On an extended recurrent Riemannian manifold

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Abstract

The object of the present paper is to introduce a type of recurrent Riemannian manifold called extended recurrent Riemannian manifold . The existence of extended recurrent Riemannian manifold is also shown by one non trivial example.

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1 Introduction:

Recurrent spaces have been of great importance and were studied by a large number of authors such as Ruse [1], Patterson [2] , Walker [3],Singh and Khan([4] and [5]) etc. In 1991, De and Guha introduced and studied generalized recurrent manifold whose curvature tensor R(X,Y)Z of type (1,3) satisfies the condition:

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where A and B are two 1-forms,B is non-zero and D denoted the operator of covariant differentiation with respect to metric tensor g. Such a space has been denoted by GK_n. In recent papers Bandyopadhyay [7], Prakasha and Yildiz [8], Khan [9] etc explored various geometrical properties by using generalized recurrent manifold on Sasakian manifold and Lorentzian α-Sasakian manifold.

Further one of the author Prasad [10] considered a non-flat Riemannian manifold (M^n, g) (n>2) whose curvature tensor R satisfies the following condition

$$(D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)g(Y,Z)X, \tag{1.2}$$

where A and B are two non-zero 1-forms and D has the meaning already mentioned. Such a manifold called by the author as semi-generalized recurrent manifold and denoted by $(SGK)_n$. Singh, Singh and Kumar[11],[12] and Chaudhary, Kumar and Singh [13] extended this notation to Lorentzian α-Sasakian manifold, P-Sasakian manifold and trans-Sasakian manifold.

The object of the present paper is to study a type of non-flat recurrent Riemannian manifold (M^n, g)(n>2) whose curvature tensor R(X,Y)Z of the type (1,3) satisfies the condition

$$(1.3) \qquad (D_U R)(X,Y)Z = A(U)R(X,Y)Z + B(U)g(Y,Z)X + C(U)g(X,Z)Y,$$

where A, B and C are three non-zero 1-forms and ρ_1, ρ_2 and ρ_3 are three vector fields such that

$$(1.4) \qquad g(U,\rho_1) = A(U),\ g(U,\rho_2) = B(U) \ and\ g(U,\rho_3) = C(U).$$

Such a manifold shall be called as an extended recurrent Riemannian manifold and 1-forms A, B and C shall be called its associated 1-forms and n-dimensional recurrent manifold of this kind shall be denoted by $(ER)_n$.
If in particular we replace C by -B then (1.3) becomes (1.1).
If in particular C=0, then (1.3) becomes (1.2).
If in particular B=0 and C=0, then the space reduced to a recurrent space according to Ruse [14] and Walker [3].

Moreover, in particular if A=B=C=0 then (1.3) becomes $(D_U R)$(X,Y)Z=0. That is , a Riemannian manifold is symmetric accordingly Kobayashi and Nomizu [15] and Desai and Amer [16]. This justifies the name "extended recurrent Riemannian manifold" for the manifold defined by (1.3) and the use of the symbol $(ER)_n$ for it.

In this paper the necessary and sufficient condition for constant scalar curvature of $(ER)_n$ is obtained. Extended recurrent manifold with cyclic Ricci tensor and Codazzi type Ricci tensor are studied. It is also show that if $(ER)_n$ admits a concurrent vector field V then V is not orthogonal to ρ_2 and ρ_3. Finally a non-trivial example of $(ER)_n$ has been constructed.

2 Preliminaries:

Let S and r denote the Ricci tensor of type (0,2) and scalar curvature respectively and Q denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, i.e.

$$(2.1) \qquad S(X,Y) = g(QX,Y),$$

for any vector field X and Y.
For(1.3), we get

$$(2.2) \qquad (D_U S)(Y,Z) = A(U)S(Y,Z) + [nB(U) + C(U)]g(Y,Z).$$

Contracting (2.2), we have

$$(2.3) \qquad dr(U) = Ur = rA(U) + n[nB(U) + C(U)].$$

3 Nature of the 1-forms A,B and C on an extended recurrent space:

From (2.3) suppose $r = 0$,then

$$(3.1) \qquad \begin{aligned} &nB(U) + C(U) = 0 \\ \Rightarrow &B(U) = -\frac{1}{n}C(U), \end{aligned}$$

