# Some results on generalized relative order $(\alpha, \beta)$ and generalized relative type $(\alpha, \beta)$ of meromorphic function with respect to an entire function

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**Abstract**

In this paper we introduce the idea of generalized relative order $(\alpha, \beta)$ and generalized relative type $(\alpha, \beta)$ of meromorphic function with respect to an entire function where $\alpha$ and $\beta$ are continuous non-negative on $(-\infty, +\infty)$ functions and then study some growth properties of entire and meromorphic functions on the basis of their generalized relative order $(\alpha, \beta)$ and generalized relative type $(\alpha, \beta)$.

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## 1.Introduction, Definitions and Notations.

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6,8,11]. We also use the standard notations and definitions of the theory of entire functions which are available in [10] and therefore we do not explain those in details. Let $f$ be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When $f$ is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of $f$ (see [6,p.4]), playing the same role as $M_f(r)$.

However, the Nevanlinna's characteristic function of a meromorphic function $f$ is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)(\overline{N}_f(r,a))$ known as counting function of $a$-points (distinct $a$-points) of meromorphic $f$ is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a) \log r$$

$$\left(\overline{N}_f(r,a)=\int_0^r \frac{\overline{n}_f(t,a)-\overline{n}_f(0,a)}{t}dt+\overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)(\overline{n}_f(r,a))$ the number of $a$-points (distinct $a$-points) of $f$ in $|z|\leq r$ and an $\infty$ -point is a pole of $f$. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of $f$ is defined as:

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta, \quad \text{where}$$

$$\log^+ x=\max(\log x,0) \text{ for all } x\geqslant 0.$$

Also we may employ $m(r,\frac{1}{f-a})$ by $m_f(r,a)$.

If $f$ is entire, then the Nevanlinna's characteristic function $T_f(r)$ of $f$ is defined as

$$T_f(r)=m_f(r).$$

Moreover, if $f$ is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of $r$. Therefore its inverse $T_f^{-1}:(T_f(0),\infty)\to(0,\infty)$ exists and is such that $\lim\limits_{s\to\infty}T_f^{-1}(s)=\infty$.

Now let $L$ be a class of continuous non-negative on $(-\infty,+\infty)$ function $\alpha$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$. For any $\alpha\in L$, we say that $\alpha\in L_1$, if $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$ and $\alpha\in L_2$, if $\alpha(\exp(cx))=(1+o(1))\alpha(\exp(x))$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$. Clearly, $L_2\subset L_1$.

Considering the above, Sheremeta [9] introduced the concept of generalized order $(\alpha,\beta)$ of an entire function. For details about generalized order $(\alpha,\beta)$ one may see [9]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order $(\alpha,\beta)$ in some different direction. For the purpose of further applications, here in this paper we rewrite the definition of the generalized order $(\alpha,\beta)$ of entire and meromorphic function in the following way after giving a minor modification to the original definition (e.g. see, [9]) which considerably extend the definition of $\varphi$-order of entire and meromorphic function introduced by Chyzhykov et al. [5].

**Definition 1.1.** (Generalized order $(\alpha,\beta)$ and generalized lower order $(\alpha,\beta)$).
Let $\alpha\in L_2$ and $\beta\in L_1$. The generalized order $(\alpha,\beta)$ and generalized lower order $(\alpha,\beta)$ of a meromorphic function $f$ denoted by $\rho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively are defined as:

$$\begin{matrix}\rho_{(\alpha,\beta)}[f]\\ \lambda_{(\alpha,\beta)}[f]\end{matrix}=\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

If $f$ is an entire function, then

$$\begin{matrix}\rho_{(\alpha,\beta)}[f]\\ \lambda_{(\alpha,\beta)}[f]\end{matrix}=\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\alpha(M_f(r))}{\beta(r)},$$

where $\alpha$, $\beta \in L_1$.

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [6]$\}$, for an entire function $f$, one may easily verify that

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha \in L_2$ and $\beta \in L_1$.

The function $f$ is said to be of regular generalized $(\alpha, \beta)$ growth when generalized order $(\alpha, \beta)$ and generalized lower order $(\alpha, \beta)$ of $f$ are the same. Functions which are not of regular generalized $(\alpha, \beta)$ growth are said to be of irregular generalized $(\alpha, \beta)$ growth.

