Subclass of Meromorphic Univalent Functions with Positive Coefficients Defined by Linear Operator

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Abstract

In this paper we introduce and study a new subclass $\sigma_p(\alpha, \beta)$ of meromorphically univalent functions defined in $E = \{z : z \in E \text{ and } 0 < |z| < 1\}$. We obtain coefficient inequalities, distortion theorems, radius of convexity, closure theorems and modified Hadamard products for the class.

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1 Introduction

Let Σ^* denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \ (a_k \geq 0) \tag{1.1}$$

which are analytic in the punctured unit disc $E = \{z : z \in E \text{ and } 0 < |z| < 1\}$ and let $g(z) \in \Sigma^*$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \tag{1.2}$$

Then the Hadamard product (or convolution) of f(z) and g (z) is given by

$$(f * g)\ (z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k. \tag{1.3}$$

A function $f(z) \in \Sigma^*$ is meromorphically starlike of order $\alpha(0 \leq \alpha < 1)$ if

$$-Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > \alpha,\ (z \in E). \tag{1.4}$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$. A function $f \in \Sigma^*$ is meromorphically convex of order α $(0 \leq \alpha < 1)$ if

$$-Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \alpha, (z \in E). \tag{1.5}$$

The class of such functions is denoted by $\Sigma_k^*(\alpha)$. The classes $\Sigma^*(\alpha)$ and $\Sigma_k^*(\alpha)$ were introduced and studied by Pommerenke [6], Miller [4] , Mogra et al [5] and others (see [1,3,7]). For a function $f(z) \in \Sigma^*$, Frasin and Darus [2] defined an operator $I^n : \Sigma^* \to \Sigma^*$ as follows.

$$I^0 f(z) = f(z)$$

$$I^2 f(z) = z(I^1 f(z))' + \frac{2}{z}$$

and for $n \in N = \{1, 2...\}$, we have

$$I^n f(z) = z(I^{n-1} f(z))' + \frac{2}{z}$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} k^n a_k z^k, (n \in N_0 = N \cup \{0\}, z \in E). \tag{1.6}$$

Now, we define a new subclass $\sigma_p(\alpha, \beta)$ of Σ^*.

Definition 1. *For $-1 \leq \alpha < 1$, and $\beta \geq 1$, we let $\sigma_p(\alpha, \beta)$ be the subclass of Σ^* consisting of functions of the form* (1.6) *and satisfying the analytic criterion*

$$Re\left\{\frac{I^{n+m} f(z)}{I^n f(z)} - \alpha\right\} > \beta \left|\frac{I^{n+m} f(z)}{I^n f(z)} - 1\right| \tag{1.7}$$

where $I^n f(z)$ in given by (1.6).

2 Coefficient Inequality

Theorem 2.1. A function f(z) of the form (1.6) is in $\sigma_p(\alpha, \beta)$ if

$$\sum_{k=1}^{\infty} k^n \left[(1+\beta)(k^m - 1)\ + 1 - \alpha\right] |a_k| \leq (1 - \alpha), -1 \leq \alpha < 1 \text{ and } \beta \geq 1$$

Proof. It suffices to show that

$$\beta\left|\frac{I^{n+m}f(z)}{I^n f(z)}-1\right|-Re\left\{\frac{I^{n+m}f(z)}{I^n f(z)}-1\right\}\leq 1-\alpha.$$

We have

$$\beta\left|\frac{I^{n+m}f(z)}{I^n f(z)}-1\right|-Re\left\{\frac{I^{n+m}f(z)}{I^n f(z)}-1\right\}$$
$$\leq(1+\beta)\left|\frac{I^{n+m}f(z)}{I^n f(z)}-1\right|$$
$$\leq\frac{(1+\beta)\sum_{k=1}^{\infty}k^n(k^m-1)\left|a_k\right|\ \left|z^k\right|}{\frac{1}{|z|}-\sum_{k=1}^{\infty}k^n\left|a_k\right|\left|z^k\right|}$$

Letting z $\to$ 1 along the real axis, we obtain

$$\frac{(1+\beta)\sum_{k=1}^{\infty}k^n(k^m-1)\left|a_k\right|}{1-\sum_{k=1}^{\infty}k^n\left|a_k\right|}$$