Hence we have the following theorem

Theorem 1. *If the scalar curvature tensor of extended recurrent is zero, then the 1-form B and C are proportional.*

Now we consider $(\mathrm{ER})_n$ is of constant scalar curvature then from (2.3), we have

$$nB(U) + C(U) = -\frac{r}{n}A(U). \tag{3.2}$$

Again if (3.2) holds, then from (2.3), we get

$$dr(U) = 0,$$
$$r = constant$$

Hence, we can state the following theorem:

Theorem 2. *An $(ER)_n$ is of constant curvature if and only if (3.2) holds.*

Now taking covariant derivative of (3.2) with respect to V, we get

$$(D_V A)(U)r + n[n(D_V B)U + (D_V C)(U)] = 0. \tag{3.3}$$

Interchanging U and V in (3.3) and then subtracting, we get

$$\begin{aligned}&[(D_V A)(U)r - (D_U A)(V)]r + n[n\{(D_V B)(U) - (D_U B)(V)\}\\&+ \{D_V C)(U) - (D_U C)(V)\}] = 0.\end{aligned} \tag{3.4}$$

Thus we have the following theorem:

Theorem 3. *In an extended recurrent space of non-zero constant scalar curvature r, the 1-forms A is closed if and only if 1-forms B and C are closed.*

Next we consider the case when the scalar curvature r is not constant. From (2.3) it follows that

$$VUr = (D_V A)(U)r + A(U)(Vr) + n[n(D_V B)(U) + (D_V C)(U)] \tag{3.5}$$

Interchanging U and V in (3.5) and then subtracting, we get

$$\begin{aligned}&[(D_V A)(U) - (D_U A)(V)]r + n[n\{(D_V B)(U) - (D_U B)(V)\} +\\&\{D_V C)(U) - (D_U C)(V)\}] + n[n\{A(U)B(V) - A(V)B(U)\} +\\&\{A(U)C(V) - A(V)C(U)\}] = 0.\end{aligned} \tag{3.6}$$

Thus we have the following theorem:

Theorem 4. *In an extended recurrent space of non-zero constant scalar curvature r, the 1-forms A, B and C cannot be closed unless the 1-forms A is collinear with the 1-forms B and C.*

Permuting equation (1.3) twice with respect to U, X, Y adding the three equation and using Bianchi's second identity, we have

$$\begin{aligned}(3.7)\quad &[A(U)R(X,Y)Z + A(X)R(Y,U)Z + A(Y)R(U,X)Z] + g(Y,Z)[B(U)X \\ &+ C(X)U] + g(X,Z)[C(U)Y + B(Y)U] + g(U,Z)[C(Y)X + B(X)Y] = 0\end{aligned}$$

Contracting (3.7) with respect to X, we get

$$\begin{aligned}(3.8)\quad &A(U)S(Y,Z) - A(Y)S(U,Z) -' R(Y,U,\rho_1,Z) + n[B(U)X \\ &+ 2C(U)]g(Y,Z) + [nC(Y) + 2B(Y)]g(U,Z) = 0.\end{aligned}$$

Using (2.1) in (3.8), we have

$$\begin{aligned}&A(U)g(QY,Z) - A(Y)g(QU,Z) - g(R(Y,U)\rho_1,Z) + n[B(U)X \\ &+ 2C(U)]g(Y,Z) + [nC(Y) + 2B(Y)]g(U,Z) = 0.\end{aligned}$$

Factoring off Z, we get

(3.9)
$$A(U)QY - A(Y)QU - R(Y,U)\rho_1 + n[B(U) + 2C(U)]Y + [nC(Y) + 2B(Y)]U = 0.$$

Contracting (3.9) with respect to Y, we have

$$(3.10)\qquad S(U,\rho_1) = \frac{1}{2}rg(U,\rho_1) + \left(\frac{n^2+2}{2}\right)B(U) + \frac{3n}{2}C(U).$$

Hence we have the following theorem:

Theorem 5. *If $(ER)_n$ satisfies second Bianchi identity then $\frac{r}{2}$ is an eigen value of Ricci S and ρ_1 is an eigen vector corresponding to the eigen value if only if $(n^2+2)B(U) + 3nC(U) = 0$.*

4 $(\text{ER})_n$ with cyclic Ricci tensor:

In this section we consider an $(\text{ER})_n$ in which the Ricci tensor is a cyclic tensor, i.e.