Now in order to refine the growth scale namely the generalized order $(\alpha, \beta)$ of a meromorphic function, we introduce the definitions of another growth indicators, called generalized type $(\alpha, \beta)$ and generalized lower type $(\alpha, \beta)$ respectively of a meromorphic function which are as follows:

**Definition 1.2.** (Generalized type $(\alpha, \beta)$ and generalized lower type $(\alpha, \beta)$).
Let $\alpha \in L_2$ and $\beta \in L_1$. The generalized type $(\alpha, \beta)$ denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type $(\alpha, \beta)$ denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of a meromorphic function $f$ having finite positive generalized order $(\alpha, \beta)$ $(a < \rho_{(\alpha,\beta)}[f] < \infty)$ are defined as:

$$\begin{matrix} \sigma_{(\alpha,\beta)}[f] \\ \overline{\sigma}_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta)}[f] \leq \sigma_{(\alpha,\beta)}[f] \leq \infty$.

Analogously, to determine the relative growth of two meromorphic functions having same non zero finite generalized lower order $(\alpha, \beta)$, one can introduced the definition of generalized weak type $(\alpha, \beta)$ and generalized upper weak type $(\alpha, \beta)$ of a meromorphic function $f$ of finite positive generalized lower order $(\alpha, \beta)$, $\lambda_{(\alpha,\beta)}[f]$ in the following way:

**Definition 1.3.** (Generalized upper weak type $(\alpha, \beta)$ and generalized weak type $(\alpha, \beta)$).
Let $\alpha \in L_2$ and $\beta \in L_1$. The generalized upper weak type $(\alpha, \beta)$ denoted by $\tau_{(\alpha,\beta)}[f]$ and generalized weak type $(\alpha, \beta)$ denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ of a meromorphic function $f$ having finite positive generalized order $(\alpha, \beta)$ $(a < \lambda_{(\alpha,\beta)}[f] < \infty)$ are defined as:

$$\begin{matrix} \tau_{(\alpha,\beta)}[f] \\ \overline{\tau}_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \tau_{(\alpha,\beta)}[f] \leq \infty$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1,2,7]) will come. Now in order to make some progress in the study of relative order,one may introduce the definitions of generalized relative order $(\alpha, \beta)$ and generalized relative lower order $(\alpha, \beta)$ of a meromorphic function with respect to another entire function in the following way:

**Definition 1.4.** (Generalized relative order $(\alpha, \beta)$ and generalized relative lower order $(\alpha, \beta)$).

Let $\alpha$, $\beta \in L_1$. The generalized relative order $(\alpha, \beta)$ and generalized relative lower order $(\alpha, \beta)$ of a meromorphic function $f$ with respect to an entire function $g$ denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\begin{matrix} \rho_{(\alpha,\beta)}[f]_g \\ \lambda_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g = z$, then Definition 1.4 reduces to Definition 1.1. If $\alpha(r) = \beta(r) = \log r$, then we get the definition of relative order of meromrophic function $f$ with respect to an entire function $g$ introduced by Lahiri et al. [7] and if $g = \exp z$ and $\alpha(r) = \beta(r) = \log r$, then $\rho_{(\alpha,\beta)}[f]_g = \rho(f)$. And if $\alpha(r) = \log^{[p]} r$, $\beta(r) = \log^{[q]} r$ and $g = z$, then Definition 1.4 becomes the classical one given in [4].

Further if generalized relative order $(\alpha, \beta)$ and the generalized relative lower order $(\alpha, \beta)$ of a meromorphic function $f$ with respect to an entire function $g$ are the same, then $f$ is called a function of regular generalized relative $(\alpha, \beta)$ growth with respect to $g$. Otherwise, $f$ is said to be irregular generalized relative $(\alpha, \beta)$ growth.with respect to $g$.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as generalized relative type $(\alpha, \beta)$ and generalized relative lower type $(\alpha, \beta)$ of meromorphic function with respect to an entire function which are as follows:

**Definition 1.5.** (generalized relative type $(\alpha, \beta)$ and generalized relative lower type $(\alpha, \beta)$).

Let $\alpha$, $\beta \in L_1$. The generalized relative type $(\alpha, \beta)$ denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type $(\alpha, \beta)$ denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of a meromorphic function $f$ with respect to an entire function $g$ having non-zero finite generalized relative order $(\alpha, \beta)$ are defined as:

$$\begin{matrix} \sigma_{(\alpha,\beta)}[f]_g \\ \overline{\sigma}_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of a meromorphic function $f$ having same non zero finite generalized relative lower order $(\alpha, \beta)$ with respect to an entire function $g$, one can introduce the definition of generalized relative upper weak type $(\alpha, \beta)$ denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type $(\alpha, \beta)$ denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of $f$ with respect to $g$ of finite positive generalized relative lower order $(\alpha, \beta)$ in the following way:

**Definition 1.6.** (Generalized relative upper weak type $(\alpha, \beta)$ and generalized relative weak type $(\alpha, \beta)$).

