^1_{n-s}}{Q^1_{m-s}}$ restricted on L_1) where polynomials P^1 and Q^1 are derived from P_n and Q_m by elimination of x or y on line L_1.

In view of Lemma1, and the numerators for wedge functions defined in equations(3.2), (3.3) and (3.4) the rational wedge function (2.1) is defined with the adjoint D_0, as the unique curve of degree two passing through the exterior intersection points of the considered element(cf. Fig.2). The normalizing constants are $k^n_i = \frac{D_0}{\left(\prod\limits_{j\neq i,i+1}^{5} l_j\right)\left(\prod\limits_{j=1}^{n-1} l_{i_j}\right)}|_i$, $k^n_{i_j} = \frac{D_0}{\left(\prod\limits_{\nu\neq i}^{5} l_\nu\right)\left(\prod\limits_{\nu\neq j}^{n-1} l_{i_\nu}\right)}|_{i_j}$ and $k^n_{c_k} = \frac{D_0}{\left(\prod\limits_{\nu=1}^{5} l_\nu\right)\gamma_{c_k}}|_{c_k}$ respectively.

5 Error in Approximation

Error in approximation by barycentric coordinates have been described in by Guessab[3]. The error[3] is defined as:

$$E(x,y) = \sum_{i=1}^{n} W_i f(x_i,y_i) - f(x,y) \tag{5.1}$$

where f is the given function to be approximated, n is the number of nodes in the considered element, (x_i,y_i) are the coordinates of the considered polygonal element.

The concept of maximum error has been given in[1] has been considered, which is defined as follows:

$$E_\infty = max_{(x,y)\in P_5}|f(x,y) - \sum_{i=1}^{n} W_i f(x_i,y_i)| \tag{5.2}$$

In the example below, $f : P_5 \longrightarrow \mathbb{R}$ has been taken as $f(x,y) = cos(xy)$. At first f has been approximated with the help of the traditional method for linear

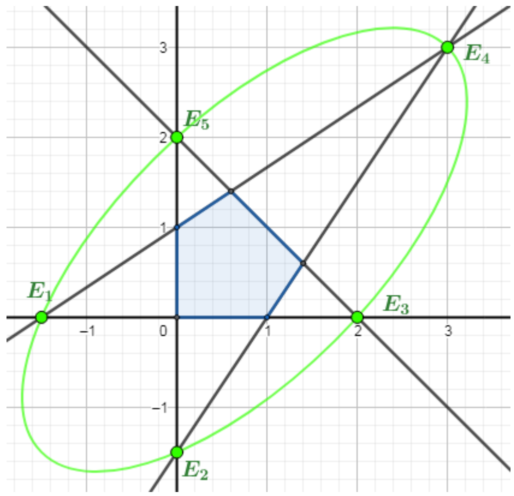

Fig. 2: Representing the unique curve of degree 2 passing through the Exterior Intersection Points

approximation, then by the method proposed in this paper, for degree two approximation. Error(5.1) and maximum error(5.2) in approximation by interpolants for degree one and degree two has been computed and it has been found that the error in approximation[2] decreases with the increase in degree of approximation.

Example 1 Let the domain $D \subseteq \mathbb{R}^2$ be discretized by pentagons(m= 5), $P_5 = (1, 2, 3, 4, 5) \in D$ such that the Cartesian coordinates of the vertices are(cf. Fig. 7):

1=(0,0), 2=(1,0), 3=(7/5,3/5), 4=(3/5,7/5), 5=(0,1) and the other nodes(side nodes and interior nodes) are $1_1 = (0, \frac{1}{3})$, $1_2 = (0, \frac{2}{3})$, $2_1 = (\frac{2}{3}, 0)$, $2_2 = (\frac{1}{3}, 0)$, $3_1 = (\frac{19}{15}, \frac{2}{5})$, $3_2 = (\frac{17}{15}, \frac{1}{5})$, $4_1 = (\frac{13}{15}, \frac{17}{15})$, $4_2 = (\frac{17}{15}, \frac{13}{15})$, $5_1 = (\frac{1}{5}, \frac{17}{15})$, $5_2 = (\frac{2}{5}, \frac{19}{15})$, and $c_1 = (\frac{1}{2}, \frac{1}{2})$(see Fig.2)

Let B^n denote the interpolant for degree-n approximation. Then for $f(x, y) = cos(x.y)$(see Figure4) and the considered element P_5.