$$(4.1)\qquad (D_X S)(Y,Z) + (D_Y S)(Z,X) + (D_Z S)(X,Y) = 0,$$

which implies

$$(4.2)\qquad dr(X) = 0.$$

From (1.3), we have

$$(4.3)\qquad dr(X) = A(X)r + n[nB(X) + C(X)].$$

Therefore from (4.2) and (4.3), we get

$$(4.4)\qquad A(X)r + n[nB(X) + C(X)] = 0.$$

From (4.1), we have

$$A(X)S(Y,Z) + A(Y)S(Z,X) + A(Z)S(X,Y) + n[B(X)g(Y,Z) + B(Y)g(X,Z) + B(Z)g(X,Y)] + [C(X)g(Y,Z) + C(Y)g(X,Z) + C(Z)g(X,Y)] = 0,$$

which yields on contraction

$$A(X)r + 2A(QX) + n(n+2)B(X) + (n+2)C(X) = 0. \tag{4.5}$$

Now in view of (4.4) and (4.5), we get

$$\begin{aligned} A(QX) &= \frac{r}{n}A(X) \\ or\ S(U,\rho_1) &= \frac{r}{n}g(X,\rho_1). \end{aligned} \tag{4.6}$$

Hence we have the following theorem:

Theorem 6. *If $(ER)_n$ has cyclic Ricci tensor, then $\frac{r}{n}$ is an eigen value of Ricci tensor S and ρ_1 is an eigen vector corresponding to the eigen value.*

5 (ER)$_n$ with Codazzi type of Ricci tensor:

In this section we consider an (ER)$_n$ in which the Ricci tensor is a Codazzi type of Ricci tensor Ferus [17]

$$(D_X S)(Y,Z) = (D_Z S)(Y,X). \tag{5.1}$$

By view of Bianchi identity and (5.1), we have

$$(divR)(X,Y)Z = 0. \tag{5.2}$$

In view of (1.3), we get on contraction

$$(divR)(X,Y)Z = A(R(X,Y)Z) + B(X)g(Y,Z) + C(Y)g(X,Z). \tag{5.3}$$

Now using (5.2) in (5.3), we get

$$A(R(X,Y)Z) + B(X)g(Y,Z) + C(Y)g(X,Z) = 0. \tag{5.4}$$

In view of (5.4), we get

$$A(QX) = -nB(X) - C(X). \tag{5.5}$$

From (2.2) and (5.1), we have

$$\begin{aligned} &A(X)S(Y,Z) - A(Z)S(Y,X) + [nB(X)+ \\ &C(X)]g(Y,Z) - [nB(Z) + C(Z)]g(X,Y) = 0. \end{aligned} \tag{5.6}$$

On contracting of (5.6), we have

$$A(X)r = A(QX) - (n-1)[nB(X) + C(X)]. \tag{5.7}$$

Using (5.5) and (5.7) in (2.3), we have

$$dr(X) = 0. \tag{5.8}$$

Again it is known [18] that in a Riemannian manifold (M^n, g) (n>3)

$$\begin{aligned}(divC)(X,Y)Z =& \frac{n-3}{n-2}[(D_X S)(Y,Z) - (D_Z S)(Y,X)]+ \\ & \frac{1}{2(n-1)}[g(X,Y)dr(Z) - g(Y,Z)dr(X)],\end{aligned} \tag{5.9}$$

where C denotes the conformal curvature.
As a consequences of (5.1) and (5.8), (5.9) reduces to

$$(divC)(X,Y)Z = 0,$$

which shows that the tensor is conservative [19].
Hence we can state the following theorem:

Theorem 7. *If in an $(ER)_n$ the Ricci tensor is a Codazzi tensor then its conformal curvature tensor is conservative.*