Let $\alpha, \beta \in L_1$. The generalized relative upper weak type $(\alpha, \beta)$ denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type $(\alpha, \beta)$ denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of a meromorphic function $f$ with respect to an entire function $g$ having non-zero finite generalized relative lower order $(\alpha,\beta)$ are defined as:

$$\begin{matrix} \tau_{(\alpha,\beta)}[f]_g \\ \overline{\tau}_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

However the main aim of this paper is to investigate some growth properties of entire and meromorphic functions using generalized relative order $(\alpha,\beta)$ and generalized relative type $(\alpha,\beta)$, of a meromorphic function with respect to an entire function.which improve and extend some earlier result (see, e.g., [3,4]). Henceforth we assume that $\alpha$, $\beta\in L_1$, $\gamma\in L_2$ and all the growth indicators are non-zero finite.

## 2. Mail Results

In this section we present the main results of the paper.

**Theorem 2.1.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0<\lambda_{(\gamma,\beta)}[f]\le\rho_{(\gamma,\beta)}[f]<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]\le\rho_{(\gamma,\alpha)}[g]<\infty$ . Then

$$\frac{\lambda_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}\le\lambda_{(\alpha,\beta)}[f]_g\le\min\left\{\frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]},\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}\right\}$$
$$\le\max\left\{\frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]},\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}\right\}\le\rho_{(\alpha,\beta)}[f]_g\le\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}.$$

**Proof.** From the definitions of $\rho_{(\gamma,\beta)}[f]$ and $\lambda_{(\gamma,\beta)}[f]$ we have for all sufficiently large values of $r$ that

$$T_f(r)\le\log(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]+\varepsilon)\beta(r))), \tag{2.1}$$

$$T_f(r)\ge\log(\gamma^{-1}((\lambda_{(\gamma,\beta)}[f]-\varepsilon)\beta(r))) \tag{2.2}$$

and also for a sequence of values of $r$ tending to infinity we get that

$$T_f(r)\ge\log(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]-\varepsilon)\beta(r))), \tag{2.3}$$

$$T_f(r)\le\log(\gamma^{-1}((\lambda_{(\gamma,\beta)}[f]+\varepsilon)\beta(r))). \tag{2.4}$$

Further from the definitions of $\rho_{(\gamma,\alpha)}[g]$ and $\lambda_{(\gamma,\alpha)}[g]$ it follows for all sufficiently large values of $r$ that

$$T_g(r)\le\log(\gamma^{-1}((\rho_{(\gamma,\alpha)}[g]+\varepsilon)\alpha(r)))$$

$$i.e.,\ T_g^{-1}(r)\ge\alpha^{-1}\Big(\frac{\gamma(\exp r)}{\rho_{(\gamma,\alpha)}[g]+\varepsilon}\Big)\text{ and} \tag{2.5}$$

$$T_g^{-1}(r) \le \alpha^{-1}\Big(\frac{\gamma(\exp r)}{\lambda_{(\gamma,\alpha)}[g] - \varepsilon}\Big). \tag{2.6}$$

Also from the definitions of $\rho_{(\gamma,\alpha)}[g]$ and $\lambda_{(\gamma,\alpha)}[g]$, we get for a sequence of values of $r$ tending to infinity we obtain that

$$T_g^{-1}(r) \le \alpha^{-1}\Big(\frac{\gamma(\exp r)}{(\rho_{(\gamma,\alpha)}[g] - \varepsilon)}\Big) \text{ and} \tag{2.7}$$

$$T_g^{-1}(r) \ge \alpha^{-1}\Big(\frac{\gamma(\exp r)}{\lambda_{(\gamma,\alpha)}[g] + \varepsilon}\Big). \tag{2.8}$$