This last expression is bounded by $1-\alpha$ if

$$\sum_{k=1}^{\infty}k^n\left[(1+\beta)(k^m-1)\ +1-\alpha\right]\left|a_k\right|\leq(1-\alpha).$$

Corollary 2.1. Let the function f (z) defined by (1.6) be in the class $\sigma_p(\alpha,\beta)$ then

$$a_k\leq\frac{(1-\alpha)}{k^n\left[(1+\beta)(k^m-1)\ +1-\alpha\right]},k\geq 1. \tag{2.1}$$

Equality holds for the functions of the form

$$f_k(z)=\frac{1}{z}+\frac{(1-\alpha)}{k^n\left[(1+\beta)(k^m-1)\ +1-\alpha\right]}z^k. \tag{2.2}$$

3 Distortion Theorems

Theorem 3.1. Let the function f(z) defined by (1.6) be in the class $\sigma_p(\alpha,\beta)$. Then for $0<|z|=r<1$,

$$\frac{1}{r}-r\leq\ |f(z)|\ \leq\ \frac{1}{r}+r \tag{3.1}$$

with equality for the function

$$f(z)=\frac{1}{z}+z \tag{3.2}$$

Proof. Suppose $f(z)$ is in $\sigma_p(\alpha, \beta)$. In view of Theorem 2.1, we have

$$(1-\alpha)\ \sum_{k=1}^{\infty} a_k \leq k^n \left[(1+\beta)(k^m-1)\ +1-\alpha\right] \leq (1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty} a_k \leq 1$$

consequently, we obtain

$$\begin{aligned} |f(z)| &= \left|\frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k\right| \leq \left|\frac{1}{z}\right| + \sum_{k=1}^{\infty} a_k\, |z|^k \\ &\leq \frac{1}{r} + r\sum_{k=1}^{\infty} a_k \\ &\leq \frac{1}{r} + r. \\ \text{Also, } |f(z)| &= \left|\frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k\right| \geq \left|\frac{1}{z}\right| - \sum_{k=1}^{\infty} a_k\, |z|^k \\ &\geq \frac{1}{r} - r\sum_{k=1}^{\infty} a_k \\ &\geq \frac{1}{r} - r. \end{aligned}$$

Hence the results (3.1) follow.

Theorem 3.2. Let the function f (z) defined by (1.6) be in the class $\sigma_p(\alpha, \beta)$. Then for $0 < |z| = r < 1$,

$$\frac{1}{r^2} - 1 \leq\ |f'(z)|\ \leq\ \frac{1}{r^2} + 1.$$

The result is sharp, the external function being of the form (2.1).

Proof : From Theorem 2.1, we have

$$(1-\alpha)\ \sum_{k=1}^{\infty} k\ a_k \leq\ k^n \left[(1+\beta)(k^m-1)\ +1-\alpha\right] \leq (1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty} k\ a_k \leq\ 1$$

Consequently, we obtain

$$\begin{aligned}|f^{'}(z)| &\leq \frac{1}{r^2}+\sum_{k=1}^{\infty} k\ a_k r^{k-1}\\ &\leq \frac{1}{r^2}+\sum_{k=1}^{\infty} k\ a_k\\ &\leq \frac{1}{r^2}+1.\\ \text{Also } |f^{'}(z)|\ &\geq \frac{1}{r^2}-\sum_{k=1}^{\infty} k\ a_k r^{k-1}\\ &\geq \frac{1}{r^2}-\sum_{k=1}^{\infty} k\ a_k\\ &\geq \frac{1}{r^2}+1.\end{aligned}$$

This completes the proof.

4 Radius of convexity

Theorem 4.1. Let the function $f(z)\ \in\ \sigma_p(\alpha,\beta)$. Then $f(z)$ is meromorphically convex of order

$\delta\ (0\leq\delta<1)$ in $0<\ |z|\ <r$, where $\gamma\,(\alpha,\beta,\delta)=\ \inf_{n\geq 1}\ \left\{\frac{(1-\delta)\ k^n[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)\ k\ (k+2-\delta)}\right\}^{1/k+1}$

The result is sharp.