6 Extended recurrent with concurrent vector field:

In this section first we suppose that the $(\mathrm{ER})_n$ admits a concurrent unit vector field V,

$$D_X V = \rho X, \tag{6.1}$$

where ρ is a non-zero constant .
By Ricci-identity

$$R(X,Y)V = 0. \tag{6.2}$$

Taking covariant derivative of (6.2), we get

$$(D_W R)(X,Y)V = -\rho R(X,Y)W \tag{6.3}$$

Also by definition of $(\mathrm{ER})_n$,we find

$$(D_W R)(X,Y)V = A(W)R(X,Y)V + B(W)g(Y,V)X + C(W)g(X,V)Y. \tag{6.4}$$

In view of (6.2),(6.3)and (6.4),we get

$$-\rho R(X,Y)W = B(W)g(Y,V)X + C(W)g(X,V)Y. \tag{6.5}$$

On contraction,we find

$$-\rho S(Y,W)=nB(W)g(Y,V)+C(W)g(Y,V). \tag{6.6}$$

Again on contraction,we get

$$-\rho r=ng(\rho_2,V)+g(\rho_3,V), \tag{6.7}$$

Since $\rho\neq 0$ and $r\neq 0$,then from (6.7),we get

$$ng(\rho_2,V)+g(\rho_3,V)\neq 0. \tag{6.8}$$

Hence we have the following theorem:

Theorem 8. *If an $(ER)_n$ admits a concurrent unit vector field V,then V is not orthogonal to ρ_2 and ρ_3, where ρ_2 and ρ_3 are the associated vector field to the 1-forms B and C.*

7 Extended conformally recurrent Riemannian manifolds:

The conformal curvature tensor C on a manifold of dimension n is defined by

$$\begin{aligned} C(X,Y)Z=&R(X,Y)Z-\frac{1}{n-2}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-\\ &g(X,Z)QY]+\frac{r}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y]. \end{aligned} \tag{7.1}$$

Taking covariant derivative of (7.1) with respect to 'U',we get

(7.2)

$$\begin{aligned} (D_UC)(X,Y)Z=&(D_UR)(X,Y)Z-\frac{1}{n-2}[(D_US)(Y,Z)X+(D_US)(X,Z)Y+\\ &g(Y,Z)(D_UQ)X-g(X,Z)(D_UQ)Y]+\\ &\frac{D_Ur}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y]. \end{aligned}$$

Using (1.3),(2.2) and (2.3),in (7.2),we get

(7.3)

$$\begin{aligned} (D_UC)(X,Y)Z=&A(U)R(X,Y)Z-\frac{1}{n-2}A(U)[S(Y,Z)X+S(X,Z)Y+\\ &g(Y,Z)QX-g(X,Z)QY]+\frac{r}{(n-1)(n-2)}A(U)[g(Y,Z)X-\\ &g(X,Z)Y]-\frac{1}{n-1}[B(U)+C(U)][g(Y,Z)X-ng(X,Z)Y] \end{aligned}$$

From (7.1) and (7.3),we get

(7.4)

$$(D_UC)(X,Y)Z=A(U)C(X,Y)Z-\frac{1}{n-1}[B(U)+C(U)][g(Y,Z)X-ng(X,Z)Y]$$

Theorem 9. *An extended conformally recurrent Riemannian manifold is conformally recurred if and only if B(U)+C(U)=0 i.e. manifold reduces to generalized reccurent.*

8 Example:

Let us consider the 3-dimensional manifold M=$\{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are standard co-ordinate of R^3.

We choose the vector fields

$$e_1 = e^{-2z}\frac{\partial}{\partial y},\ e_2 = \frac{\partial}{\partial z},\ e_3 = e^{-2z}\frac{\partial}{\partial x} \tag{8.1}$$

which is linearly independently at each point of M.

Let g be the Riemannian metric denoted by

$$g(e_i, e_j) = \left\{ \begin{matrix} 1, & i = j \\ 0, & i \neq j \end{matrix} \right\} \tag{8.2}$$

Then from equation (8.1), we have

$$[e_1, e_2] = 2e_1, [e_1, e_3] = 0, [e_2, e_3] = -2e_3. \tag{8.3}$$

By Koszul's formula

$$\begin{aligned} 2g(D_X Y, Z) =& Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) \\ &- g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned} \tag{8.4}$$

Using (8.2) and (8.3) in (8.4), we get

$$\begin{array}{lll} D_{e_1}e_1 = -2e_2, & D_{e_1}e_2 = 2e_1, & D_{e_1}e_3 = 0, \\ D_{e_2}e_1 = 0, & D_{e_2}e_2 = 0, & D_{e_2}e_3 = 0, \\ D_{e_3}e_1 = 0, & D_{e_3}e_2 = 2e_3, & D_{e_3}e_3 = -2e_2. \end{array} \tag{8.5}$$