Now from (2.3) and in view of (2.5), for a sequence of values of $r$ tending to infinity we get that

$$\begin{aligned} \alpha(T_g^{-1}(T_f(r))) &\ge \alpha(T_g^{-1}(\log(\gamma^{-1}((\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r))))) \\ i.e., \ \alpha(T_g^{-1}(T_f(r))) &\ge \alpha\Big(\alpha^{-1}\Big(\frac{\gamma(\exp\log(\gamma^{-1}((\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r))))}{(\rho_{(\gamma,\alpha)}[g] + \varepsilon)}\Big)\Big) \end{aligned}$$

$$i.e., \ \alpha(T_g^{-1}(T_f(r))) \ge \frac{(\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r)}{(\rho_{(\gamma,\alpha)}[g] + \varepsilon)}$$

$$i.e., \ \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \ge \frac{\rho_{(\gamma,\beta)}[f] - \varepsilon}{\rho_{(\gamma,\alpha)}[g] + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}[f]_g \ge \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}. \tag{2.9}$$

Analogously from (2.2) and in view of (2.8) it follows that

$$\rho_{(\alpha,\beta)}[f]_g \ge \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}. \tag{2.10}$$

Again from (2.2) and in view of (2.5) we obtain that

$$\lambda_{(\alpha,\beta)}[f]_g \ge \frac{\lambda_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}. \tag{2.11}$$

Now in view of (2.6) we have from (2.1) for all sufficiently large values of $r$ that

$$\begin{aligned} \alpha(T_g^{-1}(T_f(r))) &\le \alpha(T_g^{-1}(\log(\gamma^{-1}((\rho_{(\gamma,\beta)}[f] + \varepsilon)\beta(r))))) \\ i.e., \ \alpha(T_g^{-1}(T_f(r))) &\le \alpha\Big(\alpha^{-1}\Big(\frac{\gamma(\exp\log(\gamma^{-1}((\rho_{(\gamma,\beta)}[f] + \varepsilon)\beta(r))))}{(\lambda_{(\gamma,\alpha)}[g] - \varepsilon)}\Big)\Big) \end{aligned}$$

$$i.e.,\ \alpha(T_g^{-1}(T_f(r))) \leq \frac{(\rho_{(\gamma,\beta)}[f]+\varepsilon)\beta(r)}{(\lambda_{(\gamma,\alpha)}[g]-\varepsilon)}$$
$$i.e.,\ \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \leq \frac{\rho_{(\gamma,\beta)}[f]+\varepsilon}{\lambda_{(\gamma,\alpha)}[g]-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\rho_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}. \tag{2.12}$$

Similarly in view of (2.7), we get from (2.1) that

$$\lambda_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}. \tag{2.13}$$

Again from (2.4) and in view of (2.6) it follows that

$$\lambda_{(\alpha,\beta)}[f]_g \leq \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}. \tag{2.14}$$

The theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14).
**Remark 2.1.** From the conclusion of the above result, one may write $\rho_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}$ and $\lambda_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}$ when $\lambda_{(\gamma,\alpha)}[g] = \rho_{(\gamma,\alpha)}[g]$. Similarly $\rho_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}$ and $\lambda_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}$ when $\lambda_{(\gamma,\beta)}[f] = \rho_{(\gamma,\beta)}[f]$.
**Theorem 2.2.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then

$$\max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\right\} \leq \sigma_{(\alpha,\beta)}[f]_g \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

**Proof.** Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_{(\gamma,\beta)}[f]$ and $\overline{\sigma}_{(\gamma,\beta)}[f]$, we have for all sufficiently large values of $r$ that

$$T_f(r) \leq \log(\gamma^{-1}(\log((\sigma_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}))), \tag{2.15}$$

$$T_f(r) \geq \log(\gamma^{-1}(\log((\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}))) \tag{2.16}$$

and also for a sequence of values of $r$ tending to infinity, we get that

$$T_f(r) \geq \log(\gamma^{-1}(\log((\sigma_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}))), \tag{2.17}$$

$$T_f(r) \leq \log(\gamma^{-1}(\log((\overline{\sigma}_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}))). \tag{2.18}$$

Similarly from the definitions of $\sigma_{(\gamma,\alpha)}[g]$ and $\overline{\sigma}_{(\gamma,\alpha)}[g]$, it follows for all sufficiently large values of $r$ that

$$T_g(r) \leq \log(\gamma^{-1}(\log((\sigma_{(\gamma,\alpha)}[g]+\varepsilon)(\exp(\alpha(r)))^{\rho_{(\gamma,\alpha)}[g]})))$$

$$i.e.,\ T_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(\exp r))}{(\sigma_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.19}$$

$$T_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(\exp r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big). \tag{2.20}$$

Also for a sequence of values of $r$ tending to infinity, we obtain that

$$T_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(\exp r))}{(\sigma_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.21}$$

$$T_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(\exp r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big). \tag{2.22}$$