We have:

$$B^1 = \frac{N^1}{D^1}$$

where, $N^1 = (-18-3y+6y^2+x^2(6+5y(-1+Cos(\frac{21}{25})))+x(-3+y(22-30Cos(\frac{21}{25}))+5y^2(-1+Cos(\frac{21}{25}))))$

$D^1 = -18+6x^2-3y+6y^2-x(3+8y))$ The error in approximation(see (5.1)) has been depicted in Figure6 and the maximum error in approximation E_{max} is:

$$E^1_{max} = 0.078487$$

$$B^2 = \frac{N^2}{D^2}$$

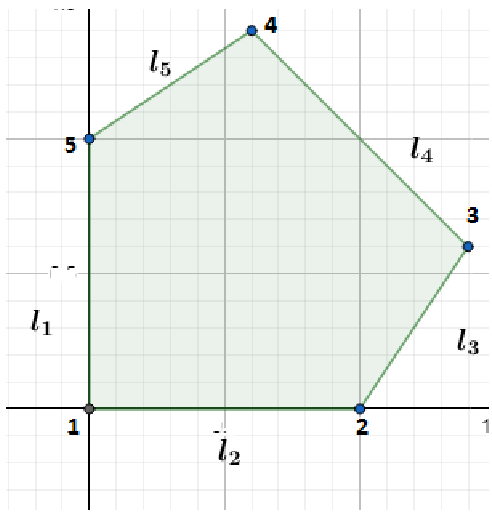

Fig. 3: Polygon of order 5

where, $N^2 = ((-18 - 3y + 6y^2 - 10x^3y(-1 + 2Cos(\frac{9}{25}) + 2Cos(\frac{21}{25}) - 3Cos(1)) + x(-3 + y(142 - 240Cos(\frac{9}{25}) + 135Cos(\frac{21}{25}) - 45Cos(1)) - 10y^3(-1 + 2Cos(\frac{9}{25}) + 2Cos(\frac{21}{25}) - 3Cos(1)) + 5y^2(-17 + 32Cos(\frac{9}{25}) - 18Cos(\frac{21}{25}) + 3Cos(1))) + x^2(6 + 5y(-17 + 32Cos(\frac{9}{25}) - 18Cos(\frac{21}{25}) + 3Cos(1)) - 5y^2(-4 + 8Cos(\frac{9}{25}) - 17Cos(\frac{21}{25}) + 13Cos(1))))$

$D^2 = (-18 + 6x^2 - 3y + 6y^2 - x(3 + 8y)))$ The error in approximation5.1, has been displayed in the figure8 The maximum error in approximation E_{max} is:

$$E^2_{max} = 0.0171157$$

6 Conclusion

The method proposed in this paper, paves a way for higher degree approximation over the domain discretized with the help of pentagons. Apart from this, the linear approximation proposed by Wachspress is also preserved in this method and the approximation of other functions is also better than the other existing interpolants for linear approximations.