The curvature tensor is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \tag{8.6}$$

Using (8.3) and (8.5) in (8.6) , we get

$$\begin{array}{lll} R(e_1,e_2)e_1 = 4e_2, & R(e_1,e_2)e_2 = -4e_1, & R(e_1,e_2)e_3 = 0 \\ R(e_2,e_3)e_1 = 0, & R(e_2,e_3)e_2 = 4e_3, & R(e_2,e_3)e_3 = -4e_2 \\ R(e_1,e_3)e_1 = 0, & R(e_1,e_3)e_2 = 0 & R(e_1,e_3)e_3 = -4e_1 \\ \multicolumn{3}{l}{R(e_1,e_1)e_1 = R(e_1,e_1)e_2 = R(e_1,e_1)e_3 = 0} \\ \multicolumn{3}{l}{R(e_2,e_2)e_1 = R(e_2,e_2)e_2 = R(e_2,e_2)e_3 = 0} \\ \multicolumn{3}{l}{R(e_3,e_3)e_1 = R(e_3,e_3)e_2 = R(e_3,e_3)e_3 = 0.} \end{array} \tag{8.7}$$

Let X,Y and Z be any three vector fields given by

$$X = a_1e_1 + b_1e_2 + c_1e_3,\quad Y = a_2e_1 + b_2e_2 + c_2e_3,\quad Z = a_3e_1 + b_3e_2 + c_3e_3, \tag{8.8}$$

where a_i, b_i and c_i are the set of all positive real numbers, i = 1, 2, 3.

(8.9)
$$\begin{aligned} R(X,Y)Z &= pe_1 + qe_2 + re_3 \\ g(Y,Z)X &= le_1 + me_2 + ne_3 \\ g(X,Z)Y &= ue_1 + ve_2 + we_3 \end{aligned}$$

where

$$\begin{aligned} &p = 4\left[a_2\left(b_1b_3 + c_1c_3\right) - a_1\left(b_2b_3 + c_2c_3\right)\right], \\ &q = 4\left[b_2\left(a_1a_3 + c_1c_3\right) - b_1\left(a_2a_3 + c_2c_3\right)\right], \\ &r = p = 4b_3\left(b_1c_2 - b_2c_1\right)], \\ &l = a_1\left(a_2a_3 + b_2b_3 + c_2c_3\right), \\ &m = b_1\left(a_2a_3 + b_2b_3 + c_2c_3\right), \\ &n = c_1\left(a_2a_3 + b_2b_3 + c_2c_3\right), \\ &u = a_2\left(a_1a_3 + b_1b_3 + c_1c_3\right), \\ &v = b_2\left(a_1a_3 + b_1b_3 + c_1c_3\right), \\ &w == c_2\left(a_1a_3 + b_1b_3 + c_1c_3\right). \end{aligned}$$

By view of the equation (8.5), (8.7), (7) and (8.9),we have

(8.10)
$$\left(D_{e_i}R\right)(X,Y)Z = D_{e_i}R(X,Y)Z - R(D_{e_i}X,Y)Z - R(X,D_{e_i}Y)Z - R(X,Y)D_{e_i}Z$$

where

$$\begin{aligned} &u_1 = 8\left[2\left(b_2c_1c_3 - a_2a_3b_1\right) - b_1c_2c_3 + a_1b_3c_2 + b_1b_2b_3\right], \\ &v_1 = 8\left[-2a_2b_1b_3 - a_2c_1c_3 + a_1c_2c_3 - a_1a_3c_2 - a_3b_1b_2\right], \\ &w_1 = 8\left[a_2b_3c_1 - a_3b_2c_1\right], u_2 = v_2 = w_2 = 0, \\ &u_3 = 0, v_3 = 8\left[-b_1b_3c_2 + b_2b_3c_1 - a_2a_3c_1 + a_1a_3c_2\right], \\ &w_3 = 8\left[a_1a_3b_2 - a_2a_3b_1\right]. \end{aligned}$$

Consequently, the manifold under consideration is not recurrent. Let us now consider 1-form non vanishes