Further from the definitions of $\tau_{(\gamma,\beta)}[f]$ and $\overline{\tau}_{(\gamma,\beta)}[f]$, we have for all sufficiently large values of $r$ that

$$T_f(r) \leq \log(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]}))), \tag{2.23}$$

$$T_f(r) \geq \log(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]}))) \tag{2.24}$$

and also for a sequence of values of $r$ tending to infinity, we get that

$$T_f(r) \geq \log(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]}))), \tag{2.25}$$

$$T_f(r) \leq \log(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]}))). \tag{2.26}$$

Similarly from the definitions of $\tau_{(\gamma,\alpha)}[g]$ and $\overline{\tau}_{(\gamma,\alpha)}[g]$, it follows for all sufficiently large values of $r$ that

$$T_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(\exp r))}{(\tau_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.27}$$

$$T_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(\exp r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big) \tag{2.28}$$

Also for a sequence of values of $r$ tending to infinity, we obtain that

$$T_g^{-1}(r) \leq \alpha^{-1}\Big( \log \Big( \frac{\exp(\gamma(\exp r))}{(\tau_{(\gamma,\alpha)}[g] - \varepsilon)} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \Big) \text{ and} \tag{2.29}$$

$$T_g^{-1}(r) \geq \alpha^{-1}\Big( \log \Big( \frac{\exp(\gamma(\exp r))}{(\overline{\tau}_{(\gamma,\alpha)}[g] + \varepsilon)} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \Big). \tag{2.30}$$

Now from (2.17) and in view of (2.27), we get for a sequence of values of $r$ tending to infinity that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \geq \Big( \frac{(\sigma_{(\gamma,\beta)}[f] - \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\tau_{(\gamma,\alpha)}[g] + \varepsilon)} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}}} \geq \Big( \frac{\sigma_{(\gamma,\beta)}[f] - \varepsilon}{\tau_{(\gamma,\alpha)}[g] + \varepsilon} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1 $\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]} \geq \rho_{(\alpha,\beta)}[f]_g$, and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \geq \Big( \frac{\sigma_{(\gamma,\beta)}[f] - \varepsilon}{\tau_{(\gamma,\alpha)}[g] + \varepsilon} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \sigma_{(\alpha,\beta)}[f]_g \geq \Big( \frac{\sigma_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}. \tag{2.31}$$

Analogously from (2.16) and (2.30), we get that

$$\sigma_{(\alpha,\beta)}[f]_g \geq \Big( \frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]} \Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \tag{2.32}$$

as in view of Theorem 2.1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]} \geq \rho_{(\alpha,\beta)}[f]_g$. Again in view of (2.20), we have from (2.15) for all sufficiently large values of $r$ that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \leq \Big( \frac{(\sigma_{(\gamma,\beta)}[f] + \varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\overline{\sigma}_{(\gamma,\alpha)}[g] - \varepsilon)} \Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \Big( \frac{\sigma_{(\gamma,\beta)}[f] + \varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g] - \varepsilon} \Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g]-\varepsilon}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$\text{(2.33)} \qquad i.e.,\ \sigma_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Thus the theorem follows from (2.31), (2.32) and (2.33).

The conclusion of the following theorem can be carried out from (2.20) and (2.23); (2.23) and (2.28) respectively after applying the same technique of Theorem 2.2 and with the help of Theorem 2.1, therefore its proof is omitted.

**Theorem 2.3.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0<\lambda_{(\gamma,\beta)}[f]<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]\leq\rho_{(\gamma,\alpha)}[g]<\infty$. Then

$$\sigma_{(\alpha,\beta)}[f]_g \leq \min\Big\{\Big(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\}.$$

Similarly in the line of Theorem 2.2 and with the help of Theorem 2.1, one may easily carry out the following theorem from pairwise inequalities numbers (2.24) and (2.27); (2.21) and (2.23); (2.20) and (2.26) respectively and therefore its proofs is omitted.

**Theorem 2.4.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0<\lambda_{(\gamma,\beta)}[f]\leq\rho_{(\gamma,\beta)}[f]<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]\leq\rho_{(\gamma,\alpha)}[g]<\infty$. Then

$$\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \leq \overline{\tau}_{(\alpha,\beta)}[f]_g \leq \min\Big\{\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\}.$$

**Theorem 2.5.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0<\rho_{(\gamma,\beta)}[f]<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]\leq\rho_{(\gamma,\alpha)}[g]<\infty$. Then

$$\overline{\tau}_{(\alpha,\beta)}[f]_g \geq \max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big\}.$$

With the help of Theorem 2.1, he conclusion of the above theorem can be carried out from (2.16), (2.19) and (2.16), (2.27) respectively after applying the same technique of Theorem 2.2 and therefore its proof is omitted.