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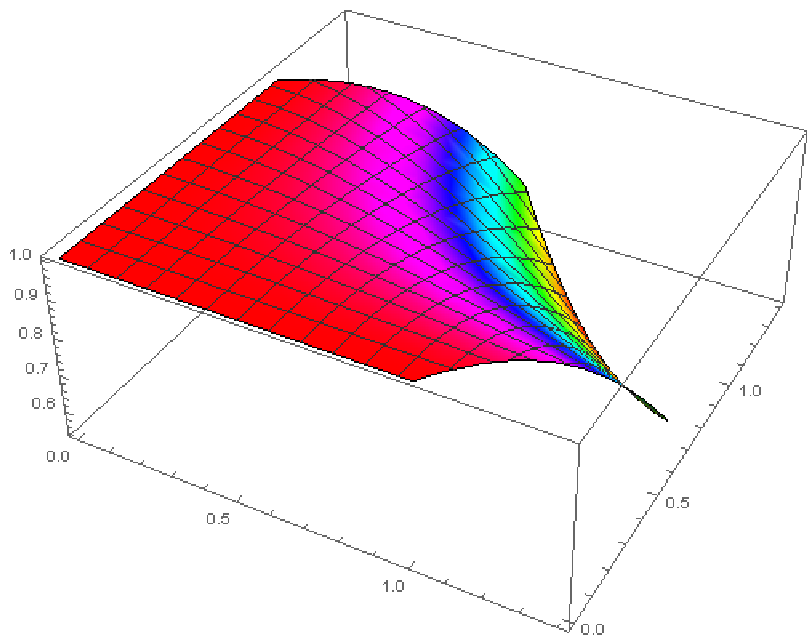

Fig. 4: f(x,y)=cos(xy)

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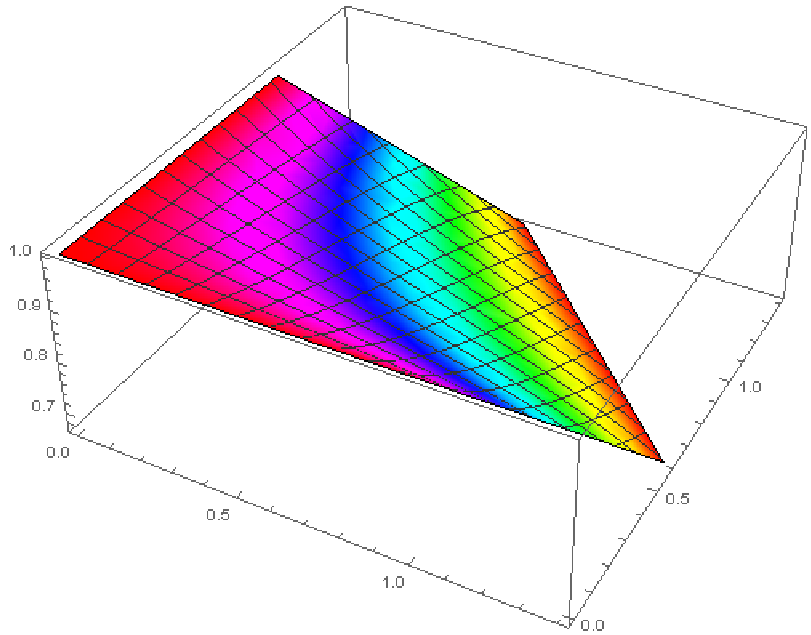