(8.11)
$$\begin{aligned} &A\left(e_i\right) = \frac{u_i(nv - mw) + v_i(nu - lw) + w_i(lv - mu)}{u(nq - mr) + v(lr - pn) + w(pm - lq)} \\ &B\left(e_i\right) = \frac{u_i(wq - vr) + v_i(ur - pw) + w_i(pv - qu)}{u(nq - mr) + v(lr - pn) + w(pm - lq)} \\ &C\left(e_i\right) = \frac{u_i(nq - mr) + v_i(lr - np) + w_i(mp - lq)}{u(nq - mr) + v(lr - pn) + w(pm - lq)}. \\ &Thus,\ we\ have \\ &A\left(e_i\right) = B\left(e_i\right) = C\left(e_i\right) = 0\ for\ i = 2. \end{aligned}$$

s.t

$$u(nq - mr) + v(lr - pn) + w(pm - lq) \neq 0.$$

From (1.3), we have

$$(D_{e_i} R)(X, Y)Z = A(e_i)R(X, Y)Z + B(e_i)g(Y, Z)X + C(e_i)g(X, Z)Y. \tag{8.12}$$

By virtue of (8.9), (8.10) and (8.11), it can be easily seen that the Riemannian manifold satisfies relation (8.12). Hence the manifold under consideration is an extended recurrent Riemannian manifold (M^3, g), which is neither recurrent nor generalized recurrent. This leads to the following theorem:

Theorem 10. *There exist an extended recurrent Riemannian manifold (M^3, g), which is neither recurrent nor generalized recurrent.*

References

[1] H.S. Ruse: On simply harmonic space, J. London Math..soc., 21,(1946).

[2] E.M. Paatterson: Some theorem on Ricci-recurrent space, J.london. math.soc., 27, (1952) 287-295.

[3] A.G.Walker: O ruse spaces of recurrent curvature,Proc.London Math. soci. 52,(1951)

[4] H. Singh and Q. Khan : On symmetric Riemannian manifolds, Novisad J. Math. Vol. 29, no.3, 1999, 301-308.

[5] H. Singh and Q. Khan :On generalized recurrent Riemannian manifolds, publ. Math. Debrecen 56 1-2,2000, 87-95.

[6] U.C. De, N. Guha and D. Kamilya: On generalized recurrent manifolds. National Academy of math. India, 9,1,1991, 1-4.

[7] Bandyaopadhyay, Mahuya: On generalized ϕ-recurrent -Sasakian manifolds, Mathematica Pannonica , vol.22(1),2011 ,19-23.

[8] D.G. Prakasha and A. Yildiz: Generalized ϕ-recurrent Lorentzian α-Sasakian manifold, Common Fac.Sci.Uni.Ank.Series A1, Vol(59)(1),2010, 53-62.

[9] Q. Khan: On generalized recurrent Sasakian manifold with special curvature tensor J(X,Y,Z), Ganita,vol 67 (2),2017, 189-194.

[10] B.Prasad:On semi-generalized recurrent manifold, Mathematica Balkanica new series Vol.14,2000, 1-4.

[11] J.P. singh, A. Singh and Rajesh Kumar:On a type of Semi generalized recurrent P-Sasakian manifold,Facta Universitatis (NIS) Ser. Math. Inform, Vol 31(1),2016,213-225.

[12] J.P. singh, A. Singh and Rajesh Kumar:On a type of Semi generalized recurrent Lorentzian α-Sasakian manifold,Bull.Cal.Math. Soc. 107(5),2015,357-370.

[13] J.Chaudhary, Rajesh Kumar, and J.P.Singh: Semi generalized ϕ-recurrent trans-Sasakian manifold,Facta Universitatis (NIS) Ser. Math. Inform, Vol 31(4),2016,863-871.

[14] H.S.Ruse: Three dimensional spaces of recurrent curvature, Pro.London Math.Soc., 50(2), 1947, 438-446.

[15] S.Kobayashi and K. Nomizu: Foundation of differential geometry, vol.1, Interscience publisher, New York, 1963.

[16] P. Desai and K.Amur:On symmetric spaces, Tensor N.S. 29, (1975), 119-124.

[17] D.A. Ferus : A remark on Codazzi tensor on constant curvature space, lecture notes math. 838, Global differential geometry and global analysis, Springer Verlag, New York, 1981.

[18] L.P. Eisenhert: Riemannian Geometry, Princeton University Press, Princetone, N.J., 1949

[19] N.J. Hicks: Notes on differential Geometry, D.Van Nostrand company Inc. Princeton, New York, 1965.