**Theorem 2.6.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0<\rho_{(\gamma,\beta)}[f]<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]\leq\rho_{(\gamma,\alpha)}[g]<\infty$. Then

$$\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \leq \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \min\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\}.$$

**Proof.** From (2.16) and in view of (2.27), we get for all sufficiently large values of $r$ that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \geq \Big(\frac{(\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\tau_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e.,\ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}}} \geq \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}[g]+\varepsilon}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1 $\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]} \geq \rho_{(\alpha,\beta)}[f]_g$, and $\varepsilon(>0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}}} \geq \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}[g]+\varepsilon}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e.,\ \overline{\sigma}_{(\alpha,\beta)}[f]_g \geq \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}. \tag{2.34}$$

Further in view of (2.21), we get from (2.15) for a sequence of values of $r$ tending to infinity that

$$\exp(\alpha(T_g^{-1}(T_f(r)))) \leq \Big(\frac{(\sigma_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\sigma_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e.,\ \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\sigma_{(\gamma,\alpha)}[g]-\varepsilon}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Again as in view of Theorem 2.1, $\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon(>0)$ is arbitrary, therefore we get from above that

$$\liminf_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\sigma_{(\gamma,\alpha)}[g]-\varepsilon}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e.,\ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}. \tag{2.35}$$

Similarly from (2.18) and (2.20), we get that

$$i.e.,\ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \tag{2.36}$$

as in view of Theorem 2.1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \rho_{(\alpha,\beta)}[f]_g$. Thus the theorem follows from (2.34), (2.35) and (2.36).

**Theorem 2.7.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then

$$\overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \min\Bigg\{\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Bigg\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (2.20) and (2.26); (2.21) and (2.23); (2.26) and (2.28); (2.23) and (2.29) respectively after applying the same technique of Theorem 2.6 and with the help of Theorem 2.1. Therefore its proof is omitted.

Similarly in the line of Theorem 2.1 and with the help of Theorem 2.1, one may easily carry out the following theorem from pairwise inequalities numbered (2.25) and (2.27); (2.24) and (2.30); (2.20) and (2.23) respectively and therefore its proof is omitted:

**Theorem 2.8.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then

$$\max\Bigg\{\Big(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Bigg\} \leq \tau_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

**Theorem 2.9.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \lambda_{(\gamma,\beta)}[f] \leq \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then

$$\tau_{(\alpha,\beta)}[f]_g \geq \max\Bigg\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Bigg\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (2.17) and (2.19); (2.16) and (2.22); (2.17) and (2.27); (2.16) and (2.30) respectively after applying the same technique of Theorem 2.6 and with the help of Theorem 2.1. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because those can be derived easily using the same technique or with some easy reasoning with the help of Remark 1 and therefore left to the readers.

**Theorem 2.10.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g](= \lambda_{(\gamma,\alpha)}[g]) < \infty$. Then

$$\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}} \leq \overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leq \min\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\}$$
$$\leq \max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\}$$
$$\leq \sigma_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

**Remark 2.2.** In Theorem 2.10, if we will replace the conditions "$0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g](= \lambda_{(\gamma,\alpha)}[g]) < \infty$" by "$0 < \rho_{(\gamma,\beta)}[f](= \lambda_{(\gamma,\beta)}[f]) < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g] < \infty$" respectively, then Theorem 2.10 remains valid with $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and $\tau_{(\alpha,\beta)}[f]_g$ replaced by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.

**Theorem 2.11.** Let $f$ be a meromorphic function and $g$ be an entire function such that $0 < \rho_{(\gamma,\beta)}[f](= \lambda_{(\gamma,\beta)}[f]) < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] < \infty$. Then

$$\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \leq \overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leq \min\Big\{\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big\}$$
$$\leq \max\Big\{\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big\}$$
$$\leq \sigma_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}.$$

**Remark 2.3.** In Theorem 2.11, if we will replace the conditions "$0 < \rho_{(\gamma,\beta)}[f] (= \lambda_{(\gamma,\beta)}[f]) < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] < \infty$" by "$0 < \lambda_{(\gamma,\beta)}[f] < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g](= \lambda_{(\gamma,\alpha)}[g]) < \infty$" respectively, then Theorem 2.11 remains valid with $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and $\tau_{(\alpha,\beta)}[f]_g$ replaced by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.

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