Fig. 5: B^1

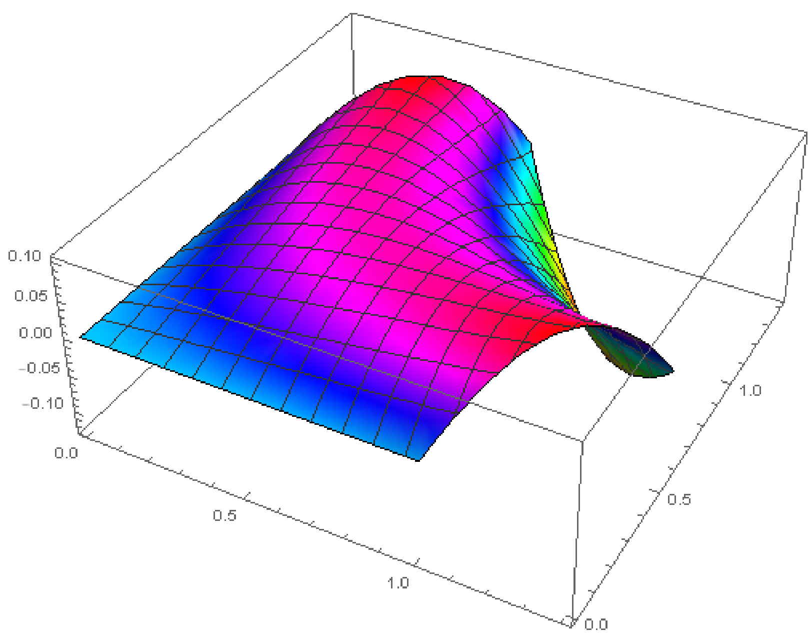

Fig. 6: Error in approximation by interpolants for degree 1 approximation

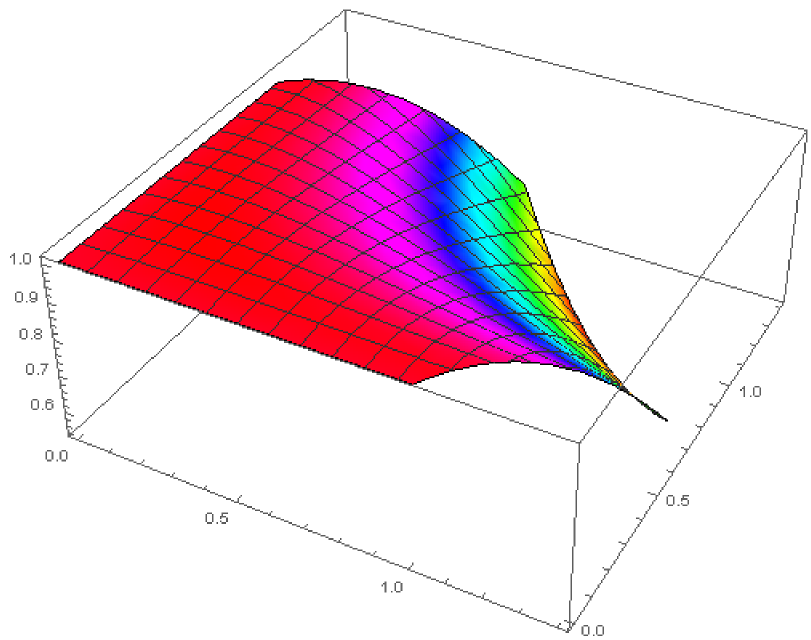

Fig. 7: B^2

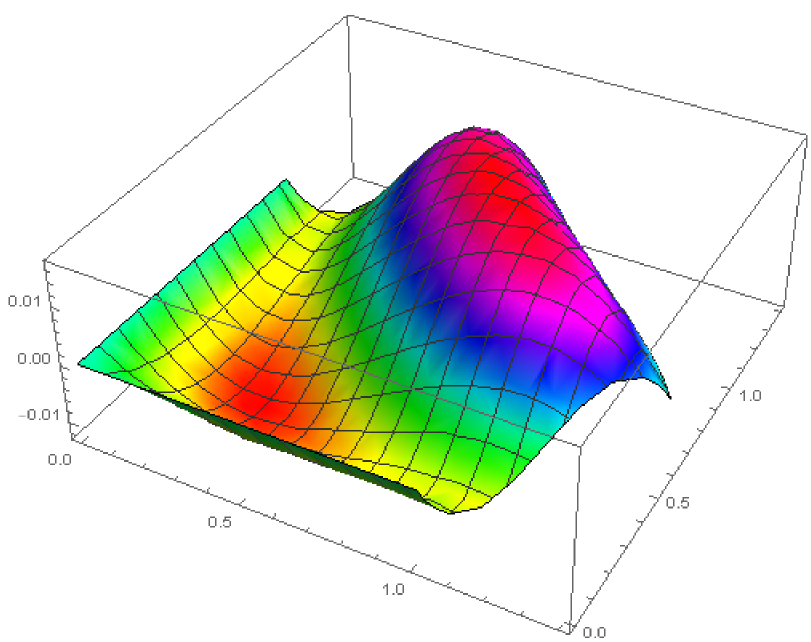

Fig. 8: Error in approximation by interpolants for degree 2 approximation