Proof : Let $f(z)$ is in $\sigma_p(\alpha,\beta)$. Then by Theorem 2.1, we have

$$\sum_{k=1}^{\infty} k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]\,|a_k|\leq(1-\alpha) \tag{4.1}$$

It is sufficient to show that

$$\left|2+\frac{zf^{''}(z)}{f^{'}(z)}\right|\leq(1-\delta)$$

for $|z|\ <\ \gamma=\gamma\ (\alpha,\beta,\delta)$, where $\gamma\ (\alpha,\beta,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf^{''}(z)}{f^{'}(z)}\right|=\left|\frac{\sum_{k=1}^{\infty}k(k+1)\ a_k z^{k-1}}{\frac{-1}{z^2}+\sum_{k=1}^{\infty}k\ a_k z^{k-1}}\right|\leq\frac{\sum_{k=1}^{\infty}k(k+1)a_k\,|z|^{k+1}}{1-\sum_{k=1}^{\infty}k\ a_k\,|z|^{k+1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{k=1}^{\infty}\frac{k(k+2-\delta)}{1-\delta}\ a_k\,|z|^{k+1}\leq 1 \tag{4.2}$$

By (4.1), it follow that (4.2) is true if

$$\frac{k\ (k+2-\delta)}{1-\delta}\ |z|^{k+1} \leq \frac{k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]}{1-\alpha},\ k\ \geq 1$$

or

$$|z|\ \leq\ \left\{\frac{(1-\delta)\ k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]}{(1-\alpha)\ k(k+2-\delta)}\right\}^{1/k+1} \tag{4.3}$$

Setting $|z|\ = \gamma(\alpha,\beta,\delta)$ in (4.3), the result follows.
The result is sharp for the function

$$f_k(z) = \frac{1}{z} + \frac{1-\alpha}{k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]} z^k,\ (k \geq 1).$$

5 Closure Theorems

Let the functions f_j (z) be defined, for j = 1, 2,... m, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k\ (a_{k,j} \geq 0) \tag{5.1}$$

Theorem 5.1. Let $f_j(z) \in \sigma_p(\alpha,\beta)$ $(j = 1,2,...,m)$. Then the function

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m} a_{k.j}\right) z^k \tag{5.2}$$

is in $\sigma_p(\alpha,\beta)$.
Proof. Since $f_j(z) \in \sigma_p(\alpha,\beta)$ $(j = 1,2,...,m)$, is follows from Theorem 2.1 that

$$\sum_{k=1}^{\infty} k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]\ a_{k.j} \leq (1-\alpha),$$

for every j = 1, 2 ...,m. Hence,

$$\sum_{k=1}^{\infty} k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]\ \left(\frac{1}{m}\sum_{j=1}^{m} a_{k,j}\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}(k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]\,a_{k,j}) \leq 1-\alpha.$$

From Theorem 2.1, if follows that $h(z) \in \sigma_p(\alpha,\beta)$.

Theorem 5.2. The class $\sigma_p(\alpha,\beta)$ is closed under convex linear combinations.
Proof. Let $f_j(z)$ $(j = 1, 2)$ defined by (5.2) be in the class $\sigma_p(\alpha,\beta)$. Then it is sufficient to show that

$$h(z) = \mu f_1(z) + (1-\mu)f_2(z)\ (0 \le \mu \le 1) \tag{5.3}$$

is in the class $\sigma_p(\alpha,\beta)$, since

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [\mu\, a_{k,1} + (1-\mu)a_{\ k,2}]\, z^k \tag{5.4}$$

then, we have from Theorem 2.1 that

$$\begin{aligned}&\sum_{k=1}^{\infty} k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]\ [\mu\, a_{k.1} + (1-\mu)a_{k,2}]\\ \le&\mu(1-\alpha) + (1-\mu)\ (1-\alpha)\\ =&1-\alpha\end{aligned}$$

so, $h(z) \in \sigma_p(\alpha,\beta)$.

Theorem 5.3. Let $f_0(z) = \frac{1}{z}$ and

$$f_k(z) = \frac{1}{z} + \frac{(1-\alpha)}{k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]} z^k (k \ge 1). \tag{5.5}$$

Then $f(z)$ is in the class $\sum_p(\alpha,\beta)$ if and only if can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z) \tag{5.6}$$

where $\mu_k \ge 0$ and $\sum_{k=0}^{\infty} \mu_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=0}^{\infty} \mu_k\ f_k(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\alpha)}{k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]} \mu_k z^k. \tag{5.7}$$

Then it follows that

$$\sum_{k=1}^{\infty} \frac{(1-\alpha)}{k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]}\ \mu_k \frac{k^n\,[(1+\beta)(k^m-1)\ +1-\alpha]}{1-\alpha}$$

$$= \sum_{k=1}^{\infty} \mu_k = 1 - \mu_0 \le 1$$

which implies that $f(z) \in \sigma_p(\alpha, \beta)$.
Conversely, assume that the function f (z) be in the class $\sigma_p(\alpha, \beta)$.
Then $a_k \leq \frac{(1-\alpha)}{k^n[(1+\beta)(k^m-1)+1-\alpha]}$
Setting $\mu_k = \frac{k^n[(1+\beta)(k^m-1)+1-\alpha]}{(1-\alpha)} a_k,\ (a_k \geq 1)$ and $\mu_0 = 1 - \sum_{k=0}^{\infty} \mu_k$
we can see that f (z) can be expressed in the form (5.6)
This completes the proof of theorem.

6 Modified Hadamard Products

For f_j (z) (j = 1,2) defined by (4.1), the modified Hadamard product of f_1(z) and f_2 (z) is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \tag{6.1}$$

Theorem 6.1. Let $f_j(z) \in \sum_p(\alpha, \beta)$ $(j = 1, 2)$. Then,

$$(f_1 * f_2(z) \in \sigma_p(\alpha, \beta) \tag{6.2}$$

The result is sharp for the function $f_j(z)$ (j = 1,2) given by

$$f_j(z) = \frac{1}{z} + z (j = 1, 2) \tag{6.3}$$

Proof. Using the technique for Schild and Silvarman [8] we need to find the largest ξ such that

$$\sum_{k=1}^{\infty} \frac{k^n [(1+\beta)(k^m - 1) \quad + 1 - \alpha]}{1 - \xi} \quad a_{k,1} \quad a_{k,2} \leq 1 \tag{6.4}$$

Since $f_j(z) \in \sum_p(\alpha, \beta)(j = 1, 2)$, we readily see that

$$\sum_{k=1}^{\infty} \frac{k^n [(1+\beta)(k^m - 1) \quad + (1 - \alpha)]}{(1 - \alpha)} \quad a_{k,1} \leq 1 \tag{6.5}$$

and

$$\sum_{k=1}^{\infty} \frac{k^n [(1+\beta)(k^m - 1) \quad + (1 - \alpha)]}{(1 - \alpha)} \quad a_{k,2} \leq 1 \tag{6.6}$$

By the Cauchy Schwarz inequality we have

(6.7) $$\sum_{k=1}^{\infty} \frac{k^n\left[(1+\beta)(k^m-1) \ \ +(1-\alpha)\right]}{(1-\alpha)} \ \sqrt{a_{k,1}, a_{k,2}} \leq 1$$

Thus it is sufficient to show that

(6.8)
$$\frac{k^n\left[(1+\beta)(k^m-1) \ +1-\alpha\right]}{1-\xi} a_{k,1} \ a_{k,2} \leq \frac{k^n\left[(1+\beta)(k^m-1) \ +1-\alpha\right]}{1-\alpha} \sqrt{a_{k,1} \ a_{k,2}}$$

or equivalently, that

(6.9) $$\sqrt{a_{k,1} \ a_{k,2}} \leq \frac{1-\xi}{1-\alpha}$$

Connecting with (6.7), it is sufficient to prove that

(6.10) $$\frac{1-\alpha}{k^n\left[(1+\beta)(k^m-1) \ +1-\alpha\right]} \ \leq \frac{1-\xi}{1-\alpha}.$$

It follows from (6.10) that

(6.11) $$\xi \leq 1-\frac{(1-\alpha)^2}{k^n\left[(1+\beta)(k^m-1) \ +1-\alpha\right]}.$$

Now defining the function $\phi(k)$ by

(6.12) $$\phi(k)=1-\frac{(1-\alpha)^2}{k^n\left[(1+\beta)(k^m-1) \ +1-\alpha\right]}$$

We see that $\phi(k)$ is an increasing function of k $(k \geq 1)$.
Therefore, we conclude that

(6.13) $$\xi \leq \phi \ (1) \ = \alpha$$

which evidently completes the proof of theorem.

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